The derivative of any algebraic expression is algebraic. First solve the problem of finding antiderivatives where the solution is a rational expression. Work backwards from the form of the solution to completely characterize those derivatives which can lead to the algebraic solution.

1 Rational Function Differentiation

Let
\[ y = \prod_{i=1}^{k} p_i(x)^{n_i} \]
be a rational function of x where the polynomials \( p_i(x) \) are squarefree and mutually relatively prime.

The derivative of \( y \) is
\[ y' = x' \sum_{i=1}^{k} n_i p_i(x)^{n_i-1} p'_i(x) \prod_{j \neq i} p_j(x)^{n_j} \]  
(1)

**Lemma 1** The expression \( \sum_{i=1}^{k} n_i p'_i(x) \prod_{j \neq i} p_j(x) \) has no factors in common with \( p_i(x) \).

Assume that the expression has a common factor \( p_h(x) \). Then

\[ p_h(x) \text{ divides } \sum_{i=1}^{k} n_i p'_i(x) \prod_{j \neq i} p_j(x) \]

Now, \( p_h(x) \) divides all terms for \( i \neq h \) and since it divides the whole sum, \( p_h(x) \) must divide the remaining term \( n_h p'_h(x) \prod_{j \neq h} p_j(x) \). But, from the above conditions, \( p_h(x) \) does not divide \( p'_h(x) \) [\( p_h(x) \) is squarefree] and \( p_h(x) \) does not divide \( p_h(x) \) for \( j \neq h \) [relatively prime condition].
2 Rational Function Integration

Now

\[ y' = x' \left( \prod_{i=1}^{k} p_i(x)^{n_i-1} \right) \sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x) \]  

(2)

There are no common factors between the sum and product terms of equation 2 because of the relatively prime condition of equation 1 and because of Lemma 1. Hence, this equation cannot be reduced and is canonical.

Split equation 2 into factors with positive and negative exponents and renumber \( i \) to be negative when \( n_i \) is negative, giving

\[ y' \prod_{i} p_i(x)^{-n_i+1} = x' \left( \prod_{i} p_i(x)^{n_i-1} \right) \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) \]  

(3)

Now to integrate equation 3 note that exponents \(-n_i+1 > 1\) because \( n_i < 0 \). Hence \( \prod_{i} p_i(x)^{-n_i+1} \) can factored (easily in fact by squarefree factorization). Now segregate the terms in the sum of equation 2 as well.

\[ \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) = \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) + \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) \prod_{k} p_k(x) \]

Substituting into equation 3 yields

\[ y' \prod_{i} p_i(x)^{-n_i+1} = \]

\[ x' \left( \prod_{i} p_i(x)^{n_i-1} \right) \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) + \]

\[ x' \left( \prod_{i} p_i(x)^{n_i-1} \right) \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) \prod_{k} p_k(x) \]  

(4)

The right side of this equation is now grouped into four polynomial terms \( AB' + A'B \) where

\[ A = \prod_{i} p_i(x)^{n_i} \]

\[ B' = \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) \]

\[ A' = \left( \prod_{i} p_i(x)^{n_i-1} \right) \sum_{i} n_i p_i'(x) \prod_{j \neq i} p_j(x) \]

\[ B = \prod_{k} p_k(x) \]
$A$ is the original numerator and $A'$ its derivative. $B$ and $B'$ can be derived from the squarefree factorization of the denominator of the integrand. $A$ and $A'$ can be recovered by a kind of long division of the right side of equation 4 by $B$ and $B'$ simultaneously. In addition to subtracting a term times $B'$ subtract the term’s derivative times $B$.

3 A First Order Differential Equation

Starting with equation 1 multiply through by $\prod_{i=1}^{k} p_i(x)$ and replace $\prod_{i=1}^{k} p_i(x)^{n_i}$ on the right side by $y$.

$$y' \prod_{i=1}^{k} p_i(x) = x' \sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x)$$  \hspace{1cm} (5)

By Lemma 1 this cannot be simplified because the two sides have no factor in common. Hence, this form is canonical.

Therefore, given an equation of form $y' q(x) = x' r(x)$, if it can be put into the form of equation 5, it can be solved as in equation 1. In order to do this we need to factor $q(x)$. This factoring can be seen as the same complexity as the partial fraction decomposition in Risch’s algorithm.

Once we have factored $q(x)$, we need to find a set of $n_i$ so that

$$\sum_{i=1}^{k} n_i p_i'(x) \prod_{j \neq i} p_j(x) = r(x)$$

. Now in order for this solution to be unique we need to show that the terms $p_i'(x) \prod_{j \neq i} p_j(x)$ are linearly independent and hence form the basis for a vector space. Let’s assume that they were not independent.

Suppose there existed a set of integers $m_i$ such that

$$\sum_{i=1}^{k} m_i p_i'(x) \prod_{j \neq i} p_j(x) = 0$$

and there exists some $m_i \neq 0$. If only one $m_i \neq 0$ then $p_i'(x) \prod_{j \neq i} p_j(x) = 0$. Since $p_j(x) \neq 0$ then $p_i'(x) = 0$. But then $p_i(x)$ would not be a polynomial in $x$. So then

$$-m_i p_i'(x) \prod_{j \neq i} p_j(x) = \sum_{h \neq i} m_h p_h'(x) \prod_{j \neq h} p_j(x)$$  \hspace{1cm} (6)

Now, $p_i(x)$ divides every term on the right side of equation 6 so $p_i(x)$ must also divide $-m_i p_i'(x) \prod_{j \neq i} p_j(x)$. But, because of squarefree, $p_i(x)$ does not divide $p_i'(x)$ and $p_i(x)$ does not not divide $p_j(x)$ when $j \neq i$. Hence, there exists a unique set of coefficients satisfying equation 5.