The Physics of Ink Marbling

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http://people.csail.mit.edu/jaffer/Marbling

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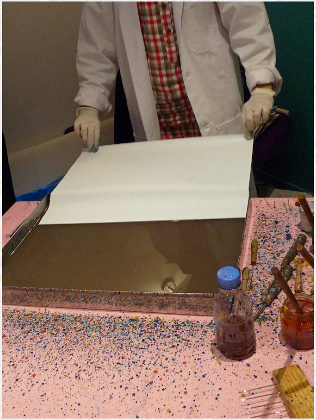


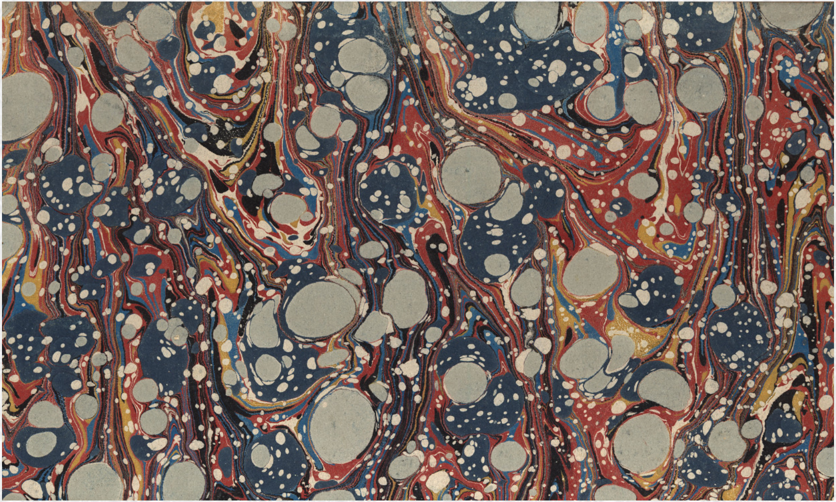
- Originated in East Asia in the 1100s or earlier.
  - ◇ Suminagashi founded by Jizemon Hiroba in 1151 in Japan.
- Appeared in Central Asia and the Islamic World in the 1400s.
  - ◇ Called Ebru in Turkish
- Spread to Europe in the 1500s.
- Necmeddin Okyay (1883-1976) of Istanbul Turkey is credited with being the first to marble floral designs.

لئن شكرتم لأزيدنكم ولئن كفرتم إن عذابي
من لم يشكر المنعم من المخلوقين لم يشكر الله
عز وجل فشكر الله ثم شكر الله على ما

