Thermodynamic Basis for Natural Convection from an Isothermal Plate

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Abstract

The combination of a heated horizontal plate with a turbine above it constitutes a non-reversible heat-engine, converting the flow of heat into mechanical work. Thermodynamic analysis of this hypothetical apparatus finds that its maximum thermodynamic efficiency is \( 0.5 \Delta T / T \), which is half of the limit for reversible heat-engines. Furthermore, when measured at the plate, both the convection and upward fluid flow will be the maximum allowed by this thermodynamic efficiency. Fluid-dynamics analysis results in a comprehensive correlation for natural convection from an upward-facing horizontal plate:

\[
\overline{N_u} = \left( 0.671 + 0.370 Ra^{1/6} \right)^2
\]

This new correlation covers laminar and turbulent convection, and averages 9% higher than measurement-fitted correlations by Lloyd and Moran for \( 22000 < Ra < 1.5 \times 10^9 \); and 4% higher than measurement-fitted correlations by Goldstein, Sparrow, and Jones for \( 1 < Ra < 10^4 \).

The same approach produces an upper bound for natural convection from a vertical plate:

\[
\overline{N_u} < \left( 0.826 + 0.387 Ra^{1/6} \right)^2
\]

For large Prandtl numbers, this matches the semi-empirical correlation reported by Churchill and Chu over its full range.

Keywords: natural convection; heat-engine; Carnot efficiency

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1. Introduction

To date, the natural convection correlations for upward-facing and vertical plates have been semi-empirical, based partly on measurements. This work provides a quantitative theoretical basis for natural convection from upward-facing and vertical plates.

Section 3 conducts a thought experiment determining the maximum extractable work from the buoyancy of heated air. It finds that the maximum heat-engine efficiency of this system is \((T - T_\infty)/(2T)\), which is half of the maximal Carnot efficiency for reversible heat-engines.

Section 4 organizes the variable groups and describes the treatment of the fluid thermal expansion coefficient \(\beta\), the only departure from convention.

Section 6 finds that the formulas for conduction (through static fluid) from a rectangular plate have different dependence on characteristic-length than the dimensional analysis of natural convection in Section 4. This manifests as more-than-predicted convection from high aspect-ratio plates at Rayleigh numbers \(Ra < 1\).

Fluid heating starts with static fluid conduction \(I_0\). Fluid heating causes buoyancy which initiates flow. Buoyant flow brings more unheated fluid into proximity with the plate, which causes more fluid heating. In the absence of obstruction, the flow and its kinetic power will increase to the maximum extent permitted by thermodynamic limits. Upward-facing horizontal is the least obstructed plate orientation; so it is the orientation most likely to be limited only by thermodynamic constraints.

Section 7 derives expressions for kinetic power flux \(I_k\) and total power flux \(I_p\), which is also the convection, above a heated upward-facing round plate. Thermodynamic efficiency \(\eta = I_k/I_p\). Solving \(I_k/I_p = \Delta T/(2T)\) for \(I_p\) produces a formula proportional to \(Ra^{1/3}\), the same dependence as Lloyd and Moran[1] use for turbulent upward natural convection.

This \(Ra^{1/3}\) term is combined with the static fluid conduction power flux \(I_0\) using the \(L^{1/2}\)-norm. The resulting formula is derived entirely from \(I_0\), dimensional analysis, and the \(L^{1/2}\)-norm.

Section 10 applies these techniques to natural convection from a vertical plate. Because the upper half of a vertical plate impedes flow from the lower half, this system does not necessarily achieve the efficiency of \(\Delta T/(2T)\). However, with an effective \(\Delta T\) which is half of the actual \(\Delta T\), the resulting upper bound matches Churchill and Chu’s[2] semi-empirical correlation at large Prandtl numbers.

The average (not local) correlations derived here apply from laminar through turbulent flows. They reveal no laminar-turbulent discontinuity.

Section 11 augments the Schulenberg correlation (1) to include conduction at very low Rayleigh numbers.

2. Prior Work

In “Natural convection heat transfer below downward facing horizontal surfaces”[3] T. Schulenberg analytically derives the correlation for convection below an isothermal strip for high Prandtl numbers. The author then interpolates this with the limiting case for low Prandtl numbers to produce a correlation for all Prandtl numbers:

\[
\text{Nu} = \frac{0.571 \, Ra^{1/5} \, Pr^{1/5}}{\left[1 + 1.156 \, Pr^{3/5}\right]^{1/3}}
\]  

While successful for downward-facing plates, Schulenberg’s sophisticated analysis does not seem applicable to the other plate orientations. Conversely, applying thermodynamic analysis to downward-facing horizontal plates is intractable because nearly all of the buoyant flow is obstructed by the plate. Schulenberg’s analysis doesn’t include conduction at low Rayleigh numbers; Section 11 extends Schulenberg’s correlation to include conduction.

In “Convective Carnot engine”[4] A. Narasimhan starts a thermodynamic analysis similar to the one here, but models the work and heat transfer as linear functions of the upward velocity. The velocity exponent was key to finding the fractional exponents of factors in \(I_k\) and \(I_p\) in the analysis here.

“Convective heat transfer law for an endoreversible engine”[5] by Huleihil and Andresen is about convection inside an endoreversible heat-engine, not convection as a heat-engine.

