On the Relative Strength of Pebbling and Resolution (Extended Abstract)

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***Abstract*—The last decade has seen a revival of interest in pebble games in the context of proof complexity. Pebbling has proven to be a useful tool for studying resolution-based proof systems when comparing the strength of different subsystems, showing bounds on proof space, and establishing size-space trade-offs. The typical approach has been to encode the pebble game played on a graph as a CNF formula and then argue that proofs of this formula must inherit (various aspects of) the pebbling properties of the underlying graph. Unfortunately, the reductions used here are not tight. To simulate resolution proofs by pebblings, the full strength of nondeterministic black-white pebbling is needed, whereas resolution is only known to be able to simulate deterministic black pebbling. To obtain strong results, one therefore needs to find specific graph families which either have essentially the same properties for black and black-white pebbling (not at all true in general) or which admit simulations of black-white pebblings in resolution.**

This paper contributes to both these approaches. First, we design a restricted form of black-white pebbling that can be simulated in resolution and show that there are graph families for which such restricted pebblings can be asymptotically better than black pebblings. This proves that, perhaps somewhat unexpectedly, resolution can strictly beat black-only pebbling, and in particular that the space lower bounds on pebbling formulas in [Ben-Sasson and Nordström 2008] are tight. Second, we present a versatile parametrized graph family with essentially the same properties for black and black-white pebbling, which gives sharp simultaneous trade-offs for black and black-white pebbling for various parameter settings. Both of our contributions have been instrumental in obtaining the time-space trade-off results for resolution-based proof systems in [Ben-Sasson and Nordström 2009].

***Keywords*-proof complexity; resolution; pebble games; pebbling formula; space; trade-off;**

I. Introduction

Pebbling is a tool for studying time-space relationships by means of a game played on directed acyclic graphs. This game models computations where the execution is independent of the input and can be performed by straight-line programs. Each such program is encoded as a graph, and a pebble on a vertex in the graph indicates that the corresponding value is currently kept in memory. The goal is to pebble the output vertex of the graph with minimal number of pebbles (amount of memory) and steps (amount of time).

Pebble games were originally devised for studying programming languages and compiler construction, but have later found a broad range of applications in computational complexity theory. The pebble game model seems to have appeared for the first time (implicitly) in [36], where it was used to study flowcharts and recursive schemata, and it was later employed to model register allocation [41], and analyze the relative power of time and space as Turing-machine resources [17], [25]. Moreover, pebbling has been used to derive time-space trade-offs for algorithmic concepts such as linear recursion [16], [43], fast Fourier transform [42], [44], matrix multiplication [44], and integer multiplication [40]. An excellent survey of pebbling up to ca 1980 is [37], and some more recent developments are covered in the author's upcoming survey [33].

The *pebbling price* of a directed acyclic graph G in the black pebble game captures the memory space, or number of registers, required to perform the deterministic computation described by G. We will mainly be interested in the the more general *black-white pebble game* modelling nondeterministic computation. Black-white pebbling was introduced in [18] and has been studied in [23], [26], [27], [29]–[31], [46] and other papers. Let us refer to vertices of a directed graph having indegree 0 as *sources* and vertices having outdegree 0 as *sinks*.

Definition 1 (Pebble game). Let G be a directed acyclic graph (DAG) with a unique sink vertex z. The *black-white pebble game* on G is the following one-player game. At any time t, we have a configuration $\mathbb{P}_t = (B_t, W_t)$ of black pebbles B_t and white pebbles W_t on the vertices of G, at most one pebble per vertex. The rules of the game are as follows:

1) If all immediate predecessors of an empty vertex v have pebbles on them, a black pebble may be placed on v. In particular, a black pebble can always be placed on a source vertex.
2) A black pebble may be removed from any vertex at any time.
3) A white pebble may be placed on any empty vertex at any time.

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4) If all immediate predecessors of a white-pebbled vertex v have pebbles on them, the white pebble on v may be removed. In particular, a white pebble can always be removed from a source vertex.

A *(complete) black-white pebbling of* G, also called a *pebbling strategy for* G, is a sequence of pebble configurations $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ such that $\mathbb{P}_0 = (\emptyset, \emptyset)$, $\mathbb{P}_\tau = (\{z\}, \emptyset)$, and for all $t \in [\tau]$, $\mathbb{P}_t$ follows from $\mathbb{P}_{t-1}$ by one of the rules above. The *time* of a pebbling $\mathcal{P} = \{\mathbb{P}_0, \ldots, \mathbb{P}_\tau\}$ is simply $\mathit{time}(\mathcal{P}) = \tau$ and the *space* is $\mathit{space}(\mathcal{P}) = \max_{0 \leq t \leq \tau}\{|B_t \cup W_t|\}$. The *black-white pebbling price* (also known as the *pebbling measure* or *pebbling number*) of G, denoted $\mathit{BW\text{-}Peb}(G)$, is the minimum space of any complete pebbling of G.