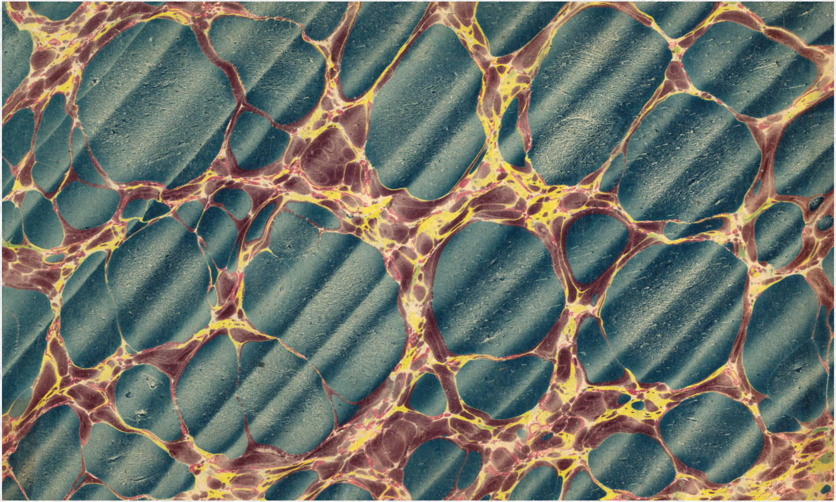
French curl - France 1735

nonpareil - Germany 1720-1770

England 1830

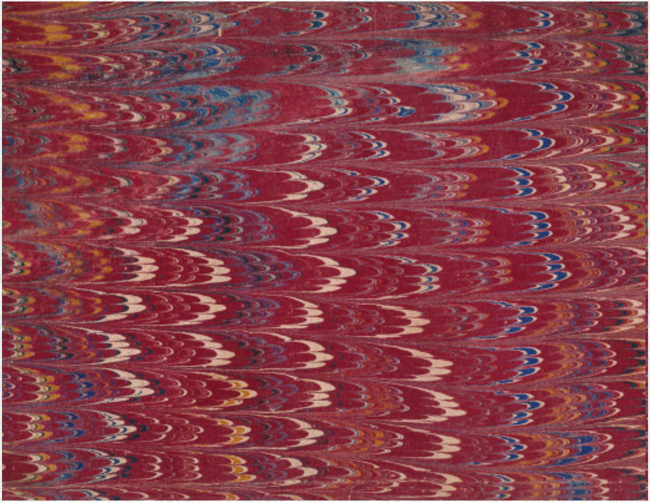


Spanish wave - Paris 1843



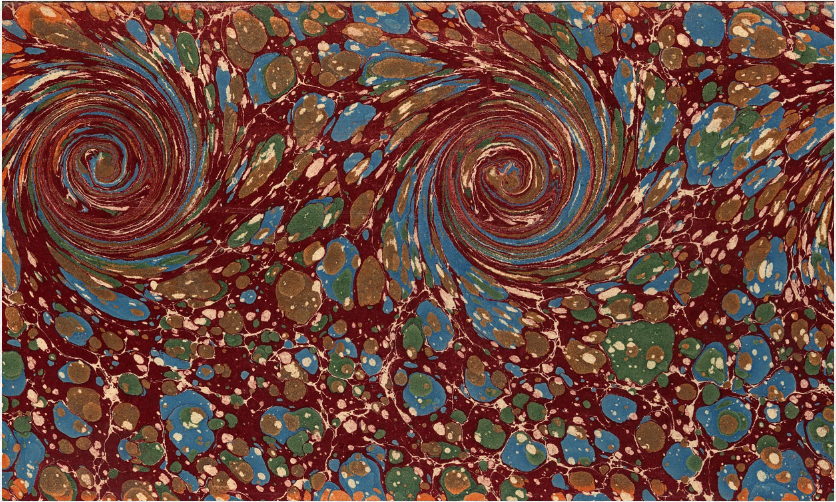
nonpareil - London 1847

double nonpareil - London 1872



bouquet - Vienna 1875

French curl - France 1880



bouquet - Germany 1899

A function $F : X \to Y$ between two topological spaces X and Y is called a homeomorphism if it has the following properties:

- F is a bijection (one-to-one and onto),
- F is continuous,
- its inverse function F^{-1} is continuous.

Each marbling operation F is a homeomorphism between a topological space and itself. F is a deformation of X, which is undone by its inverse F^{-1}

- F preserves all topological properties of X
- The composition of two homeomorphisms F_1 and F_2 is a homeomorphism $F_2 \circ F_1$ with inverse $F_1^{-1} \circ F_2^{-1}$.

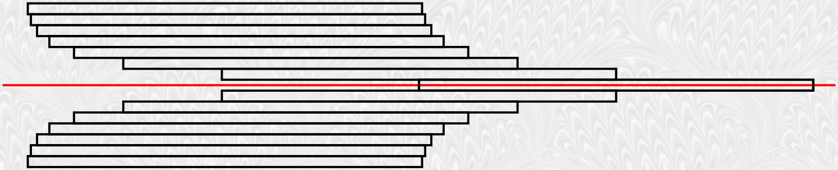
- *Forward Rendering*: Points on the boundaries of color regions are mapped with composite $F_n \circ \ldots \circ F_2 \circ F_1$, then the regions outlined by their mapped points are filled with their color; detail is preserved at all resolutions.

- *Reverse Rendering*: Each point on the screen is transformed with the inverse composite map $F_1^{-1} \circ F_2^{-1} \circ \ldots \circ F_n^{-1}$ to find the color of its original location.

- Rendering with either method is orders of magnitude faster than direct numerical simulation of the Navier-Stokes equations at many instants of time.

- Tank is arbitrarily large; straight stroke is arbitrarily long.
- Inks are incompressible Newtonian fluids; uniform viscosity.
- Flow is stable; interested only in initial and final positions.

→ Flow is uniformly parallel to the stroke line.
→ Laminar flow.
→ Can replace perpetual travel of point along line with simultaneous finite travel along line.



- Inertial forces insignificant compared with viscous forces.

→ Displacement proportional to velocity.

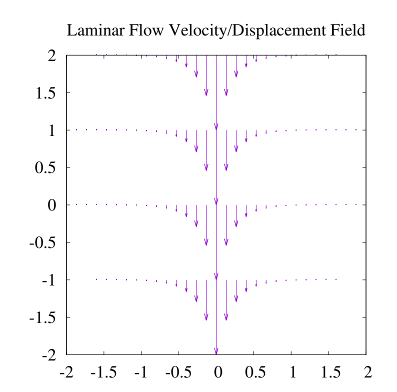
→ Each layer's travel is proportional to adjacent layer travel.

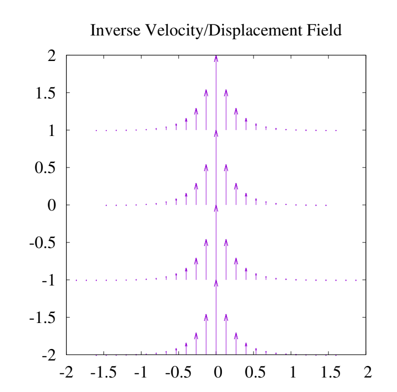
→ Separation of variables: displacement parallel to line depends only on perpendicular distance from line.

$$F_x = 0 \quad F_y = \frac{U}{\exp(|x|/L)} \quad L = \frac{\nu}{|U|}$$

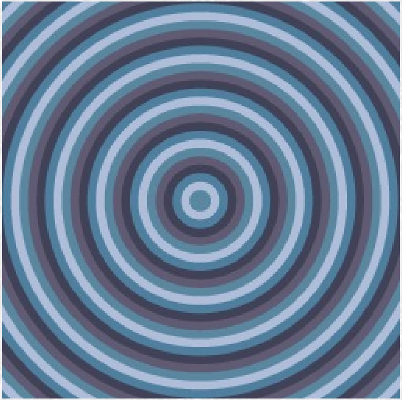
→ Parallel strokes are independent; displacements add linearly.