Atmospheric convection is of great interest to planetary science; but is considerably more complicated than the convection modeled here. Whereas the present model concerns only heat transfer close to the plate, atmospheric models treat heating at the surface, cooling in the upper atmosphere, radiative transfer, phase changes of moisture, quasi-equilibrium, rotating frame of reference, and viscous losses.
Natural convection as a heat-engine: A theory for CAPE[6] by Rennó and Ingersoll characterizes an atmosphere by its convective available potential energy (CAPE). Their system is analyzed as a cyclic heat-engine versus the continuous process analyzed here.


Both of the papers look to the Carnot efficiency limit for reversible heat-engines. But Section 3 of the present work shows that the efficiency limit for heated air is actually half of the Carnot limit.

3. Unenclosed Heat-Engine

Consider a column of still, dry air under the influence of Earth’s gravity. At equilibrium the column will have a uniform temperature \( T_\infty \) and a pressure profile \( P \) which is exponentially decreasing with altitude \( z \):

\[
P(z) = P_0 \exp \left( -\frac{z g M}{RT_\infty} \right)
\]

A small cube of air has density \( \rho = \frac{M P}{RT_\infty} \). A heated cube of air has density \( \rho_h = \frac{M P}{R(T_\infty + \Delta T)} \). The buoyancy force acting on the heated cube of volume \( V \) is:

\[
F_B = (\rho - \rho_h) g V = \frac{g V M P \Delta T}{RT_\infty (T_\infty + \Delta T)}
\]

\( \Delta Q = c_p \rho V \Delta T \) is the quantity of heat (energy) required to raise the temperature of the cube from \( T_\infty \) to \( T = T_\infty + \Delta T \). If \( 0 < \Delta T \ll T_\infty \), then the buoyancy force acting on the heated cube of volume \( V \) is\(^1\):

\[
F_B \approx \left( \frac{M P}{\rho RT} \right) \frac{g \Delta Q}{c_p T} = \frac{g \rho V \Delta T}{T}
\]

As an air parcel rises, its state changes. Temperature, volume, and pressure are three variables representing two degrees of freedom constrained by the ideal gas law. Concerning convection in the troposphere, Fermi’s Thermodynamics[8] teaches:

Since air is a poor conductor of heat, very little heat is transferred to or from the expanding air, so that we may consider the expansion as taking place adiabatically.

The temperature of a parcel of dry air will drop 9.8 K per kilometer of altitude gain (which is \( g/c_p \)). So in a column of initially uniform temperature, a heated parcel will rise until its temperature drops to ambient. From conservation of mass \( V \rho = V_0 \rho_0 \).

The maximum work which could be extracted from the rising plume is the integral of the upward force \( F_B \) with respect to height above the plate \( z \):

\[
W = \int_0^{\Delta T} \left( \frac{c_p}{g} \Delta T - \frac{z g}{c_p} \right) \frac{g \rho_0 V_0}{T} \, dz
\]

\[
= \frac{g \Delta Q}{c_p T} \int_0^{\Delta T} \left( 1 - \frac{z g}{c_p \Delta T} \right) \, dz
\]

\[
= \frac{g \Delta Q}{c_p T} \frac{\Delta T}{2 g} = 0.5 \frac{\Delta Q \Delta T}{T}
\]

The thermodynamic efficiency of the combination of the plate and turbine as a heat-engine is:

\[
\eta = \frac{W}{\Delta Q} = 0.5 \frac{\Delta T}{T} \tag{2}
\]

As must be the case, \( \eta \) is not greater than the maximum efficiency of a reversible (Carnot) engine, \( \Delta T/T \).

\(^1\) \( F_B \approx g \Delta Q/(c_p T) \) being independent of \( V \) and \( \Delta T \) implies that it can be considered as the buoyancy of a parcel of heat, no matter how dispersed it is.
4. Dimensional Analysis

Nusselt’s dimensional analysis of natural convection (from Lienhard and Lienhard[9] with $I_p$ added) chooses its four variable groups as ($\overline{Nu}$ is Nusselt number; $Pr$ is Prandtl number):

$$\overline{Nu} \equiv \frac{h L}{k} = \frac{I_p L}{k \Delta T}, \quad Pr \equiv \frac{\nu}{\alpha}, \quad \Pi_3 \equiv \frac{L^3}{\nu^2 g} = \frac{L g}{(\nu/L)^2}, \quad \Pi_4 \equiv \beta \Delta T$$

From which come the dimensionless Grashof (Gr) and Rayleigh numbers (Ra):

$$\Pi_3 \Pi_4 = Gr = \frac{\beta \Delta T g L^3}{\nu^2}, \quad GrPr \equiv Ra = \frac{\beta \Delta T g L^3}{\alpha \nu}$$

$T$ is the plate surface temperature and $T_\infty$ is the fluid temperature far from the plate.

$$\Delta T = T - T_\infty \geq 0$$

If $\beta = 1/T_\infty$ (customary for natural convection from a vertical plate in a gas), then:

$$\Pi_4 = \frac{T - T_\infty}{T_\infty} \quad 0 \leq \Pi_4 < +\infty \quad \eta < \frac{1}{2 \frac{\Pi_4}{1 + \Pi_4}} \quad (3)$$

If $\beta = 1/T$, then:

$$\Pi_4 = \frac{T - T_\infty}{T} \quad 0 \leq \Pi_4 < 1 \quad \eta < \frac{1}{2 \Pi_4} \quad (4)$$

$\Pi_4$ in equation (4) has the simplest relationship to the efficiency limit (2) for the unenclosed heat-engine of Section 3; it is used in the following derivations.

