Skin-Friction and Forced Convection from an Isothermal Rough Plate

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Abstract

The current models for skin-friction drag from a plate with a rough surface are derived by analogy to flow within pipes having rough interiors. But this analogy fails at low Reynolds number (Re) flow rates because the boundary-layer must compress into the center of the pipe, while the plate boundary-layer is unbounded. A significant discrepancy from the pipe analogy at low Re was found by the rough plate experiments of Pimenta, Moffat, and Kays (1975) and by experiments conducted by the present author (2019).

An additional problem is that the roughness parameter in pipe analogy theories is tied to the drag measurements of flows inside Nikuradse’s assortment of sand-roughened pipes (1933). Prandtl and Schlichting (1934) caution that their theory applies only to sand-roughness. More useful would be a theory based on direct measurements of roughness profiles.

The present work derives the formula for a plate’s skin-friction drag coefficient given its root-mean-squared height-of-roughness, and Re bounds from its spatial frequency spectrum.

The present theory is in close agreement with the Mills-Hang (1983) theory, the Pimenta et al measurements, and the experiments conducted by the author over their respective Re ranges.

Keywords: rough turbulence; skin-friction; profile roughness; forced-convection; self-similarity

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1. **Introduction**

Skin-friction drag is the pressure opposing a flow due to viscous dissipation of the turbulence generated by that flow along a surface. The skin-friction drag of a rough surface is important to the fluid dynamics of vehicles and turbines. The related phenomenon of forced convection (from a rough surface) has application to modeling weather and the thermal behavior of buildings.

In 1934 Prandtl and Schlichting published *Das Widerstandsgesetz rauher Platten (The Resistance Law for Rough Plates)* [1], which brilliantly infers a relation for skin-friction resistance for rough plates from their analysis of Nikuradse’s measurements of sand glued inside pipes (“sand-roughness”). At the conclusion of the paper they write:

“The resistance law just derived for rough plates has chiefly validity for a very specific type of roughness, namely a smooth surface to which sand grains have been densely attached and where the Nikuradse pipe results have been taken as the basis.…

A single roughness parameter (the relative roughness) will in all likelihood no longer answer the purpose in continued investigations of the roughness problem.”

Hama [2] lays out three challenges to the pipe-plate analogy:

“Now there is no obvious reason why pipe flow and boundary-layer flow should be identical or even similar. First, a pressure gradient is essential for flow through a pipe but not along a plate. Second, pipe flow is confined and perforce uniform, while flow along a plate develops semi-freely and bears no such a priori guarantee of displaying similar velocity profiles at successive sections. Finally, the diameter and roughness size are the only geometrical dimensions of established flow in pipes, whereas at least three linear quantities are necessary to characterize the boundary layer.”

Hama attempts to confirm the pipe-plate analogy with measurements of wire screens affixed to flat plates, but concludes that confirmation is uncertain for all but large Reynolds numbers (Re).

The roughness in prior works [1] [2] [3] [4] is reported in Nikuradse’s sand-roughness metric $k_S$, the height of “coarse and tightly placed roughness elements such as for example coarse sand grains glued on the surface”.

A machined analog to sand-roughness, the plate tested by Pimenta, Moffat, and Kays [3] was composed of 11 layers of densely packed metal balls 1.27 mm in diameter “arranged such that the surface has a regular array of hemispherical roughness elements.” Pimenta et al write that, while the agreement of their data with the Prandtl-Schlichting plate model is “rather good” in the fully rough regime, their apparatus does not have the same behavior as “Nikuradse’s sand-grain pipe flows in the transition region”.

By taking the wake component of the velocity profile into account, Mills and Hang [4] created a model which improved the match to the Pimenta et al data at large Re, but did not address the discrepancy for low Re flows.

The “fully rough regime” refers to a mode of flow in pipes with rough interiors which has larger skin-friction coefficients than similar flow inside smooth pipes. For sand-roughness, the fully rough regime holds at flow rates larger than a transition value; and skin-friction coefficients are insensitive to flow rates larger than that transition value. This same behavior for rough plates is supported by the measurements from Pimenta et al. But at slower flow rates, Hama’s challenges remain unresolved.

These difficulties motivate a new theoretical analysis of flow along plate roughness.
2. Prior work

The pipe analogy having some success in the fully rough regime (large Re), there are several formulas to compare with those to be developed in the present work.

In *Boundary-layer theory* [5] Prandtl and Schlichting give formulas for fully rough (large Re) local ($c'_f$) and average ($c_f$) skin-friction coefficient for a rough flat plate in terms of its sand-roughness $k_S$ and the distance $x$ along the plate in the direction of flow up to its characteristic-length $L$:

$$c'_f = \left(2.87 + 1.58 \log_{10} \frac{x}{k_S}\right)^{-2.5} \quad x \leq L$$

$$c_f = \left(1.89 + 1.62 \log_{10} \frac{L}{k_S}\right)^{-2.5} \quad 10^2 < \frac{L}{k_S} < 10^6$$

Mills and Hang [4] present a formula (3) which is more accurate than the Prandtl-Schlichting formula (1) on the local skin-friction coefficient measurements from Pimenta et al [3]. Their local and average skin-friction coefficient formulas are:

$$C_f = \left(3.476 + 0.707 \ln \frac{x}{k_S}\right)^{-2.46} \quad x \leq L$$

$$C_D = \left(2.635 + 0.618 \ln \frac{L}{k_S}\right)^{-2.57} \quad 750 < \frac{L}{k_S} < 2750$$

White [6] gives a formula for fully rough local (not average) skin-friction coefficient:

$$C_f = \left(1.4 + 3.7 \log_{10} \frac{x}{k_S}\right)^{-2} \quad \frac{x}{k_S} > \frac{1000}{1000} \Rightarrow \frac{L}{k_S} > 1000$$

Prandtl and Schlichting [1], and Mills and Hang [4] both derived the average formulas (2) and (4) respectively from the local formulas by numerical integration:

$$C_D \left(\frac{L}{k_S}\right) = \frac{k_S}{L - L_P} \int_{L_P/k_S}^{L/k_S} C_f(x) \, dx$$

But the local formulas have a singularity at $x/k_S = 0$ and can go negative when $0 < x/k_S < 1$. The lower limit of integration should be greater than or equal to 1, but is not revealed in their papers. The average formulas are quite sensitive to the lower limit because the largest value of the local formulas occurs there, a region for which $x/k_S \gg 1$ does not apply.

For the Prandtl-Schlichting formula (1), with a lower bound of 1 (and initial $dx/k_S = 2$), integration of the local $c'_f$ is within ±0.5% of the average $c_f$ in formula (2) over the range 200 $< x/k_S < 200000$.

For the Mills-Hang formula (3), with a lower bound of 1.6 (and initial $dx/k_S = 0.01$), integration of the local $C_f$ is within ±0.5% of the average $C_D$ in formula (4) over the range 200 $< x/k_S < 200000$.