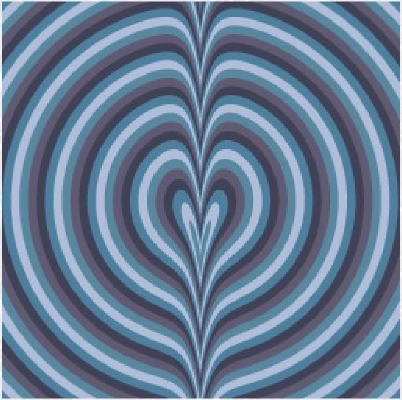
Linear Stroke Field



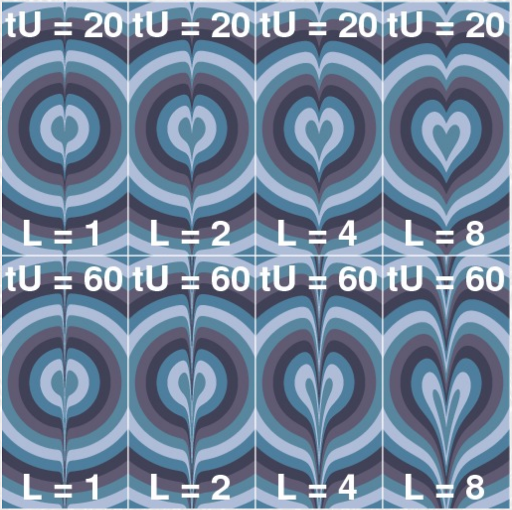


$$F_x = 0 \quad F_y = \frac{U}{\exp(|x|/L)} \quad L = \frac{\nu}{|U|}$$

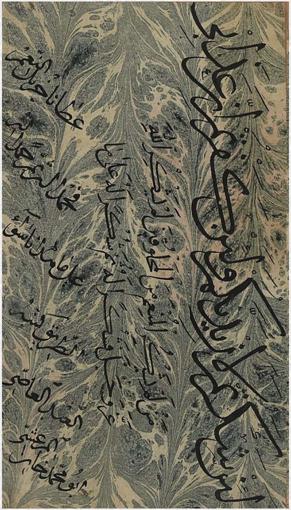




$$F_x = 0 \quad F_y = \frac{U}{\exp(|x|/L)} \quad L = \frac{\nu}{|U|}$$


tU = 20
tU = 20
tU = 20
tU = 20
L = 1
L = 2
L = 4
L = 8
tU = 60
tU = 60
tU = 60
tU = 60
L = 1
L = 2
L = 4
L = 8

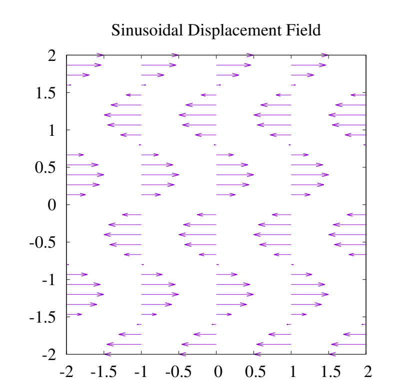
Persian Calligraphy Background

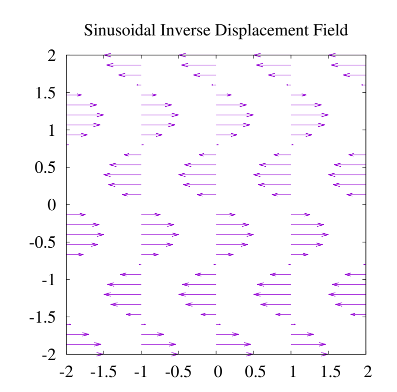


A Rake in Action

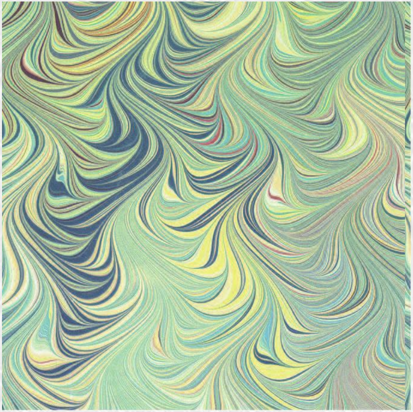
https://www.youtube.com/watch?v=igr6Znc8aek