5. Characteristic-Length

For a horizontal upward-facing plate, Fig. 14(f) from Fujii and Imura[10] shows that convection draws fluid from the sides of the plate into a central plume. Thus the length of the horizontal flows where the fluid comes closest to the plate is controlled by the distance from the edges to the center of the plate.

The characteristic-length $L^*$ is the area-to-perimeter ratio. Lloyd and Moran[1] measured upward convection from rectangles having aspect ratios up to 10:1. They write:

“It is immediately obvious that within the scatter of the data, approximately ±5 percent, the data from all planforms are correlated through the use of $L^*$, including the nonsymmetric right triangle planform.”

Goldstein, Sparrow, and Jones[11] also found that $L^*$ unified their measurements with aspect ratios up to 7:1.

For a $D \times H$ rectangular plate with $D \leq H$ and aspect ratio $r$:

$$A = H D \quad r = \frac{H}{D} \quad H = \sqrt{A r} \quad D = \sqrt{A/r}$$

$$L^* = \frac{A}{2D + 2H} = \frac{A}{2\sqrt{A r} + 2\sqrt{A/r}} = \frac{\sqrt{A r}}{2r + 2} = \frac{\sqrt{A}}{2r + 2}$$
6. **Conduction Shape Factor**

When \( \Delta T > 0 \) is very small, the fluid movement will be negligible, and \( I_p = I_0 \) will be heat conduction through the fluid. \( \text{Nu}_0 \) and \( I_0 \) are the conduction Nusselt number and conduction power flux for the upward-facing plate with characteristic-length \( L^* \). The power flowing from an object into an infinite solid (in this case static fluid) is \( q = S k \Delta T \), where \( k \) is thermal conductivity and \( S \) is the conduction shape factor (having length unit).

\[
I_0 A = \text{Nu}_0 A \frac{k \Delta T}{L^*} = q = S k \Delta T \quad \text{Nu}_0 = \frac{S L^*}{A} = \frac{S}{\sqrt{A}} \sqrt{\frac{\pi}{2 r + 2}}
\]

For one face of a rectangular plate with aspect ratio \( r \geq 1 \), ASHRAE[12] gives a shape factor \( S \):

\[
S = \frac{\pi \sqrt{A r}}{\ln (4 r)} \quad \text{Nu}_0 = \frac{\pi}{\ln (4 r)} \frac{r}{2 r + 2} < 0.567 \quad (5)
\]

Incropera, DeWitt, Bergman, and Lavine[13] gives a dimensionless shape factor \( q_{SS}^2 = 0.932 \) for both faces of a rectangular plate. For one face:

\[
q = \frac{q_{SS}^2}{2} k \Delta T \frac{A_s}{L_C} \quad L_C = \sqrt{\frac{A_s}{4 \pi}} \quad A_s = 2 A
\]

\[
I_0 A = \text{Nu}_0 A \frac{k \Delta T}{L^*} = q = \frac{q_{SS}^2}{2} k \Delta T \sqrt{4 \pi A_s} \quad \text{Nu}_0 = \frac{q_{SS}^2}{2} \frac{L^*}{A} \frac{\sqrt{3 \pi A_s}}{A} = 1.17 \frac{\sqrt{\pi}}{1 + r} < 0.584 \quad (6)
\]

Incropera, DeWitt, Bergman, and Lavine[13] also give a shape factor for one side of a circular disk \( S = 2 D \).

\[
\text{Nu}_0 = \frac{S L^*}{A} = \frac{8 L^*}{\pi (2 L^*)^2} = \frac{2}{\pi} \approx 0.637 \quad (7)
\]

For rectangular plates, conduction \( q = q_{SS}^2 k \Delta T \sqrt{4 \pi A_s} \) depends on area \( A_s \), but not on aspect ratio. Natural convection depends on aspect ratio via the plate’s characteristic-length \( L^* \).

Figure 1 shows that trying to equate either rectangular conduction model with convection makes \( \text{Nu}_0 \) dependent on aspect ratio in a way which can’t be reduced to \( L^* \). The dependence on aspect ratio appears at Rayleigh numbers \( \text{Ra} < 1 \), where conduction dominates convective heat transfer.

In trying to find a convection upper bound, the plate with the highest shape factor should be used. That shape is a round disk with Nusselt number \( 2/\pi \) from equation (7).

![Figure 1 Nusselt number for conduction vs. aspect ratio](image-url)
The assumption of negligible movement is in conflict with the assumption that the system is in a steady-state. It may take a long time to reach steady-state, but if the Rayleigh number \( Ra > 0 \), a Newtonian fluid will move.