Churchill [7] compares eight formulas from various sources with the data from Pimenta et al [3] and doesn’t find any to be superior to the Mills-Hang formula (3). It does, however, have different methods for computing the average (mean) skin-friction $C_m$ from local $C_f$ for smooth and rough surfaces respectively:

$$C_m = C_f \frac{1 - 4.516 \sqrt{C_f}}{1 - 7.965 \sqrt{C_f} + 21.52 C_f}$$

$$C_m = C_f \frac{1 - 4.516 \sqrt{C_f}}{1 - 7.965 \sqrt{C_f}}$$

Prandtl and Schlichting [1], and Mills and Hang [4] incorrectly applied smooth plate averaging (6) to rough plates, but had no average measured data to compare with.

The utility of all of the formulas would be much greater if they were defined in terms of a roughness metric applicable to any surface. The most commonly used traceable roughness metrics are root-mean-squared (RMS) and arithmetic-mean height-of-roughness.

Afzal, Seena, and Bushra [8] fit 5.333 for the RMS to sand-roughness conversion factor and 6.45 for arithmetic-mean to sand-roughness in pipes.

Flack, Schultz, Barros, and Kim [9] measured skin-friction from grit-blasted surfaces in a duct. They write “The root-mean-square roughness height ($k_{rms}$) is shown to be most strongly correlated with the equivalent sand-roughness height ($k_S$) for the grit-blasted surfaces.”
3. Rough turbulence

Forced flow along a flat plate with a rough surface is different in character from forced flow along a smooth surface because the peaks of roughness protrude through what would otherwise be a viscous sub-layer adjacent to the plate. Lienhard and Lienhard [10] teach: “Even a small wall roughness can disrupt this thin sublayer, causing a large decrease in the thermal resistance (but also a large increase in the wall shear stress).”

If boundary-layers are disrupted by roughness, then it is better to consider the effect of roughness on flow as a series of disruptions than as a continuous boundary-layer. This is the approach taken by the present work.

In order to guarantee that the turbulence generated by flow along a rough surface is rough turbulent, there should be disruptive roughness along the whole length of the flow over the plate. The turbulent boundary-layer depth along a smooth plate increases as $x^{4/5}$, where $x$ is the distance from its leading edge. Roughness whose envelope height increases linearly with $x$ can disrupt a smooth boundary layer repeatedly along the whole plate.

Although vortexes can arise in two-dimensional systems, turbulence is a three-dimensional phenomena. Its random velocity fluctuations must be isotropic at all but the coarsest length scales. In order that the turbulence velocity differences from the bulk flow resulting from interactions with the plate roughness are evenly distributed, the roughness should be at least weakly isotropic; rotating the plate should not substantially affect the behavior of the system. A plate surface composed of parallel ridges and valleys would not meet this criterion.

4. Roughness

Let “profile roughness” be a function $z(x)$ where $0 \leq x \leq L$ is distance from the leading edge in the direction of flow. Functions being single-valued, neither tunnels nor overhangs are allowed.

Consider the roughness profile $z(x)$. Its mean height $\overline{z}$ and root-mean-squared height-of-roughness $\epsilon$ are:

$$\overline{z} = \frac{1}{L} \int_0^L z(x) \, dx$$
$$\epsilon = \frac{1}{L} \int_0^L [z(x) - \overline{z}]^2 \, dx$$

(8)

The roughness function $z(x)$ need not be continuous to be integrable. So local flow properties may not be well-defined anywhere. For these reasons, the present analysis focuses on the average skin-friction coefficient $f_C$.

A profile roughness function $z(x)$ has “self-similar roughness” with (integer) branching factor $n \geq 2$ if the RMS height-of-roughness of $z(x)$ over each (evenly divided) sub-interval $x_i < x < x_{i+1}$ for $0 \leq i < n$ at a succession of scales converging to zero. A consequence of this definition of self-similar roughness is that the ratio of the length of the interval $L = x_n - x_0$ to its RMS height-of-roughness $\epsilon$ will be invariant over its succession of scales converging to zero.

Self-similarity is of interest because the average skin-friction coefficient $f_C$ is a function of $L/\epsilon$. If the roughness is self-similar, then $L/\epsilon$ and $f_C(L/\epsilon)$ will be constant over the span of the roughness profile.

Of particular interest are self-similar roughness profiles that are permutations of the linear ramp $z(x) = x$ from $x = 0$ to $x = w$. Every elevation from 0 to $w$ occurs exactly once in a ramp-permutation. Because the RMS height-of-roughness calculation depends only on the $z$ values and not their relation to $x$, for all ramp-permutation profile roughness:

$$\epsilon = \sqrt{\frac{1}{w} \int_0^w [x - w/2]^2 \, dx} = \frac{w}{\sqrt{12}}$$

(9)

This raises the question of whether the linear ramp, which is the identity permutation of a linear ramp, can produce rough turbulence from a steady flow. A ramp doesn’t meet the isotropy requirement of Section 3 because a plate with a ramp profile in both directions has 0 profile roughness perpendicular to its gradient.

1 Note that $z(x_i)$ values contribute to the interval height-of-roughness, but not to any sub-interval height-of-roughness.
5. Self-similar profile roughness

**Figure 1** Gray-code profile roughness

The integer Gray-code sequence $G(x, w)$ with $0 \leq x < w = 2^i$ shown in Figure 1 has an RMS height-of-roughness $\epsilon = w/\sqrt{12}$ from equation (9) and is a self-similar roughness (bisected; $n = 2$) as seen by its recurrence:

$$G(x, w) = \begin{cases} 
    x, & \text{if } w = 1; \\
    w + G(w - 1 - (x \mod w), w/2), & \text{if } \lfloor x/w \rfloor = 1; \\
    G(x \mod w, w/2), & \text{otherwise.}
\end{cases} \tag{10}$$

**Figure 2** Wiggliest self-similar profile roughness

The integer sequence $W(x, w)$, which reverses direction at each bifurcation, produces the wiggliest possible self-similar roughness with $0 \leq x < w = 2^i$ and is shown in Figure 2. Being a ramp-permutation, it has an RMS height-of-roughness $\epsilon = w/\sqrt{12}$:

$$W(x, w) = \begin{cases} 
    x, & \text{if } w = 1; \\
    \lfloor x/w \rfloor w + W(w - 1 - (x \mod w), w/2), & \text{otherwise.}
\end{cases} \tag{11}$$

**Figure 3** Randomly reversing bifurcation profile roughness

Figure 3 shows an integer sequence generated by recursive-descent with random reversing at each bifurcation. Being a ramp-permutation, it has an RMS height-of-roughness $\epsilon = w/\sqrt{12}$; its roughness is approximately self-similar.