Sinusoidal Displacement

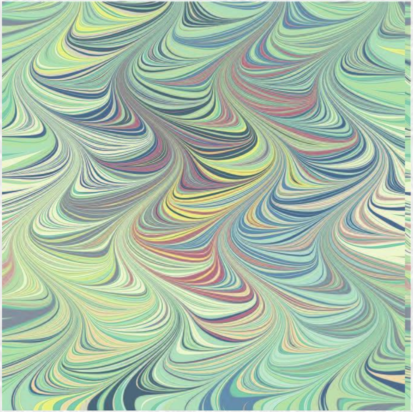




$$F_x = \zeta \sin \frac{2\pi y}{p} \qquad F_y = 0$$



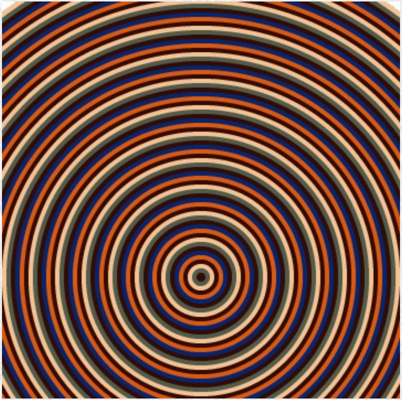
Diane Maurer-Mathison

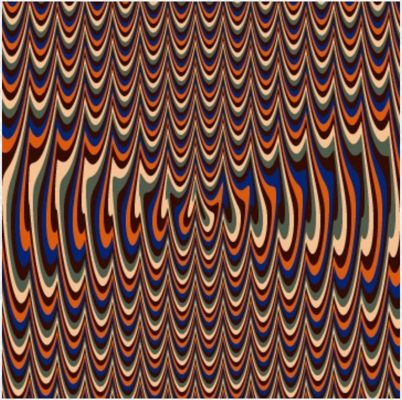


Mathematical Marbling

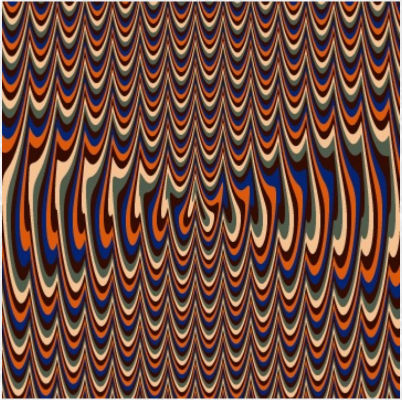
Physical Marbling

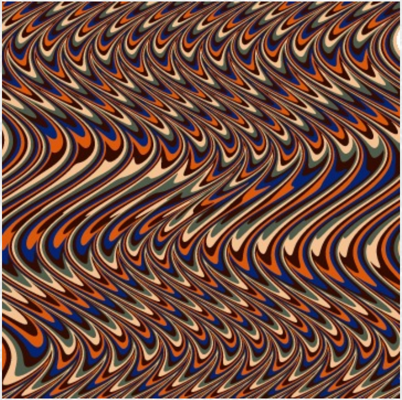
Mathematical Marbling



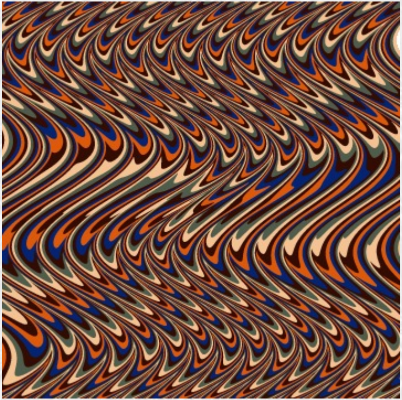


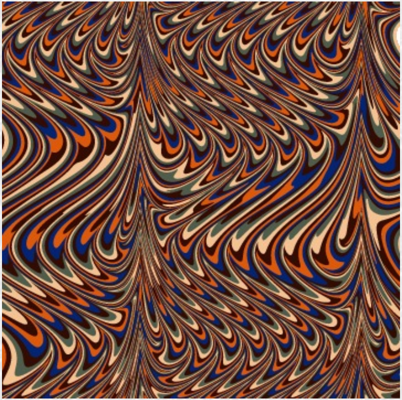
Horizontal Sinusoidal Displacement



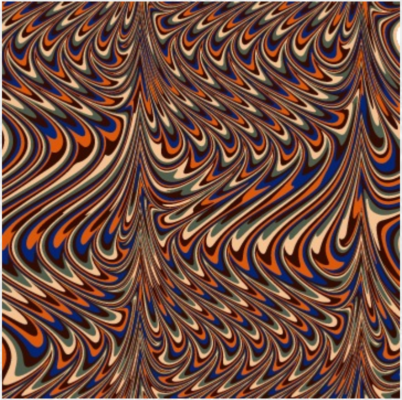


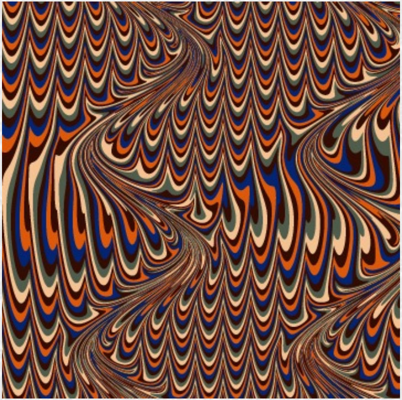
Stroke 2 upward



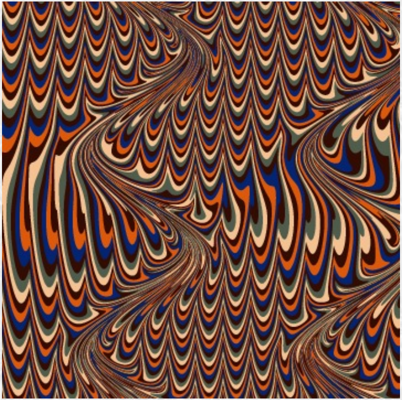


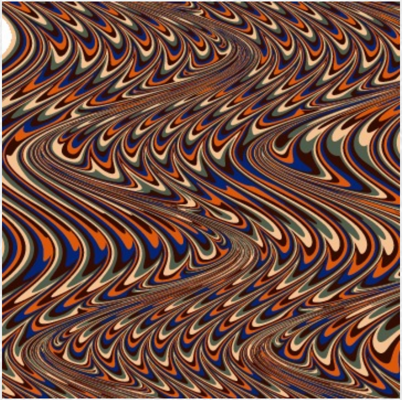
Undo Horizontal Displacement



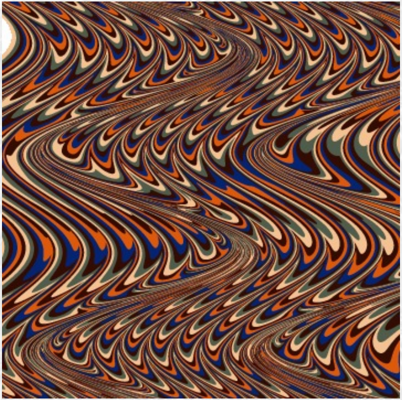


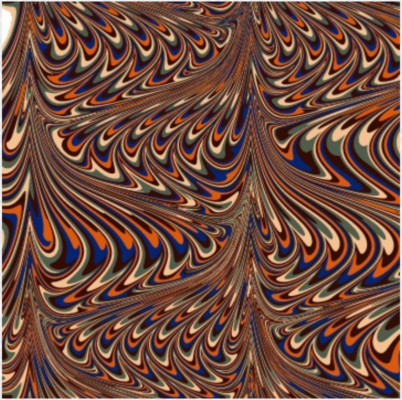
Opposite Horizontal Displacement



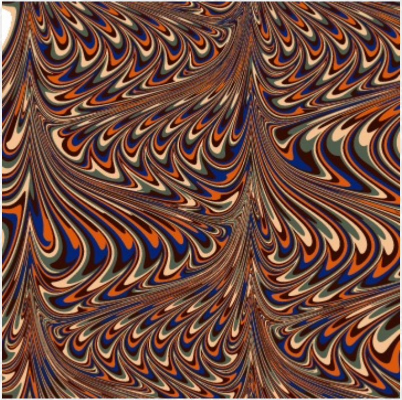