With upward convection drawing fluid from the sides of the plate into a central plume, \( \Delta T \) at the center of the plate will be reduced along with its conduction. Scaling the equation (7) shape factor by \( \sqrt{1/2} \) has the same effect as halving the convecting area:

\[
\text{Nu}^* = \frac{\text{Nu}_0}{\sqrt{2}} = \frac{\sqrt{2}}{\pi} \approx 0.450
\]

This \( \text{Nu}^* \) value is used for the constant component in upward-facing convection in Section 7.

### 7. The Isothermal Plate as a Heat-Engine

Consider a continuously heated horizontal upward-facing round plate. The characteristic-length \( L = L^* \), the area-to-perimeter ratio which is half of its radius. At some distance above the plate, the rising fluid is no longer accelerating upward. Place a (perfect) turbine intake at this separation plane such that the kinetic energy of the upward flow is completely recovered by the hypothetical turbine.

The system being in continuous operation, instead of energies \( \Delta Q \) and \( W \), power fluxes (W/m²) are of interest. The powers per unit area are \( I_k \) for the kinetic flux of the fluid and \( I_p \) for the plate total. The thermodynamic efficiency \( \eta = I_k/I_p \).

Kinetic flux \( I_k \) will increase with increasing thermal expansion coefficient \( \beta \), temperature difference \( \Delta T \), gravitational acceleration \( g \), and viscosity \( \nu \) and decrease with increasing specific heat (at constant pressure) \( c_p \) and thermal diffusivity \( \alpha = k/(\rho c_p) \). Introduce a new dimensionless group:

\[
\Pi_5 = \frac{\beta^2 \Delta g L \nu^2}{c_p \alpha^2} = \frac{\beta g L}{c_p} \Pi_4 \text{Pr}^2
\]

\( \rho A u \) is the mass passing through the separation plane per second where \( u \) is the average upward velocity of fluid through the separation plane. The kinetic power of the flow will be twice the mass (because it is buoyant flow) times half of the velocity squared. Fig. 14(f) from Fujii and Imura[10] shows that upward-facing convection draws fluid from the sides of the plate into a central plume. This horizontal flow experiences a change in momentum which should be included in \( I_k \).

The depth of fluid moved along the plate is proportional to \( u L/(2 \nu) \). The division by 2 is because the length of the horizontal flow inward is the radius, \( 2 L^* \):

\[
I_k = \frac{u L}{2 \nu} \frac{\rho A u u^2}{\Pi_5 A} = \frac{\rho L u^4}{2 \nu \Pi_5}, \quad u = \left[ \frac{2 \nu I_k \Pi_5}{\rho L} \right]^{1/4}
\]

Movement brings more fluid into proximity with the plate. The amount of fluid heated will increase with \( u \) and decreasing kinematic viscosity \( \nu \) resulting in the \( u L/(2 \nu) \) term again:

\[
I_p = k \frac{\Delta T}{L} \text{Nu}^*_0 \left[ 1 + \frac{u L}{2 \nu} \right] = k \frac{\Delta T}{L} \text{Nu}^*_0 \left[ 1 + \frac{L}{2 \nu} \left[ \frac{2 \nu I_k \Pi_5}{\rho L} \right]^{1/4} \right]
\]

When \( u L/\nu \gg 1 \) two variable groups with power flux units (W/m²) arise from equation (10):

\[
\Psi_p = \frac{k \Delta T}{L}, \quad \Psi_k = \frac{\rho \nu^3}{\Pi_5 L^3}, \quad I_p = \Psi_p \text{Nu}^*_0 \left[ \frac{I_k}{8 \Psi_k} \right]^{1/4}
\]

With \( \beta = 1/T \), the upper bound of efficiency of the plate in converting plate flux \( I_p \) to kinetic flux \( I_k \) can be found by combining equations (11) and (4):

\[
\eta = \frac{I_k}{I_p} < \frac{\Pi_4}{2} \quad I_k < \frac{\Pi_4}{2} I_p = \frac{\Psi_p \Pi_4 \text{Nu}^*_0}{2} \left[ \frac{I_k}{8 \Psi_k} \right]^{1/4}
\]

\text{\(^2\)} The combination of 1 and \( u L/(2 \nu) \) is actually more complicated than addition. It is corrected later in this section.
Dividing both sides by $I_k^{1/4}$, then raising both sides to the 4/3 power solves for $I_k$:

$$I_k^{3/4} < \frac{\Psi_p \Pi_4 Nu_0^*}{2} \left(\frac{1}{8 \Psi_k}\right)^{1/4} I_k < \left[\frac{\Psi_p \Pi_4 Nu_0^*}{2} \left(\frac{1}{8 \Psi_k}\right)^{1/4}\right]^{4/3} = \Psi_p \left[\frac{Nu_0^*}{2}\right]^{4/3} \frac{\Pi_4}{\Psi_k}$$

(12)

Increasing $I_k$ causes increasing $I_p$ which causes increasing $I_k$ to the maximum allowed by inequality (12). Substituting $I_k$ from (12) into equation (11):

$$I_p = \Psi_p Nu_0^* \left[\frac{I_k}{8 \Psi_k}\right]^{1/4} = \Psi_p Nu_0^* \left[\frac{\Psi_p \Pi_4 Nu_0^*}{2} \left(\frac{1}{8 \Psi_k}\right)^{1/4}\right]^{1/3} = \Psi_p \left[\frac{Nu_0^*}{2}\right]^{4/3} \frac{\Psi_p \Pi_4}{\Psi_k}^{1/3}$$