There are $2^w$ distinct self-similar ramp-permutation roughness profiles. Profiles chosen randomly have $w$ bits of sequence entropy ($w/2$ bits of Shannon entropy). There are only two distinct ramp profiles and only two distinct wiggliest profiles; their sequence entropies are 1 bit.
6. Roughness travel

![Figure 4: Travel along profile roughness](image)

In order to convert some of the flow to rough turbulence, parcels of fluid must move in directions not parallel to the bulk flow. Such movement would result from deflection of flow by the vertical spans of discrete profile roughness; the amount of turbulence produced would grow with the height-of-roughness.

For integer ramp-permutation roughness profile \( Y(x, w) \), the sum of the lengths of all horizontal segments is \( w - 1 = 2^\iota - 1 \). The sum of the absolute value of the length of each vertical segment is:

\[
\sum_{x=0}^{2^\iota-2} |Y(x, 2^\iota) - Y(x + 1, 2^\iota)|
\]  

(12)

If a parcel of fluid were to trace the ramp-permutation roughness profile \( Y(x, w) \) from \( x = 0 \) to \( x = w - 1 \), then \( w - 1 \) is the horizontal distance it would travel, while formula (12) is the vertical distance.

Shown in Figure 4 is the vertical to horizontal travel ratio versus \( \iota \), the base-2 logarithm of \( w \). The slope is 1/2 for the Gray-code and random-reversal cases, 0 for the linear ramp, and 2/3 for the wiggliest roughness \( W \). The vertical segments in Figure 2 are indeed longer than the vertical segments in Figures 1 and 3.

A wiggliest roughness profile \( W(x, w) \) is an extreme case; it reverses vertical direction at each increment of \( x \). For each wiggliest roughness profile there are many more random bifurcation roughness profiles. Going forward, \( W(x, w) \) will be excluded as an outlier.

From Figure 4, the vertical to horizontal travel ratio for the Gray-code and random-reversal sequences deviates little from:

\[
\frac{\iota}{2} = \frac{\log_2 w}{2}
\]  

(13)

Formula (13) needs to be normalized by the height-of-roughness \( \epsilon \). Turning to dimensional analysis, the argument to \( \log_2 \) must be dimensionless, involve \( \epsilon \), and be greater than 1 so that the logarithm is positive. Replacing \( w \) with \( L/\epsilon \), the ratio of vertical to horizontal travel is then:

\[
\frac{2}{\log_2(L/\epsilon)}
\]  

(14)

From equation (9), the maximum peak-to-valley height is \( \sqrt{12} \) times the RMS height-of-roughness \( \epsilon \). So the vertical to horizontal ratio should be scaled:

\[
\frac{1}{\sqrt{12} \log_2(L/\epsilon)} = \frac{1}{\sqrt{3} \log_2(L/\epsilon)}
\]  

(15)

Newberry and Savage [11] demonstrate that some self-similar systems which are modeled using continuous power-law probability distributions (such as the Pareto distribution) are better modeled using discrete power-law distributions.
The present work uses their idea in reverse. Conversion of flow into turbulence by contact with discrete self-similar roughness having been modeled above, the turbulence generation by a random self-similar roughness will be inferred using a random variable.

The base-2 logarithm is an artifact of the discrete roughness functions from Section 5. The analogous mean field theory is to treat \( Z = |\Delta Y| \) as a continuous random variable having a Pareto distribution where the frequency of \( Z \) is inversely proportional to \( Z^2 \):

\[
1 \sqrt{3} \int_{\varepsilon}^{L} \frac{\sqrt{3} Z}{Z^2} dZ = \frac{1}{\sqrt{3} \ln(L/\varepsilon)}
\]  

(16)

In this generalization of formula (15) to isotropic self-similar roughness, the surface height-of-roughness \( \varepsilon \), defined in (17), is used in formula (16) instead of the profile height-of-roughness \( \varepsilon \).

\[
\tau = \frac{1}{A} \int_{A} z dA \quad \varepsilon = \sqrt{\frac{1}{A} \int_{A} [z - \bar{z}]^2 dA}
\]  

(17)

Note that \( \varepsilon(Y \times Y) = \sqrt{2} \varepsilon(Y) \). However, the average path length a parcel takes through \( Y \times Y \) is similarly increased. So the ratio of \( L \) to height-of-roughness remains the same.

7. Skin-friction

For a rough flat surface subjected to a steady flow parallel to its surface which is sufficiently fast to generate rough turbulence, the skin-friction drag is the pressure opposing the flow due to viscous dissipation of that turbulence.

The skin-friction coefficient \( f_C \) is the ratio of the shear stress \( \tau \), which is primarily the skin-friction drag when \( L/\varepsilon \gg 1 \), to the flow’s kinetic energy density \( \rho v^2/2 \), where \( v \) is the bulk flow velocity and \( \rho \) is the fluid density.

\[
f_C = \frac{\tau}{\rho v^2/2}
\]  

(18)

\( f_C \) is dimensionless; both \( \tau \) and \( \rho v^2/2 \) have units of pressure, kg/(m·s²).

Scaling by the vertical to horizontal travel ratio from formula (16) converts a horizontal velocity \( v \) to a friction velocity \( v^* \), from which \( \tau \) is derived:

\[
v^* = \frac{v}{\sqrt{3} \ln(L/\varepsilon)} \quad \tau = \frac{\rho v^*}{2} = \frac{\rho v^2}{6 \ln^2(L/\varepsilon)} \quad \frac{L}{\varepsilon} \gg 1
\]  

(19)

For a given \( Re \) there must be some roughness ratio \( L/\varepsilon \) so large that it behaves as a smooth surface rather than a rough surface. The roughness Reynolds number \( Re_\varepsilon \) is the dimensionless friction velocity with characteristic-length \( L \):

\[
Re_\varepsilon = \frac{v^* \varepsilon}{\nu} = \frac{v}{\sqrt{3} \ln(L/\varepsilon)} \frac{\varepsilon}{\nu} = \frac{Re}{\sqrt{3} [L/\varepsilon] \ln(L/\varepsilon)}
\]  

(20)

At all \( L/\varepsilon \gg 1 \) scales, the crossover between mostly rough turbulent and mostly smooth turbulent flow will have the same \( Re_\varepsilon \) value. \( Re_\varepsilon \) is linear in velocity; the half-power \( Re_\varepsilon = \sqrt[4]{2} \). Combining formula (20) with \( Re_\varepsilon = \sqrt[4]{2} \) relates \( Re \) to \( L/\varepsilon \) at the crossover:

\[
Re = \sqrt{\frac{3}{2} \frac{L}{\varepsilon} \ln \frac{L}{\varepsilon}}
\]  

(21)

Combining equations (18) and (19) produces a formula for \( f_C \) dependent only on \( L/\varepsilon \), with the \( Re \) lower bound from equation (21):

\[
f_C = \frac{1}{3 \ln^2(L/\varepsilon)} \quad \frac{L}{\varepsilon} \gg 1 \quad Re > \sqrt{\frac{3}{2} \frac{L}{\varepsilon} \ln \frac{L}{\varepsilon}}
\]  

(22)