A *black pebbling* is a pebbling using black pebbles only, i.e., having $W_t = \emptyset$ for all t. The *(black) pebbling price* of G, denoted $\mathit{Peb}(G)$, is the minimum space of any complete black pebbling of G.

In the last decade, there has been renewed interest in pebbling in the context of proof complexity.[1] A (non-exhaustive) list of proof complexity papers using pebbling in one way or another is [2], [5], [6], [8]–[13], [19], [21], [22], [24], [32], [35], [39]. The way pebbling results have been used in proof complexity has mainly been by studying so-called *pebbling contradictions* (also known as *pebbling formulas* or *pebbling tautologies*). These are CNF formulas encoding the pebble game played on a DAG G by postulating the sources to be true and the sink to be false, and specifying that truth propagates through the graph according to the pebbling rules. The idea to use such formulas seems to have appeared for the first time in [28], and they were also studied in [13], [38] before being explicitly defined in [12].

Definition 2 (Pebbling contradiction). Suppose that G is a DAG with sources S and a unique sink z. Identify every vertex $v \in V(G)$ with a propositional logic variable v. The *pebbling contradiction* over G, denoted Peb_G, is the conjunction of the following clauses:

- for all $s \in S$, a unit clause s (*source axioms*),
- For all non-sources v with immediate predecessors $pred(v)$, the clause $\bigvee_{u \in pred(v)} \overline{u} \vee v$ (*pebbling axioms*),
- for the sink z, the unit clause $\overline{z}$ (*target* or *sink axiom*).

Let $f_d : \{0,1\}^d \mapsto \{0,1\}$ be any any nonconstant Boolean function. Then the *substitution pebbling contradiction with respect to* f_d is the CNF formula $Peb_G[f_d]$ obtained by substituting $f_d(x_1, \ldots, x_d)$ for every variable x and expanding the result to conjunctive normal form in the canonical way.

If the graph G has n vertices and maximal indegree ℓ, $Peb_G[f_d]$ is easily verified to be an unsatisfiable formula over dn variables with less than $2^{d(\ell+1)} \cdot n$ clauses of size at most $d(\ell+1)$ (i.e., a $(d(\ell+1))$-CNF formula as defined shortly). An example illustrating Definition 2 is given in Figure 1.

Let us also briefly recall some proof complexity definitions for completeness. A *literal* is either a propositional logic variable or its negation, denoted x and $\overline{x}$, respectively. A *clause* $C = a_1 \vee \cdots \vee a_k$ is a set of literals. A clause containing at most k literals is called a k*-clause*. A *CNF formula* $F = C_1 \wedge \cdots \wedge C_m$ is a set of clauses. A k*-CNF formula* is a CNF formula consisting of k-clauses. We say that F *implies* C, denoted $F \vDash C$, if any truth value assignment satisfying F must also satisfy C.

Definition 3 (Resolution [1]). A sequence of *clause configurations* (sets of clauses) $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ is a *resolution refutation* of a CNF formula F if $\mathbb{C}_0 = \emptyset$, $\mathbb{C}_\tau$ contains the contradictory *empty clause* 0 without any literals, and for all $t \in [\tau]$, $\mathbb{C}_t$ is obtained from $\mathbb{C}_{t-1}$ by one of the following rules:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for some $C \in F$ (an *axiom clause*).

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for some $C \in \mathbb{C}_{t-1}$.

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{D\}$ for some D inferred from $C_1, C_2 \in \mathbb{C}_{t-1}$ by the *resolution rule*, i.e., $D = C_1 \cup C_2 \setminus \{x, \overline{x}\}$ for some variable x such that $x \in C_1 \setminus C_2$ and $\overline{x} \in C_2 \setminus C_1$.

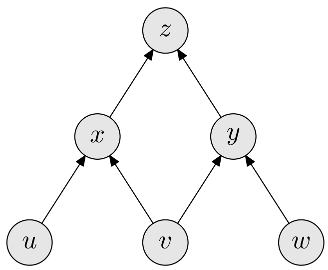
Definition 4 (Length and space). The *length* $L(\pi)$ of a resolution derivation π is the total number of axiom downloads and inferences made in π, i.e., the total number of clauses counted with repetitions.

The *clause space* $Sp(\mathbb{C})$ of a clause configuration $\mathbb{C}$ is $|\mathbb{C}|$, i.e., the number of clauses in $\mathbb{C}$, and the *total space* $TotSp(\mathbb{C})$ is $\sum_{C \in \mathbb{C}} |C|$, i.e., the total number of literals in $\mathbb{C}$ counted with repetitions. The clause space (total space) of a derivation π is the maximal clause space (total space) of any clause configuration $\mathbb{C} \in \pi$.