Both $I_p$ and $I_k$ have factors $\sqrt[3]{\Psi_p \Pi_4/\Psi_k}$. How does $\Psi_p \Pi_4/\Psi_k$ relate to the dimensionless Rayleigh number ($Ra$)?

$$\frac{\Psi_p \Pi_4}{\Psi_k} = \frac{k \Delta T L^3 \beta g L \nu^2}{c_p \alpha^2} = \frac{\beta \Delta T g L^3}{\alpha \nu} = Ra$$

$$I_k = \Psi_p \left[\frac{Nu_0^*}{2}\right]^{4/3} Ra^{1/3} \Pi_4 \quad I_p = \Psi_p \left[\frac{Nu_0^*}{2}\right]^{4/3} Ra^{1/3} = \Psi_p \frac{Nu}{\Psi_k} \quad Nu = \left[\frac{Nu_0^*}{2}\right]^{4/3} Ra^{1/3} \approx 0.137 Ra^{1/3}$$

If conduction and convection were independent non-interacting processes, then 1 and $u L/(2 \nu)$ would simply add as in equation (10). If they were competing processes then they would combine as the $L^p$-norm where $p > 1$. But conduction is an integral part of natural convection processes, so $Nu_0^*$ and $[Nu_0^*/2]^{4/3}Ra^{1/3}$ combine as the $L^{1/2}$-norm:

$$\bar{Nu} = \left\| Nu_0^*, \left[\frac{Nu_0^*}{2}\right]^{4/3} Ra^{1/3} \right\| \approx \left(\sqrt{Nu_0^*} + \left[\frac{Nu_0^*}{2}\right]^{4/6} Ra^{1/6}\right)^2 \approx \left(0.671 + 0.370 Ra^{1/6}\right)^2$$

(13)

---

3 Churchill and Chu[2] also use $p = 1/2$ for convection from a vertical plate.

$$\|x, y\|_p = (|x|^p + |y|^p)^{1/p} \quad \|x, x\|_2 = \sqrt{2} x \quad \|x, x\|_1 = 2 x \quad \|x, x\|_{1/2} = 4 x$$
8. Comparison With Measurements

A potential difficulty in comparing equation (13) with existing measurements is that \( \beta = 1/T \) may not have been used for the measurements. Figure 2 shows that, at large \( \Delta T \), \( \Pi_4 \) and Ra computed with \( \beta = 1/T_\infty \) will be greater than the Ra computed with \( \beta = 1/T \), causing the measured Nu values to appear less than predicted. But the maximum dependence of Nu on \( \beta \) is \( \beta^{1/3} \), reducing the severity of this discrepancy.

![Figure 2: \( \Pi_4 = \beta \Delta T \)](image)

However, both sources of the correlations in Table 1 made mass transfer measurements of concentrations. Lloyd and Moran\([1]\) measured electrochemical mass transfer.\(^4\) Goldstein, Sparrow, and Jones\([11]\) measured sublimation from solid naphthalene in air. Figure 3 plots equation (13) with the Table 1 correlations; only correlation (17) is turbulent.

![Figure 3: Natural convection from upward-facing plate](image)

\(^4\) Lloyd and Moran write “One of the advantages of using the present mass transfer technique over heat transfer experiments for high Prandtl number fluids is the fact that the property variations for the mass transfer experiments are essentially zero, whereas the corresponding heat transfer experiments exhibit large variation in properties.”
Table 1  Upward convection correlations

<table>
<thead>
<tr>
<th>Range</th>
<th>Correlation</th>
<th>Source</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 &lt; Ra &lt; 200$</td>
<td>$Nu = 0.96 Ra^{1/6}$</td>
<td>Goldstein, Sparrow, and Jones[11]</td>
<td>(14)</td>
</tr>
<tr>
<td>$200 &lt; Ra &lt; 10^4$</td>
<td>$Nu = 0.59 Ra^{1/4}$</td>
<td>Goldstein, Sparrow, and Jones[11]</td>
<td>(15)</td>
</tr>
<tr>
<td>$2.2 \times 10^4 &lt; Ra &lt; 8 \times 10^6$</td>
<td>$Nu = 0.54 Ra^{1/4}$</td>
<td>Lloyd and Moran[1]</td>
<td>(16)</td>
</tr>
<tr>
<td>$8 \times 10^6 &lt; Ra &lt; 1.5 \times 10^9$</td>
<td>$Nu = 0.15 Ra^{1/3}$</td>
<td>Lloyd and Moran[1]</td>
<td>(17)</td>
</tr>
</tbody>
</table>

Correlation (13) for the round disk averages 9% higher than measurement-fitted correlations by Lloyd and Moran[1] for $22000 < Ra < 1.5 \times 10^9$; and 4% higher than measurement-fitted correlations by Goldstein, Sparrow, and Jones[11] for $1 < Ra < 10^4$. The Figure 3 trace for a square plate averages 2% lower than Lloyd and Moran[1] and 6% lower than Goldstein, Sparrow, and Jones[11]. While it is possible that the experiments did not reach their maximum possible efficiency, it is not possible that they exceeded it. This rules out the square plate correlation as the efficiency limit.