---

2 Prandtl and Schlichting [1] calculate \( \tau \) as \( \tau = \rho v^2 \), not \( \tau = \rho v^*^2/2 \). This results in Prandtl-Schlichting \( c_f \) being twice \( f_C \) of the present work. \( f_C \) is scaled correctly for the Chilton-Colburn analogy (46); \( c_f \) is not.
Crossover equation (21) is an approximate boundary because, with the great variety of self-similar roughness profiles, the peaks which disrupt the boundary layer can occur in many positions. However, knowing that the smooth turbulent skin-friction coefficient equals the rough turbulent coefficient near the boundary, and knowing the boundary’s aggregate relation to $Re$, suggests that the smooth turbulent skin-friction coefficient can be inferred from formulas (20) and (22) with $Re_\varepsilon = 1$. It’s not as simple as half of $f_C(L/(2\varepsilon))$ because $f_C$ is a nonlinear function:

$$f_C = \frac{\sqrt{2}}{3} \left[ \frac{1}{3} \ln\left(\frac{L}{e\varepsilon}\right) \right]$$

$$Re = \frac{\sqrt{3} L}{\varepsilon} \ln\left(\frac{L}{\varepsilon}\right)$$

Formula (23) can be solved using the Lambert $W$ function, the inverse of $x \ln x$:

$$f_C = \frac{\sqrt{2}}{3} \ln^{-2} \left( \frac{W_0(Re/\sqrt{3})}{e} \right) = \frac{\sqrt{2}}{3} \left[ \ln \left( \frac{W_0(Re/\sqrt{3})}{e} \right) - 1 \right]$$

**Figure 5** $f_C$ versus $Re$ of smooth plate

Churchill [7] compares smooth turbulent formulas from various sources with measured data from Smith and Walker (1959), and Spalding and Chi (1964). The measured data, formula values, and the “present work” smooth turbulence formula (23) are shown in Figure 5. The “present work” formula (23) has 0.75% RMS percentage deviation from the “Smith-Walker and Spalding-Chi measurements”; the second closest is “Churchill Eq. 24” with 1.5%.

Eliminating $L/\varepsilon$ from formula (23), the implicit equation is:

$$\ln \left( \frac{Re \sqrt{f_C}}{\sqrt{3} f_C + \sqrt{2}} \right) = \frac{\sqrt{3} f_C + \sqrt{2}}{\sqrt{3} f_C}$$

(25)
8. Spectral roughness
There is a deep connection between the discrete Fourier transform \( X_j \) of a discrete roughness profile \( Y(x, w) \) and its RMS height-of-roughness \( \epsilon \):

\[
X_j = \sum_{x=0}^{w-1} Y(x, w) e^{-2\pi i x / w} \quad \epsilon = \sqrt{\frac{1}{w} \sum_{j=1}^{w-1} |X_j|^2}
\]

Note that the constant term, \( X_0 \) (which is the mean value of \( Y \)), is not included in the sum for \( \epsilon \).

Because each discrete Fourier coefficient is a Riemann sum over rectangles sampled at their left corners, the discrete Fourier transform underestimates the area under the integrals and introduces a phase shift. The compensated spectrum is:

\[
S_j = X_j \frac{w}{2j} \sin \frac{\pi j}{w} \exp \frac{\pi i j}{w} \quad 0 < j \leq \frac{w}{2}
\]

Because of sampling noise, such compensation is not warranted for physical roughness measurements.

\[
\epsilon / w = 12^{-1/2} \approx 0.289
\]

**Figure 6** wiggliest and ramp spectra
Figure 6 shows the \( |S_j|/w \) spectra of the wiggliest (in Figure 2) and ramp profiles. The amplitudes of the wiggliest and ramp spectra are identical; the difference in their profiles is the phase of the complex-valued \( S_j \); the ramp phases are all 90° or all −90°, while the wiggliest phases are an unequal mixture of +90° and −90°.

Figure 7 shows the \( |S_j|/w \) spectrum of the Gray-code roughness profile (from Figure 1) and the average of the amplitudes of the Fourier spectra of 187 instances of 128-point random-bifurcation profiles. The similarity between these spectra indicates that Gray-code roughness is representative of self-similar ramp-permutation roughness. The phases of the Gray-code spectrum are evenly distributed among 0°, 180°, 63.43°, and 116.57°.

The similarity length scale is proportional to \( w/j \). In order to show the spectral structure of roughness across scale, \( j S_j/w^2 \) normalizes the spectrum relative to \( S_{j_P} \), where \( j_P \) is the index of the largest \( |X_j| \).

\[
\epsilon / w = 12^{-1/2} \approx 0.289
\]

**Figure 7** Gray and random spectra

**Figure 8** wiggliest and ramp spectra

**Figure 9** Gray and random spectra
Figures 8, 9, and 11 show \( j S_j/w^2 \) spectra corresponding to the \( S_j/w \) spectra in Figures 6, 7, and 10 respectively. The traces for the ramp, wiggliest, and Gray-code profiles in Figures 8 and 9 show that these profiles are self-similar; the amplitudes repeat across the spatial frequencies. It is remarkable that the ramp and wiggliest profiles in Figure 8 have identical spectral amplitudes given how differently these two profiles appear. But both are ramp-permutation outliers not covered by the present theory.

For repetitions with a small period, \( L_P/\varepsilon \) may be less than 3, for which \( \ln(L_P/\varepsilon) \) behaves badly. Figure 12 shows that 3.14 \((L_P/\varepsilon)^{8/7}\) is a good approximation to \( \text{Re}_P \) over a wide range of \( L_P/\varepsilon \).
Figure 12  Reₚ smooth-rough turbulent threshold

Would like to show that Reₚ transition occurs where \( \delta_2 (L \text{ Re}/L) = \varepsilon \).

\[
Re_x = \sqrt{\frac{3}{2}} \frac{L \text{ Re}}{\varepsilon} \ln \frac{L \text{ Re}}{\varepsilon} \quad Re_P = \left( \frac{L}{L \text{ Re}} \right) Re_x
\]

\[
\frac{L \text{ Re}}{\varepsilon} = W_0 \left( Re_x \sqrt{\frac{2}{3}} \right)
\]

Continuing the exact analysis of smooth turbulent skin-friction from Section 7, where \( \varepsilon \) is free, the momentum thickness at Reₚ is:

\[
\delta_2 = \frac{L}{9 \sqrt{3} \text{ Re} \ varepsilon} \quad Re_x = \sqrt{\frac{1}{3}} \frac{L}{\varepsilon} \ln \frac{L}{\varepsilon} \quad \delta_2 = \frac{L}{9} \frac{W_0 (Re_x/\sqrt{3})}{Re \sqrt{3}}
\]