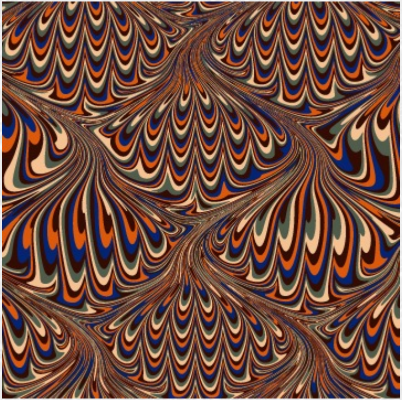
Stroke 2 Upward



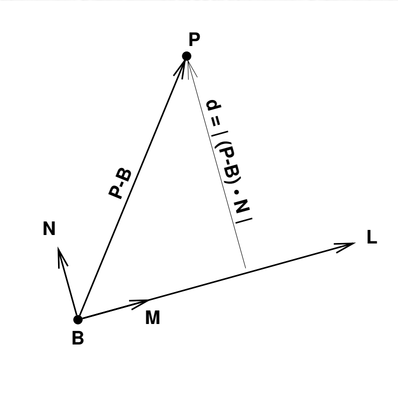


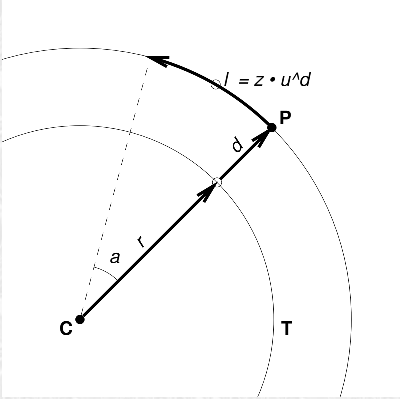
Undo Horizontal Displacement

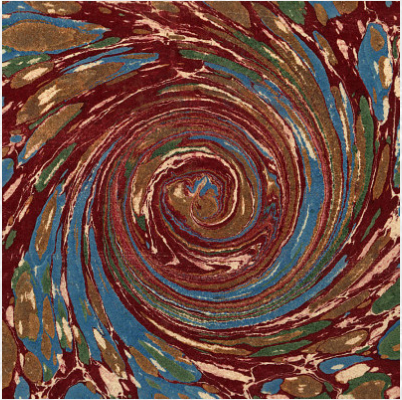




Bouquet Pattern


P
d = |(P-B) • N|
P-B
N
L
M
B

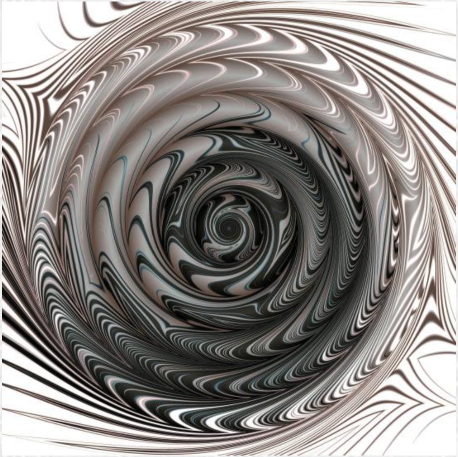

l = z • u^d
P
d
r
a
C
T



Physical Marbling

Mathematical Marbling

Circular Design with Transfer Effects



http://people.csail.mit.edu/jaffer/Marbling/Transfer-Effects

$\theta \propto \frac{e^{-r}}{r}$

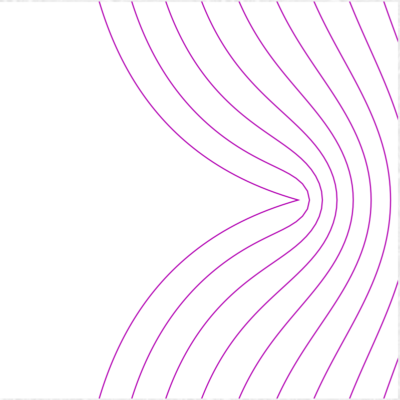
$\theta \propto \frac{1}{r}$

Short Stroke Marbling

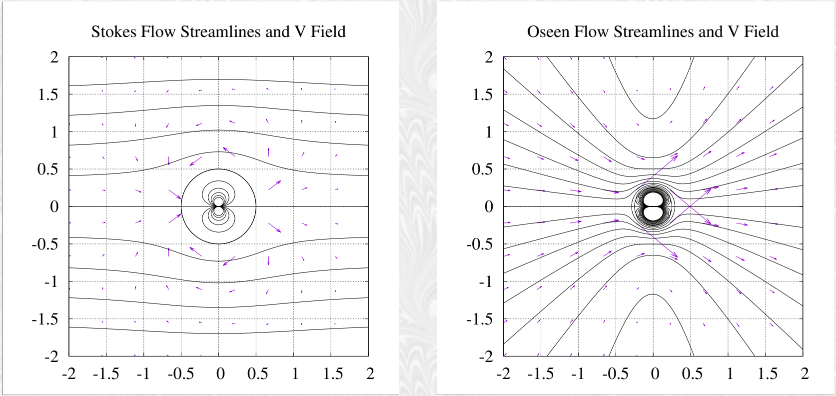
Çiçekli Ebru by Necmeddin Okyay

Floral Ebru by Necmeddin Okyay

Short Stroke Deformation



Stokes Flow versus Oseen Flow



Flows induced by circular disk moving to the right.