9. Upward Velocity and Temperature

Combining equation (9) with equation (12) gives the upward velocity $u$ averaged over the plate area:

$$u = \left[ \frac{\nu^2 g \Pi^4 \Pr^3}{L} \right]^{1/4} \left[ \frac{Nu^*}{2} \right]^{1/3} Ra^{1/12}$$

When $Ra \gg 1$, the thermal power flowing through the separation plane is:

$$(I_p - I_k) A = \rho c_p u A (T_s - T_\infty)$$

where $T_s$ is the fluid temperature at the separation plane. $T_s - T_\infty$ is the average temperature increase for upward flowing fluid:

$$T_s - T_\infty = I_p \frac{(1 - \Pi_4/2)}{\rho c_p u}$$
10. **Vertical Plate**

The vertical plate characteristic-length $L$ is the vertical length of the plate. $\text{Nu}_0$ for the vertical plate will be the same no matter what the width of the plate. Collapsing the width of the plate and the conductive solid it is embedded in results in $\text{Nu}_0$ being half of the two-dimensional conduction from a line segment of length $L$ in a uniformly conductive plane.

Consider $\text{Nu}_0$ for one face of a $4L^* \times 4L^*$ upward-facing plate from equation (6):

$$\text{Nu}_0 = \frac{q_{SS}^*}{2} \frac{L^*}{A} \frac{\sqrt{4\pi A_s}}{A} = 0.932 \frac{\sqrt{2\pi}}{4}$$

The conduction from an upward-facing $4L^* \times 4L^*$ plate can be considered the two-dimensional conduction $\text{Nu}_0^*$ crossed with itself at right angles; the factor of 4 in equations (18) is the ratio of vertical to upward-facing characteristic-lengths.

$$\text{Nu}_0 = \sqrt{\frac{\text{Nu}_0^*}{4}} \quad \text{Nu}_0^* = 4 \text{Nu}_0^2 \approx 1.36$$

(18)

The round upward-facing plate had flow toward its center. Its conduction $\Delta T$ (via the shape factor) was consequently scaled by $\sqrt{1/2}$ in equation (8). For a vertical plate, fluid heated by the plate flows upward along its surface. The temperature profile of fluid near the plate increases in temperature with distance from the lower edge. Thus the local $\Delta T$ decreases with distance from the lower edge. This results in the effective $\Delta T$ (and shape-factor) for vertical plate conduction being scaled by 1/2.

$$\text{Nu}_0^* = \frac{\text{Nu}_0^2}{2} = \frac{4 \text{Nu}_0^2}{2} = 0.932^2 \frac{\pi}{4} \approx 0.682$$

(19)

The effective $\Delta T$ for vertical convection is also scaled by 1/2 by making substitutions:

$$\Delta T \rightarrow \Delta T/2 \quad \Pi_4 \rightarrow \Pi_4/2 \quad \Pi_5 \rightarrow \Pi_5/2$$

The thickness of the boundary layer at the top edge of the plate will be proportional to $\delta = u L^2/\nu$. The mass passing through the separation plane per second will be $\rho L u \delta = \rho L^3 u^2 / \nu$. As before, the kinetic power of the flow will be twice the mass times half of the velocity squared. The kinetic and plate power fluxes are:

$$I_k = \frac{2 \rho L^3 u^2}{\nu \Pi_5 A} = \frac{2 \rho L \nu^4}{\nu \Pi_5} u = \left[ \frac{\nu I_k \Pi_5}{2 \rho L} \right]^{1/4}$$

$$I_p = \frac{k \Delta T}{2 L} \text{Nu}_0^* \left[ 1, \frac{u L}{\nu} \right]_{1/2} = \frac{k \Delta T}{2 L} \text{Nu}_0^* \left[ 1, \frac{L}{\nu} \left[ \frac{\nu I_k \Pi_5}{2 \rho L} \right]^{1/4} \right]_{1/2}$$

With $u L / \nu \gg 1$, collect factors into $\Psi_p$ and $\Psi_k$ from definitions (11):

$$I_p = \frac{\Psi_p}{2} \text{Nu}_0^* \left[ \frac{I_k}{2 \Psi_k} \right]^{1/4}$$

(20)

The upper bound of efficiency of the plate in converting plate flux $I_p$ to kinetic flux $I_k$ is:

$$\eta = \frac{I_k}{I_p} < \frac{\Pi_4}{2} \quad I_k < \frac{\Pi_4}{2} I_p = \frac{\Psi_p \Pi_4 \text{Nu}_0^*}{4} \left[ \frac{I_k}{2 \Psi_k} \right]^{1/4}$$

Dividing both sides by $I_k^{3/4}$, then raising both sides to the 4/3 power solves for $I_k$:

$$I_k^{3/4} < \frac{\Psi_p \Pi_4 \text{Nu}_0^*}{4} \left[ \frac{1}{2 \Psi_k} \right]^{1/4} \quad I_k < \left[ \frac{\Psi_p \Pi_4 \text{Nu}_0^*}{4} \right]^{4/3} \left[ \frac{1}{2 \Psi_k} \right]^{1/3} = \Psi_p \frac{\text{Nu}_0^{4/3}}{4} \left[ \Psi_p \Pi_4 \Psi_k \right]^{1/3} \frac{\Pi_4}{2}$$