From Schlichting [5], the momentum thickness of laminar and smooth turbulent boundary-layers are \( \delta_2 = 0.664 x \text{ Re}^{-1/2} \) and \( \delta_2 = 0.036 x \text{ Re}^{-1/5} \) respectively. Equivalently, the laminar and smooth turbulent boundary layer thicknesses are:

\[
\delta_2 = 0.664 \sqrt{\text{ Re} x} \frac{L}{\text{ Re} x} \quad \delta_2 = 0.036 \text{ Re}^{4/5} \frac{L}{\text{ Re} x}
\]

Figure 13 shows that smooth turbulent \( \delta_2 \) from equation (30) and \( \delta_2 = 0.036 \text{ Re}^{4/5} L/\text{ Re} \) from equation (31) are close at the laminar to turbulent crossover. These boundary layers would have equal thickness at:

\[
0.664 \sqrt{\text{ Re} x} = 0.036 \text{ Re}^{4/5} x \quad Re_x = \left( \frac{0.664}{0.036} \right)^{10/3} \approx 16579
\]

Figure 13  momentum thickness
Figure 13 shows the momentum thickness of laminar and smooth turbulent flows over a 1 m long plate with Re = 1 × 10^6. These two curves divide the graph into four regions. When the point at \((\text{Re}_x, \varepsilon)\) is in regions I or IV, then it wouldn’t significantly disrupt laminar flow; so the flow over the plate will be laminar. When \((\text{Re}_x, \varepsilon)\) is in region II, then its height is sufficient to disrupt both laminar and smooth turbulent flow; so the flow over the plate will be rough turbulent.

When \((\text{Re}_x, \varepsilon)\) is in region III, then the roughness \(\varepsilon\) is sufficient to disrupt laminar flow, but not high enough to disrupt smooth turbulent flow; so the flow over the plate will be smooth turbulent. With \(\delta_2 = \varepsilon\) and \(x = L_P\), solve for the laminar upper-bound \(\text{Re}_\lambda\) and smooth turbulent upper-bound \(\text{Re}_\sigma\):

\[
\text{Re}_\lambda = \left(\frac{0.664}{\varepsilon}\right)^2 L_P L
\]

\[
\text{Re}_\sigma = \left(\frac{0.036}{\varepsilon}\right)^5 \frac{L^4}{L_P L}
\]

Equating \(\text{Re}_\lambda\) and \(\text{Re}_\sigma\):

\[
\frac{L_P}{\varepsilon} = \left(\frac{0.664^2}{0.036^5}\right)^{1/3} \approx 193.9
\]

When \(L_P/\varepsilon < 193\), the flow over the whole plate transitions directly from laminar to rough turbulence at \(\text{Re}_\lambda\). When \(L_P/\varepsilon > 194\), then \(\text{Re}_\sigma > \text{Re}_\lambda\) and the flow over the whole plate transitions from laminar to smooth turbulence around \(\text{Re}_\lambda\), and to rough turbulence around \(\text{Re}_\sigma\). Lienhard [12] models a gradual transition from laminar to smooth turbulence for smooth plates. But for periodic roughness with \(L_P \ll L\), the roughness in the leading patch controls the flow over the rest of the plate (which has the same height-of-roughness).

Note that \(\text{Re}_\sigma\) and \(\text{Re}_\lambda\) in formulas (33,34) depend only on the geometry of the plate. If all lengths of a plate are scaled, \(\text{Re}_\sigma\) and \(\text{Re}_\lambda\) don’t change. Similarly, local \(\text{Re}_x = x \text{Re}/L\) will be independent of scaling.

Because the boundary-layer depth shrinks with increasing free-stream velocity, rough turbulence is generally the case for rough plates at large \(\text{Re}\). Surprisingly, smooth turbulence can also appear at large \(\text{Re}\).

10. **Periodic smoothness**

In the discussion appended to Hama [2], Dr. S. F. Hoerner points out:

“...there is hardly any physical or natural surface condition which is truly equal or similar to sand roughness. One conclusion from sand experiments has been the expectation that from then on every rough surface should have a constant terminal drag coefficient. As early as 1924, it has been demonstrated (by Hopf and Fromm in Zeitschr. Angew. Math. Mech., 1923:329-339) that certain types of roughness do not show any constant coefficients.”

Consider a smooth flat plate etched with a square grid of narrow grooves which is subjected to a moderate flow parallel to its surface. The boundary-layer near the leading edge will be disrupted by the grooves perpendicular to the flow. If, at the scale of a smooth (square) patch, the smooth turbulent boundary-layer depth at the trailing edge of the patch has a momentum thickness greater than \(\varepsilon/2\), then every patch from that point to the trailing edge of the plate will have smooth turbulent \(C(\sigma) \propto \text{Re}^{-1/5}\).

Let \(L_T\) be the length of a single patch in the direction of flow. The length variables involved in plate convection are \(L\), \(L_T\), \(L_P\), \(L_P - L_T\), \(x\), \(\varepsilon\), and \(\delta_2\). If the smooth turbulent momentum thickness of the patch is greater than \(\varepsilon/2\) and crosses the \(L_P - L_T\) gap, then the downstream flow will be smooth turbulent. Addressing this gap are dimensionless factors forcing the rough turbulence upper-bound \(\text{Re}_\ell\) towards 0 as the gap shrinks.

\[
\text{Re}_\ell = \frac{L_P}{L_T} \frac{L_P - L_T}{0.036 L_P} \frac{0.036 L_P}{L} \frac{\varepsilon/4}{L}\left(\frac{2\varepsilon}{0.036 L_P}\right)^{5/4} = \frac{L/8}{0.036 L_T} \left(\frac{2\varepsilon}{0.036 L_P}\right)^{9/4}
\]

All surfaces are rough. A patch behaves smoothly if its smooth turbulent boundary-layer isn’t disrupted by the roughness \(\varepsilon_q\) of the patch. Treating the patch as a plate with characteristic-length \(L_T\), repeat length
and roughness $\varepsilon_q$, equation (34) yields $\text{Re}_\sigma = (L_T/L_q) \left(0.036 L_q/\varepsilon_q\right)^5$. In order for the smooth turbulent boundary-layer to flow over the roughness $\varepsilon_q$, $\text{Re}_\ell$ must be less than $\text{Re}_\sigma$ converted to characteristic-length $L$ by a factor of $L/L_T$:

$$\text{Re}_\ell < \frac{L}{L_q} \left[ \frac{0.036 L_q}{\varepsilon_q} \right]^5$$

(37)

The local flow transitions from rough to smooth turbulence at $\text{Re}_x = \text{Re}_\ell$ only when formula (37) is satisfied. Expressing this as a constraint on $\varepsilon_q$, smooth turbulence will emerge at $\text{Re}_x > \text{Re}_\ell$ only if:

$$\varepsilon_q < \frac{L_T}{71.3} \left( \frac{L_T}{L_q} \right)^{0.2} \left( \frac{L_P}{\varepsilon} \right)^{0.45}$$

(38)

11. **Local skin-friction**

While the local skin-friction coefficient $f_c$ is not well defined for self-similar roughness, it is tractable for periodic roughness by avoiding the singularity at the leading edge of the plate. This is done by using $x_S = L_P/\varepsilon = 5.333 L_P/k_S$ instead of $0$ as the lower bound of integration.