$$\nabla \cdot \vec{F}(r,\theta) = \frac{1}{r}\frac{\partial r F_r}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} = 0$$

$$\lim_{r\to\infty} \left\|\vec{F}(r,\theta)\right\| = 0 \qquad \vec{F}(0,0) = [U,0]$$

$$F_r(0,\theta) = U\cos\theta \qquad F_\theta(0,\theta) = -U\sin\theta$$

$$F_r(r,\theta) = U \cos\theta \exp\frac{-r}{L} \quad F_\theta(r,\theta) = \left(\frac{r}{L} - 1\right) U \sin\theta \exp\frac{-r}{L}$$

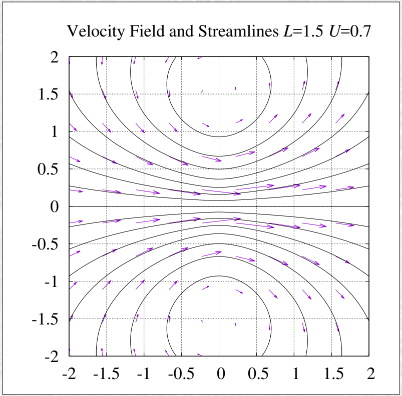
$$L = \nu/U \qquad r = \sqrt{x^2 + y^2}$$

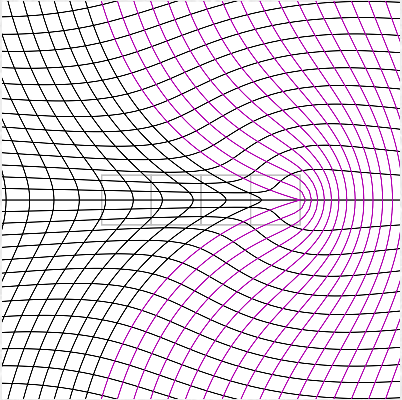
$$F_x = U\frac{rL - y^2}{rL\exp(r/L)} \quad F_y = U\frac{xy}{rL\exp(r/L)}$$

Stream Function ψ:

$$\psi(x,y) = \frac{Uy}{\exp(r/L)}$$

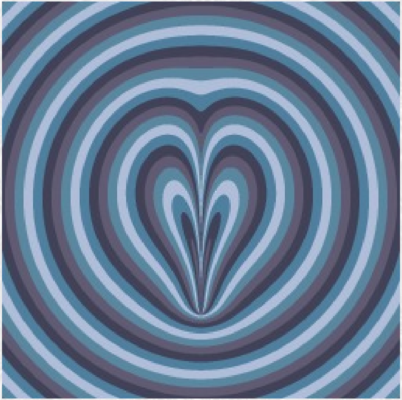
Pure Oseen Flow



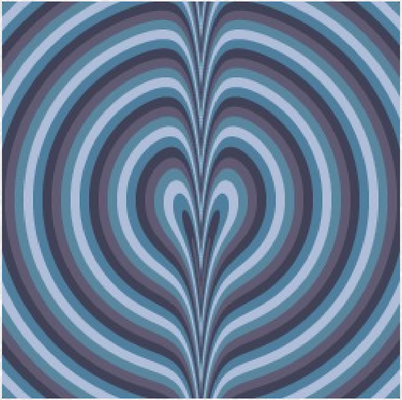


$$\psi(x,y) = \frac{Uy}{\exp(r/L)} \quad F_x = \frac{\partial \psi}{\partial y} \quad F_y = -\frac{\partial \psi}{\partial x}$$