(21)
Because the upper half of the plate obstructs flow from the lower half, the system will convect less than the same plate in an upward-facing orientation. At best an upper bound can be found for this system. Let $I_k$ be the maximum allowed by inequality (21). Substituting $I_k$ from (21) into equation (20):

$$ I_p = \frac{\Psi_p}{2} \frac{Nu_0^*}{2 \Psi_k} \left[ \frac{I_k}{2 \Psi_k} \right]^{1/4} = \frac{\Psi_p}{2} \frac{Nu_0^*}{\Psi_k} \left[ \frac{\Psi_p \Pi_4 Nu_0^*}{4} \right]^{1/3} = \frac{\Psi_p}{2} \frac{Nu_0^4/3}{\Psi_k} \left[ \frac{\Psi_p \Pi_4}{\Psi_k} \right]^{1/3} $$

$$ I_k = \frac{\Psi_p}{4} \frac{Nu_0^4/3}{Ra^{1/3}} \Pi_4/2 \quad I_p = \Psi_p \frac{Nu_0^4/3}{4} \frac{Ra^{1/3}}{Nu} = \frac{Nu_0^4/3}{4} \frac{Ra^{1/3}}{Nu} \approx 0.150 Ra^{1/3} $$

$$ \frac{Nu}{Nu_0^*} < \left\| \frac{Nu_0^4/3}{4} \frac{Ra^{1/3}}{Nu} \right\|^{1/2} = \left( \sqrt{Nu_0^*} + \frac{Nu_0^{4/6}}{2} \frac{Ra^{1/6}}{Nu} \right)^2 \approx \left( 0.826 + 0.387 Ra^{1/6} \right)^2 \quad (22) $$

With some algebraic manipulation, the correlation for natural convection from a vertical plate reported by Churchill and Chu[2] is equivalent to:

$$ Nu = \left\{ \begin{array}{ll}
0.825 + \frac{0.387 Ra^{1/6}}{1 + (0.492/Pr)^{9/16}}^{8/27} & 1 \leq Ra \leq 10^{12}
\end{array} \right. \quad (23) $$

Over the full range of $Pr > 0$, the denominator can only reduce the magnitude of $Nu$ in equation (23):

$$ 1 < \left[ 1 + (0.492/Pr)^{9/16} \right]^{8/27} $$

So correlation (23) satisfies inequality (22).

11. **Downward Convection**

For a downward-facing rectangular plate the characteristic-length $L_R$ is half of the shorter plate side. For square plates $L_R = L/2 = 2 L^*$ and $A = 4 L^2_R$. The outward creep beneath the downward-facing plate stays in contact with the plate until it reaches an edge; its temperature profile is hardly different from pure conduction. So its conduction is not scaled, which cancels the factor of $1/2 = L_R/L$ from the change in characteristic-length; hence $Nu_0^*$ from equation (19) is used for downward convection also. The convective flow from upward-facing and vertical plates brings unheated fluid into contact with the plate, which is responsible for amplifying the convection. Fig. 14(c) from Fujii and Imura[10] shows the unheated fluid rising from below the center of the downward-facing plate where it will be slowly warmed by conduction through the fluid above it. So the conduction $Nu_0^*$ should be simply added to Schulenberg’s convection correlation (1):

$$ \overline{Nu} = Nu_0^* + \frac{0.571 Ra^{1/5} Pr^{1/5}}{1 + 1.156 Pr^{3/5}}^{1/3} = 0.682 + \frac{0.544 Ra^{1/5}}{1 + (0.785/Pr)^{3/5}}^{1/3} \quad (24) $$

This sum is consistent with measurements from the Convection Machine[14], although the lowest nonzero Nusselt number it can reliably produce and measure is about 7.5.
12. Discussion

The departure from convection conventions here is $\beta = 1/T$ from equation (4) instead of $\beta = 1/T_{\infty}$ from equation (3). This choice makes $\Pi_4 = 2\eta$. With thermodynamic efficiency $\eta$ a factor of $Ra$, it is possible to directly relate efficiency and convection. Choosing $\beta = 1/T$ might enable the derivation of natural convection correlations for other shapes.

Even with this unconventional choice for $\beta$, it was possible to derive an upper bound for convection from a vertical plate having the same asymptotic dependence on $\beta$ and $\Pi_4$ as that from Churchill and Chu[2].

For the upward-facing and vertical plates, in order to detect a 10% discrepancy in $Nu$ due to the choice of $\beta = 1/T$ versus $\beta = 1/T_{\infty}$ in air at temperature $T_{\infty} = 27^\circ C$, a plate temperature $T > 126^\circ C$ would be required. Although this paper is neutral on the issue of $\beta$ choice for downward-facing plates, their $Nu \propto \beta^{1/5}$ dependence would require $T > 210^\circ C$ to change $Nu$ by 10%.

The development of the upward-convection formula posits that, in the absence of obstructions, convection will increase to the maximum allowed by thermodynamic laws. Measurement apparatus nearly always has a finite extent which obstructs upward flow. It therefore shouldn’t be surprising that actual upward-convection measurements are a few percent less than the formula predicts. It is still possible that intrinsic factors prevent achievement of the predicted thermodynamic efficiency. But the comparison in Section 8 would seem to limit the magnitude of such inefficiencies.