The average skin-friction drag $f_C$ of a continuous boundary-layer is calculated from local drag coefficient $f_c$ by formula (6), which can be rewritten as:

$$f_C(x) = \left( \int_{x_S}^x f_c(x) \frac{dx}{x - x_S} \right) / f_c(x)$$

(39)

But periodic roughness disrupts the boundary layer repeatedly. Instead of $f_c$ accruing linearly, it should go as the square. This leads to a formula for $f_C(x)$ given $f_c(x)$:

$$\frac{f_C(x)}{f_c(x)} = \left( \left( \int_{x_S}^x f_c(x) \frac{dx}{x - x_S} \right) / f_c(x) \right)^2$$

(40)

At $x/k_S = 750$, formula (40) has values in excess of 20% larger than formula (6) used by Prandtl and Schlichting [1] and Mills and Hang [4]. The average skin-friction coefficient was of minor importance in those works, and wasn’t compared with measured data.

Applying formula (40) to Mills-Hang local formula (3) yields:

$$C_D^2 / C_f$$

(41)

The transform for local friction $f_c$ given average friction $f_C$ is:

$$\frac{f_c(x)}{f_C(x)} = \left( \int_{x_S}^x f_c(x) \frac{dx}{x - x_S} \right)^2$$

(42)

Applying formula (42) to the present work’s average skin-friction formula (22) produces a formula for local skin-friction coefficient where $x_S = L_P/\varepsilon$ and $x_S \leq x \leq L/\varepsilon$:

$$f_c(x) = \frac{\left( \ln x + 2 \left( x_S/x - 1 \right) \right)^2}{3 \ln^4 x} \quad \frac{L}{\varepsilon} \geq x > x_S = \frac{L_P}{\varepsilon} \geq 1$$

(43)

The local skin-friction coefficient for smooth turbulent flow can be derived from equations (24) and (44):

$$f_c = \frac{d(\text{Re} f_{C_S} \text{Re})}{d\text{Re}} = \frac{\sqrt{3}}{3} \left( W_0(x) + 1 \right) \ln \left( W_0(x)/e \right) - 2 \left( W_0(x) + 1 \right) \ln^3 \left( W_0(x)/e \right) \quad \text{Re}_x = \frac{\text{Re} \sqrt{3}}{3} > 2 e$$

(45)
12. Forced convection

The Chilton-Colburn analogy (46) relates friction factors to turbulent forced convective heat transfer, expressed as the average Nusselt number ($\overline{Nu}$), in terms of the flow’s Reynolds number (Re) and the fluid’s Prandtl number (Pr).

$$\overline{Nu} = \frac{fC_2}{2} \frac{Re}{Pr^{1/3}}$$  \hspace{1cm} (46)

Combining equation (46) with equation (22) produces a formula for forced convection:

$$\overline{Nu} = \frac{Re Pr^{1/3}}{6 \ln^2 \left( \frac{L}{\varepsilon} \right)}$$ \hspace{1cm} (47)

13. Experimental results

What can be learned from physical measurements of skin-friction or forced-convection from rough plates?

If measurements of plates with isotropic self-similar roughness were close to the predictions of skin-friction equation (22) or convection equation (47), then those measurements would support the present formulas for self-similar roughness only. If measurements (over a range of Re values) of diverse plates with periodic isotropic roughness were close to the present predictions, then that would be compelling evidence that the present formulas reflect a physical law of turbulent flow along isotropic surface roughness.

The Convection Machine [13] bi-level plate surface was composed of (676) square 8.28 mm × 8.28 mm × 6 mm posts spaced on 11.7 mm centers over a 0.305 m square plate. The area of the top of each post was 0.686 cm², half of its 1.37 cm² cell. The RMS height-of-roughness and also the arithmetic-mean height-of-roughness were 3 mm. A periodic equal-area bi-level architecture provides the largest RMS height-of-roughness possible in a 6 mm peak-to-valley span. This surface has no self-similarity.

![Figure 14 Convection from rough plate; $\varepsilon = 3$ mm; $L = 0.305$ m](image)

The points labeled “$\Delta T = 11$ K measured” are the measured $\overline{Nu}$ values in Figure 14; the points labeled “without natural” discount the natural convection from the measured values using the mixed convection model described in Jaffer [14].

Applying convection formula (47) to the bi-level plate geometry yields $\overline{Nu} = 0.0078 Re Pr^{1/3}$. The corresponding trace “rough turbulent $L$” in Figure 14 matches the “without natural” points ±2% for $2300 < Re < 50000$. 

14
At $\text{Re} > 50000$ the “smooth turbulent $L_T$” trace (having slope 4/5) shows that convection is from a smooth turbulent flow, but with a smaller characteristic-length ($L_T \approx 8.2 \text{ mm}$) than “smooth turbulent $L$”. The transition at $\text{Re}_T = 50035$ is computed by equation (36). Constraint (38) indicates that flow will transition from rough turbulence to smooth turbulence at $\text{Re}_T > \text{Re}_T = 50035$ if the post surface roughness $\varepsilon_q < 0.4 \text{ mm}$.

With $L_P/\varepsilon < 193$, formula (33) predicts that flow over the whole plate transitions directly from laminar to rough turbulence at $\text{Re}_\lambda \approx 175$, too small to test directly in the Convection Machine.

The periodic bi-level plate behaving compatibly with formula (47), which was derived from an analysis of self-similar roughness, supports the claim that formulas (22) and (47) are intrinsic to turbulent flow along isotropic roughness and not specific to self-similar roughness.
14. Fully rough regime

Figure 15  average friction coefficient of sphere-roughened plate

Figure 15 compares average skin-friction formulas. “0.5 $C_D$” is the Mills-Hang formula (4); “0.5 $C_m(C_f)$” are the Churchill formulas (7); “0.5 $C_D^2/C_f$” is from formula (41). “0.5 $C_D^2/C_f$” is within ±2.5% of “present work” equation (22) over Mills-Hang’s range $750 < L/k_S < 2750$ ($4000 < L/\varepsilon < 14650$).

This close match in the completely rough regime supports the present theory with the measurements from Pimenta et al [3] as modeled by Mills and Hang [4].

Figure 16  local skin-friction coefficient

Figure 16 plots the local skin-friction coefficients from Prandtl-Schlichting (1), Mills-Hang (3), White (5) and the present theory (43) in the fully rough regime (large Re).

At large $x/k_S$ the inset graph in Figure 16 shows that the local formulas from Mills-Hang, Prandtl-Schlichting, and White differ from the present work, growing to −8%, −13%, and −16% relative to $f_c$ formula (43) at $L/k_S = 10^6$. Note that only Prandtl-Schlichting included such a high ratio in their range.
15. **Transitional rough regime**

Afzal, Seena, and Bushra [8] (also Schlichting [5]) relate that the turbulent flow inside commercial pipes behaves differently from the flow inside Nikuradse’s sand coated pipes in the transitional rough regime. While the skin-friction coefficients for commercial pipes are monotonically decreasing with increasing Re on the Moody diagram ($f_C$ versus Re), in the diagram for Nikuradse’s pipes the coefficient trace for each roughness reaches its minimum just to the right of the smooth skin-friction line, a behavior they term “inflectional”.