Comparison with Line Stroke



Short Stroke

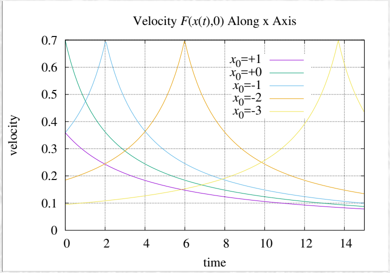


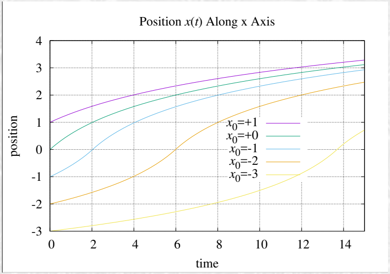
Line Stroke

Displacement Field for $y = 0$

$$\beta = \exp\frac{|x_0|}{L} - \frac{tU}{L}$$

$$x_f(t) = \begin{cases} L\ln\left(\exp\frac{x_0}{L} + \frac{tU}{L}\right), & \text{if } x_0 \geq 0; \\ -L\ln(\beta), & \text{if } \beta > 0; \\ L\ln(2-\beta), & \text{otherwise.} \end{cases}$$





Displacement Field for $y \neq 0$

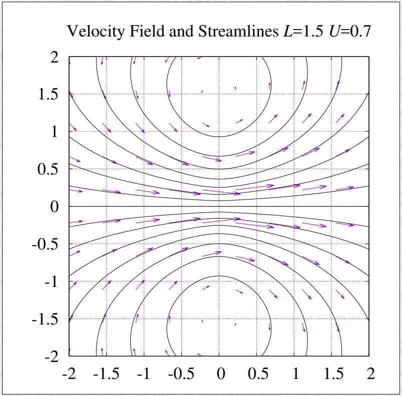
$$x_\psi(y) = \pm\sqrt{\left(L \ln \frac{Uy}{\psi}\right)^2 - y^2} \qquad y \neq 0$$

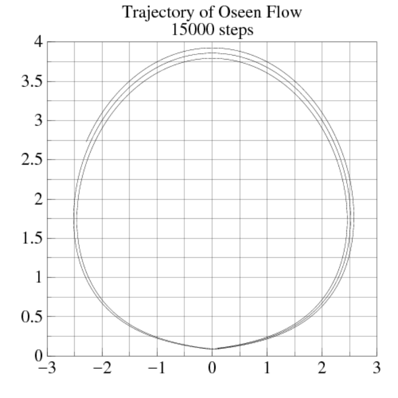
$$\begin{aligned} w(y)^2 &= F_x(x_\psi(y), y)^2 + F_y(x_\psi(y), y)^2 \\ &= U^2 \frac{[L^2 + y^2]/L^2 - 2y^2 \Big/ \left(L\sqrt{L^2 \ln(yU/\psi)^2 + y^2}\right)}{\exp\left(2\sqrt{L^2 \ln(yU/\psi)^2 + y^2}/L\right)} \end{aligned}$$

$$\zeta = \sqrt{L^2 \ln(yU/\psi)^2 + y^2}$$

$$\int \frac{dy}{w(y)} = \int \frac{L\zeta \exp(\zeta/L)\, dy}{U\sqrt{(L^2 + y^2)\,\zeta^2 - 2L\zeta y^2}}$$

Trajectory of Velocity Field

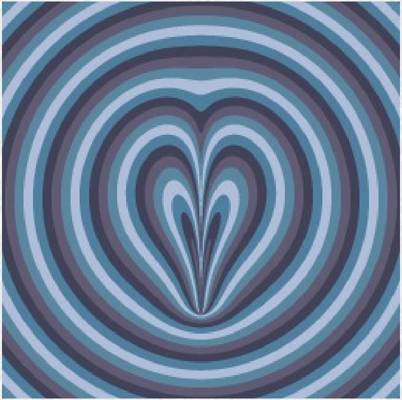




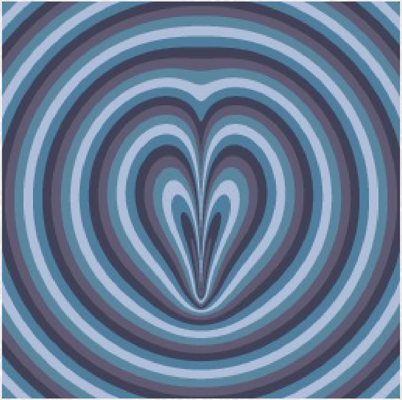
$$\psi(x, y) = \frac{Uy}{\exp(r/L)} \quad F_x = \frac{\partial \psi}{\partial y} \quad F_y = -\frac{\partial \psi}{\partial x}$$