It was unexpected that the only coefficients which needed to be introduced by the analysis were $1/2$ and $\sqrt{1/2}$. Both were for the reduction in heat transfer due to heating of the fluid in contact with the plate. Where the flow adjacent to the plate was radial, it was $\sqrt{1/2}$; where the flow was parallel it was $1/2$. Both are equivalent to the effective plate area being half of the heated plate area.

The mismatch between conduction characteristic-length and convection characteristic-length only occurs with non-circular upward-facing plates. In this case, conduction can be greater than convection predicted by correlation (13) at Rayleigh numbers less than one. In air with $\Delta T = 4K$ such small Rayleigh numbers occur above plates with sides shorter than 5 mm.

The $L^p$-norm shows up repeatedly in fluid-mechanics and heat-transfer. Proving that the $L^{1/2}$-norm correctly blends the static and dynamic components of upward and vertical natural convection would be an exciting theoretical advance.

13. Conclusion

Dimensional analysis and the thermodynamic constraints on heat-engine efficiency combine to give a theoretical derivation of the correlation for upward convection from an isothermal plate. This new correlation is consistent with widely cited measurements over 9 decades of Rayleigh numbers.

$$Nu = \left(0.671 + 0.370 Ra^{1/6}\right)^2$$

A similar approach to convection from a vertical plate derives an upper bound for vertical plate convection which, for large Prandtl numbers, matches Churchill and Chu’s semi-empirical correlation over 12 decades of Rayleigh numbers.

$$\overline{Nu} < \left(0.826 + 0.387 Ra^{1/6}\right)^2$$

For downward convection, adding a conduction term extends Schubenberg’s formula to arbitrarily small Rayleigh numbers.

$$\overline{Nu} = 0.682 + \frac{0.544 Ra^{1/5}}{\left[1 + (0.785/Pr)^{3/5}\right]^{1/3}}$$
14. Nomenclature

\[ A = D H = \text{plate area (m}^2\text{)} \]
\[ A_s = \text{conduction plate area (m}^2\text{)} \]
\[ c_p = \text{fluid specific heat at constant pressure (J/(kg} \cdot \text{K))} \]
\[ D = \text{plate width (m)} \]
\[ g = \text{gravitational acceleration (m/s}^2\text{)} \]
\[ H = \text{plate height (m)} \]
\[ h = \text{plate average convective surface conductance (W/(m}^2 \cdot \text{K))} \]
\[ I_0 = \text{conduction power flux (W/m}^2\text{)} \]
\[ I_k = \text{kinetic power flux (W/m}^2\text{)} \]
\[ I_p = \text{plate power flux (W/m}^2\text{)} \]
\[ k = \text{fluid thermal conductivity (W/(m} \cdot \text{K))} \]
\[ L = \text{plate characteristic-length (m)} \]
\[ L^* = \text{area-to-perimeter ratio (m)} \]
\[ L_R = \text{downward-facing characteristic-length (m)} \]
\[ M = \text{air molar mass (kg)} \]
\[ N_u_0 = \text{conduction Nusselt number} \]
\[ N_u = \text{average Nusselt number} \]
\[ P = \text{air pressure (N/m}^2\text{)} \]
\[ Pr = \text{Prandtl number} \]
\[ q = \text{conduction power (W)} \]
\[ q_{SS} = \text{dimensionless conduction shape factor} \]
\[ R = \text{air gas constant (J/(kg} \cdot \text{K))} \]
\[ R_a = \text{Rayleigh number} \]
\[ r = H/D = \text{plate aspect ratio} \]
\[ S = \text{conduction shape factor (m)} \]
\[ T = \text{plate temperature (K)} \]
\[ T_\infty = \text{bulk fluid temperature (K)} \]
\[ u = \text{fluid velocity (m/s)} \]
\[ V = \text{air volume (m}^3\text{)} \]
\[ W = \text{work (J)} \]

Greek Symbols

\[ \Delta Q = \text{heat (J)} \]
\[ \Delta T = T - T_\infty = \text{temperature difference between plate and fluid (K)} \]
\[ \alpha = k/(\rho c_p) = \text{fluid thermal diffusivity (m}^2\text{/s)} \]
\[ \beta = \text{fluid thermal expansion coefficient (1/K)} \]
\[ \delta = \text{boundary layer thickness (m)} \]
\[ \eta = I_k/I_p = \text{thermodynamic efficiency} \]
\[ \nu = \text{fluid kinematic viscosity (m}^2\text{/s)} \]
\[ \Pi_3, \Pi_4, \Pi_5 = \text{dimensionless variable groups} \]
\[ \Psi_k = \text{kinetic power flux (W/m}^2\text{)} \]
\[ \Psi_p = \text{plate power flux (W/m}^2\text{)} \]
\[ \rho = \text{fluid density (kg/m}^3\text{)} \]

Subscripts

\[ 0 = \text{conduction} \]
\[ k = \text{kinetic} \]
\[ p = \text{plate} \]
\[ R = \text{downward-facing plate} \]
\[ \infty = \text{bulk fluid} \]
Superscripts

* = upward-facing plate
' = vertical plate

15. References