The Prandtl-Schlichting plate model inherited the inflectional curve from Nikuradse’s pipes. The average coefficient of resistance ($c_f$) Moody diagram from Prandtl and Schlichting [1] (also Schlichting [5]) shows $c_f$ following a $-\frac{1}{5}$ slope smooth turbulent line to a 7% dip spread over a decade of Re just to the right of the smooth turbulent line before leveling out further to the right. An inflectional curve is shown in Figure 17.

The present work’s local and average skin-friction formulas (43) and (22) predict no variation in $f_C$ with Re from a periodic roughness producing rough turbulence ($\max(Re_L, Re_f) < Re < Re_f$).

The rough surface tested by Pimenta et al [3] was 11 layers of densely packed metal balls 1.27 mm in diameter “arranged such that the surface has a regular array of hemispherical roughness elements.” They used sand-roughness $k_S = 0.79$ mm.

Being repeating patterns, both sphere-roughened and sand-roughened surfaces are self-dissimilar.

Prandtl and Schlichting put the boundary between the transitional rough and fully rough regimes for a plate at sand-roughness Reynolds number $Re_k = \nu * k_S / \nu = 70.8$ where $\nu$ is the friction velocity from equations (19) and $\nu$ is the fluid kinematic viscosity. Pimenta et al assign $Re_k = 65$. Solving for Re:

$$\frac{Re_k}{\nu} = \frac{v}{\sqrt{3} \ln(L/\varepsilon)} \frac{5.333 \varepsilon}{\nu} = \frac{5.333 Re}{\sqrt{3} \ln(L/\varepsilon)}$$

$$Re \approx 0.325 Re_k \frac{L}{\varepsilon} \ln \frac{L}{\varepsilon}$$

Figure 16 shows the local skin-friction coefficient versus $x/k_S$ for the Pimenta et al sphere-roughened plate at three rates of flow, $Re_k = 41.6$, 68.5, and 103. The averages of these local coefficients from $450 < x/k_S < 2800$ are 0.00247, 0.00252, and 0.00253 respectively. These averages are within 2.5% of each other, less than the 5% spread predicted by Prandtl and Schlichting. The “Mills-Hang” trace is the fully rough local $C_f/2$; its close proximity to the “P.M.K” traces shows the lack of significant behavioral difference between the regimes.

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3 $C_f/2$ at $x/k_S = 452$ was measured only at $Re_k = 41.6$; it is included in all three averages.
Figure 17 is a Moody diagram of the 3 mm and 1 mm RMS roughness bi-level plate measurements, with natural convection discounted (as was done for Figure 14), then converted to friction coefficients $f_C$ using the Chilton-Colburn analogy (46). Only measurements for $Re < Re_\lambda$ are shown. The vertical arrows show the Prandtl-Schlichting and Pimenta et al regime boundaries calculated for 3 mm RMS roughness by equation (49). The $L/k_S = 19.1$ trace is the $c_f = 0.0320$ value calculated by Prandtl-Schlichting formula (2) for 3 mm RMS roughness.

An assumption of pipe analogy theories is that rough skin-friction coefficients will not be less than smooth turbulent coefficients $f_{C\sigma}$ in all turbulent flow regimes. Note that using $c_f/2$ instead of $c_f$ (while leaving $f_{C\sigma}$ unchanged) would violate that assumption.

The “measured” points are within 2% of the "$L/\varepsilon = 102$" line at $f_C \approx 0.0156$. The points follow neither the “inflectional” nor the “monotonic” curves. And clearly, some coefficients are less than $f_{C\sigma}$. A point on “$L/\varepsilon = 305$” shows that $f_C$ can be less than $f_{C\sigma}$, even at Re values a decade smaller than the smooth turbulent intercept (although it is much closer to $f_C = 0.0102$ than its ±7% expected measurement uncertainty).

16. Discussion

Recapping the development:

- In rough turbulence the surface roughness disrupts boundary layer flow over a plate.
- Self-similar roughness has a constant characteristic-length to RMS height-of-roughness ratio $L/\varepsilon \gg 1$ at a succession of scales converging to zero.
- Flat surfaces with isotropic self-similar roughness have skin-friction coefficients dependent only on $L/\varepsilon$: $f_C = (1/3) \ln^{-2} (L/\varepsilon)$
- Periodic roughness is not self-similar. The effective repeat length $L_P = L/j_P$, where $j_P \gg 1$ is the index of the discrete Fourier transform coefficient $X_j$ with the largest amplitude.
- Experiments with periodic isotropic roughness (bi-level and sphere-roughened plates) find that the average (not local) skin-friction coefficient is within ±2.5% of $f_C$ over wide ranges of Re. That flows from such dissimilar rough surfaces hew so closely to the present formulas is compelling evidence that these formulas reflect physical laws of turbulent flow along isotropic surface roughness.
- The transition from laminar or smooth turbulent flow to rough turbulent flow along a plate with isotropic periodic roughness occurs when $\varepsilon$ exceeds the larger of the (laminar or smooth turbulent) momentum thicknesses of a smooth plate having length $L_P/2$.
- When $L_P/\varepsilon < 193$ the flow over the whole plate transitions from laminar flow to rough turbulence at $Re_\lambda$.
- This transition to rough turbulence is an order of magnitude lower than predicted by pipe analogy theories. $f_C$ is constant above $Re_\lambda$; $f_C$ measurements of the bi-level plate are flat within 2% through the boundaries between hydraulically smooth, transitional, and fully rough regimes of pipe analogy theories.
- Downstream from a disruption, a turbulent boundary layer restarts from zero thickness over a flat, smooth patch. If this boundary layer grows large enough to bridge the rough bits outside of the smooth patches, then the flow downstream will be smooth turbulent. A formula for this threshold $Re_\varepsilon$ was developed; the convection model using it matches the measurements for the 1 mm and 3 mm roughness bi-level plates within their expected measurement uncertainties.

Self-similar roughness disrupts emerging boundary layers along its entire length, generating the most turbulence (as revealed by its skin-friction coefficient) possible from a given height-of-roughness. Periodic roughness generates the same amount of turbulence for $\max(Re_\lambda, Re_\sigma) < Re < Re_\varepsilon$.

The present work finds that rough turbulence over a plate is a simpler phenomena than flow having smooth turbulence. Rough turbulent flow has a constant skin-friction coefficient dependent only on the ratio of characteristic-length to RMS height-of-roughness.