Drive from Start

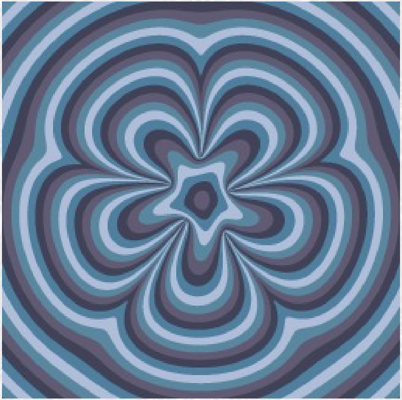
Drive from Midpoint



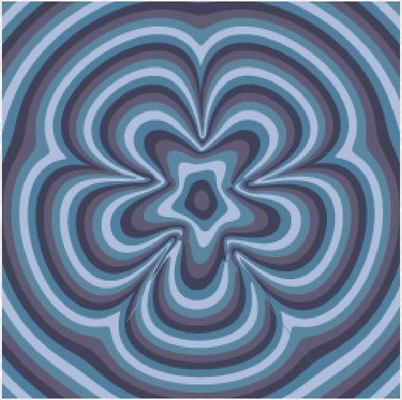
Drive from Start



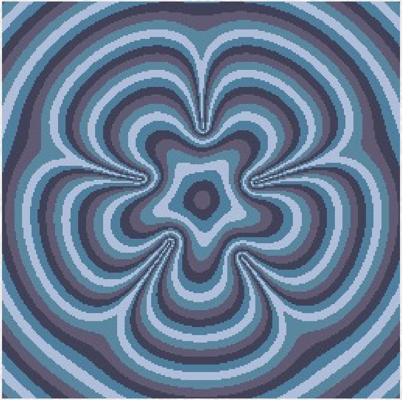
Drive from Midpoint



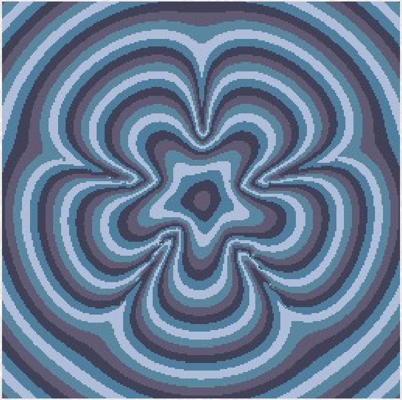
Drive from Start



Drive from Midpoint



Inverse Drive from End



Inverse Drive from Midpoint

Solid Marbling

- Except for transfer effects, the two-dimensional mathematical marbling techniques have straightforward three-dimensional analogs.
- Three dimensional short stroke solution:

$$F_r(r, \theta, \varphi) = U \cos\theta \exp \frac{-r}{L}$$

$$F_\theta(r, \theta, \varphi) = \left(\frac{r}{2L} - 1 \right) U \sin\theta \exp \frac{-r}{L}$$

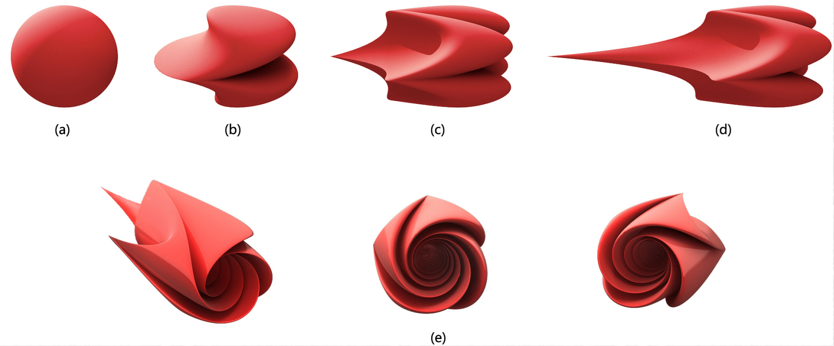
Oak Flooring

Birds Eye Maple

By Fluzwup - Own work, CC0,

https://commons.wikimedia.org/w/index.php?curid=22072699

Solid Marbled Teapot



- Are marbling homeomorphisms relevant to biological morphogenesis?

- Are there other solutions to the point Oseen flow differential equation?
- Does the point Oseen solution lead to an improved formula for drag from small spheres?
- Use contour-walking algorithm instead of riding gradients for short stroke.
- Model inks moved with puffs or streams of air.
- Folded paper transfer effects. http://marbleart.us/Moire.htm

- *Mathematical Marbling*, IEEE Computer Graphics and Applications, Nov.-Dec. 2012 (vol. 32 no. 6) pp 26-35
- Mathematical Marbling Video https://youtu.be/ZgVbIaKhC_4
- *Solid Mathematical Marbling*, IEEE Computer Graphics and Applications, vol. 37, no. 2, pp. 90-98, Mar.-Apr. 2017
- *Marbling-based creative modelling*, Vis. Comput., 33, 6-8 (June 2017), 913-923.
- *Oseen Flow in Ink Marbling*, arXiv:1702.02106 [physics.flu-dyn]
- http://people.csail.mit.edu/jaffer/Marbling