Equation (46) is the original (1933) form of the Chilton-Colburn analogy. Lienhard [12] demonstrates that these particular Re and Pr exponents are not correct for smooth turbulent flow in all fluids. In particular, the Pr exponent should be 0.6 in gasses. The rough turbulent convection on the Convection Machine matches theory within its ±2% expected measurement uncertainty using $Pr^{1/3}$. With $Pr = 0.71$ (for air), $Pr^{0.6}$ is
nearly 9% larger than \( \Pr^{1/3} \), far exceeding the ±2% expected measurement uncertainty.

This \( \Pr^{1/3} \) form of the analogy is consistent with rough turbulence’s repeated boundary-layer disruptions being less dependent on fluid properties (not captured by \( \Pr^{1/3} \)) than is the generation of smooth turbulence.

For periodic isotropic roughness, the \( L \gg L_P \) condition guarantees that the transition to rough turbulence occurs in the leading band of periodic roughness, which controls the rest of the plate. When \( L_P/\varepsilon > 194 \), \( \text{Re}_\lambda \) is the critical Re flow.

The smooth turbulent threshold \( \text{Re}_\sigma < 1 \) for both bi-level plates and the rough surface tested by Pimenta et al [3]. The laminar threshold \( \text{Re}_\lambda \approx 1577 \) for the \( \varepsilon = 1 \) mm bi-level plate. \( \text{Re} = 1577 \) is too slow to test in the Convection Machine wind-tunnel. For a bi-level plate having 0.5 mm height-of-roughness \( \text{Re}_\lambda = 6310 \), but free-stream turbulence of the wind-tunnel might still prevent laminar flow.

17. Conclusions

- The turbulent skin-friction coefficient for a smooth plate is:

\[
f_{C\sigma} = \frac{\sqrt{2}}{3} \ln^{-2} \left( \frac{W_0 (\text{Re}/\sqrt{3})}{\varepsilon} \right) = \frac{\sqrt{2}}{3} \left[ \ln \left( \frac{W_0 (\text{Re}/\sqrt{3})}{\varepsilon} \right) - 1 \right]^2
\]

where \( W_0 \) is the smallest positive branch of the Lambert \( W \) function, the inverse of \( x \ln x \).

- For a periodic or self-similar plate surface with isotropic RMS height-of-roughness \( \varepsilon > 0 \), which is producing rough turbulence in a steady flow of strength \( \text{Re} \), the skin-friction coefficient and average forced convection formulas are:

\[
f_C = \frac{1}{3 \ln^2 (L/\varepsilon)} \quad \text{Nu} = \frac{\text{Re} \Pr^{1/3}}{6 \ln^2 (L/\varepsilon)} \quad \frac{L}{\varepsilon} \gg 1
\]

- For periodic isotropic roughness with period \( L_P \ll L \), when \( L_P/\varepsilon < 193 \), the flow over the whole plate transitions directly from laminar to rough turbulence at \( \text{Re}_\lambda \). When \( L_P/\varepsilon > 194 \), then \( \text{Re}_\sigma > \text{Re}_\lambda \) and the flow over the whole plate transitions from laminar to smooth turbulence at \( \text{Re}_\lambda \), and to rough turbulence at \( \text{Re}_\sigma \).

\[
\text{Re}_\lambda = \left( \frac{0.664}{\varepsilon} \right)^2 L_P L \quad \text{Re}_\sigma = \left( \frac{0.036}{\varepsilon} \right)^5 L_P^4 L
\]

- For periodic isotropic roughness with period \( L_P \ll L \) and flat patches in each cell with length \( L_T < L_P \) in the direction of flow, patch roughness \( \varepsilon_q \ll \varepsilon \), and patch period \( L_q < L_T \), the flow will be rough turbulent (to which the formulas apply) when \( \text{Re} > \max(\text{Re}_\lambda, \text{Re}_\sigma) \) and:

\[
\text{Re}_x < \text{Re}_\ell = \frac{L/8}{0.036 L_T} \left( \frac{2 \varepsilon}{0.036 L_P} \right)^{9/4} \quad \text{Re}_\ell < \frac{L}{L_q} \left[ \frac{0.036 L_q}{\varepsilon_q} \right]^5
\]
18. Nomenclature

\[ \bar{Nu} = \text{average Nusselt number (convection)} \]

\[ Pr = \text{fluid Prandtl number} \]

\[ Re = \text{Reynolds number of flow parallel to the plate} \]

\[ Re_L = \text{local Re rough-to-smooth turbulence threshold} \]

\[ Re_k = v^* k_S/\nu = \text{roughness Reynolds number} \]

\[ Re_L = \text{laminar Reynolds number upper-bound} \]

\[ Re_{\sigma} = \text{smooth turbulent Reynolds number upper-bound} \]

\[ Re_x = x \bar{Re}/L = \text{local Reynolds number} \]

\[ C_f, C_D = \text{Mills-Hang local, average friction coefficient} \]

\[ C_m = \text{Churchill mean (average) friction coefficient} \]

\[ C_f/2 = \text{Pimenta et al local friction coefficient} \]

\[ c_f, c_f = \text{Prandtl-Schlichting local, average friction coefficient} \]

\[ f_c, J_c = \text{local, average friction coefficient} \]

\[ G(x, w) = \text{Gray-code profile function} \]

\[ j_P = \text{index of largest } X_j \]

\[ k_S = \text{sand-roughness (m)} \]

\[ L = \text{plate characteristic-length (m)} \]

\[ L_P = \text{effective roughness repeat length (m)} \]

\[ L_q = \text{smooth patch roughness repeat length (m)} \]

\[ L_T = \text{average periodic flat length in direction of flow (m)} \]

\[ n = \text{branching factor of profile roughness function} \]

\[ S_j = \text{compensated discrete Fourier transform coefficient} \]

\[ v = \text{bulk fluid velocity (m/s)} \]

\[ v^* = \text{friction velocity (m/s)} \]

\[ w = 2^i = \text{integer power of two} \]

\[ W(x, w) = \text{wiggliest integer self-similar profile} \]

\[ Y(x, w) = \text{integer ramp-permutation self-similar profile} \]

\[ X_j = \text{discrete Fourier transform coefficient} \]

\[ x, y = \text{distance (m)} \]

\[ z(x) = \text{profile roughness function (m)} \]

\[ \bar{z} = \text{mean height of roughness (m)} \]

\[ Z = \text{profile roughness function random variable (m)} \]

Greek Symbols

\[ \delta_2 = \text{momentum thickness (m)} \]

\[ \epsilon = \text{RMS profile height-of-roughness (m)} \]

\[ \varepsilon = \text{RMS surface height-of-roughness (m)} \]

\[ \varepsilon_q = \text{periodic smooth patch height-of-roughness (m)} \]

\[ \omega = \log_2 w = \text{positive integer} \]

\[ \nu = \text{fluid kinematic viscosity (m}^2/\text{s)} \]

\[ \rho = \text{fluid density (kg/m}^3) \]

\[ \tau = \text{fluid shear stress (N/m}^2) \]

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19. References