Taking the minimum over all refutations of a CNF formula F, we define $L(F \vdash 0) = \min_{\pi : F \vdash 0}\{L(\pi)\}$, $Sp(F \vdash 0) = \min_{\pi : F \vdash 0}\{Sp(\pi)\}$, and $TotSp(F \vdash 0) = \min_{\pi : F \vdash 0}\{TotSp(\pi)\}$ as the length, clause space, and total space of refuting F in resolution, respectively.

Given any black-only pebbling $\mathcal{P}$ of G, it is straightforward to simulate this pebbling in resolution to refute the corresponding pebbling contradiction $Peb_G[f_d]$ in length $\mathrm{O}(\mathit{time}(\mathcal{P}))$ and total space $\mathrm{O}(\mathit{space}(\mathcal{P}))$. This was perhaps first noted in [8] for the simple Peb_G formulas, but the simulation extends readily to any formula $Peb_G[f_d]$, with the constants hidden in the asymptotic notation depending on f_d and the maximal indegree of G. In the other direction, it was recently shown in [11] (strengthening results in [9]) that if f_d has the right properties—for instance, if it is the exclusive or function or the threshold function evaluating to true if k out of d variables are true for $1 < k < d$—then

[1]We remark that the pebble game studied in this paper should not be confused with the (very different) *existential pebble games* that have also been used in proof complexity, for instance, in the papers [3], [4], [7].



(a) Pyramid graph Π_2 of height 2.

$$
\begin{aligned}
& u \\
\wedge\ & v \\
\wedge\ & w \\
\wedge\ & (\overline{u} \vee \overline{v} \vee x) \\
\wedge\ & (\overline{v} \vee \overline{w} \vee y) \\
\wedge\ & (\overline{x} \vee \overline{y} \vee z) \\
\wedge\ & \overline{z}
\end{aligned}
$$

(b) Pebbling contradiction Peb_{Π_2}.

$$
\begin{aligned}
& (u_1 \vee u_2) & \wedge\ & (\overline{v}_2 \vee \overline{w}_1 \vee y_1 \vee y_2) \\
\wedge\ & (v_1 \vee v_2) & \wedge\ & (\overline{v}_2 \vee \overline{w}_2 \vee y_1 \vee y_2) \\
\wedge\ & (w_1 \vee w_2) & \wedge\ & (\overline{x}_1 \vee \overline{y}_1 \vee z_1 \vee z_2) \\
\wedge\ & (\overline{u}_1 \vee \overline{v}_1 \vee x_1 \vee x_2) & \wedge\ & (\overline{x}_1 \vee \overline{y}_2 \vee z_1 \vee z_2) \\
\wedge\ & (\overline{u}_1 \vee \overline{v}_2 \vee x_1 \vee x_2) & \wedge\ & (\overline{x}_2 \vee \overline{y}_1 \vee z_1 \vee z_2) \\
\wedge\ & (\overline{u}_2 \vee \overline{v}_1 \vee x_1 \vee x_2) & \wedge\ & (\overline{x}_2 \vee \overline{y}_2 \vee z_1 \vee z_2) \\
\wedge\ & (\overline{u}_2 \vee \overline{v}_2 \vee x_1 \vee x_2) & \wedge\ & \overline{z}_1 \\
\wedge\ & (\overline{v}_1 \vee \overline{w}_1 \vee y_1 \vee y_2) & \wedge\ & \overline{z}_2 \\
\wedge\ & (\overline{v}_1 \vee \overline{w}_2 \vee y_1 \vee y_2) & &
\end{aligned}
$$

(c) Substitution pebbling contradiction $Peb_{\Pi_2}[\vee_2]$ with respect to binary logical or.

Figure 1. Example of pebbling contradiction with substitution for the pyramid graph Π_2.

any resolution refutation π of $Peb_G[f_d]$ can be translated into a black-white pebbling of G with time and space upper-bounded by the length and space of π, respectively (adjusted for small multiplicative constants depending on the maximal indegree of G).

There is an obvious gap in these reductions between pebbling and resolution. To interpret a resolution refutation of a pebbling contradiction in terms of a pebbling of the underlying graph, the full power of black-white pebbling is needed to make the reduction work. If we want to translate pebblings of graphs into refutations of the corresponding pebbling contradictions, however, we only know how to do this for the weaker black pebble game.

To see why resolution has a hard time simulating black-white pebblings, let us start by discussing a black-only pebbling $\mathcal{P}$. We can easily mimic such a pebbling in a resolution refutation of $Peb_G[f_d]$ by deriving that $f_d(v_1, \ldots, v_d)$ is true whenever the corresponding vertex v in G is black-pebbled. We end up deriving that $f_d(z_1, \ldots, z_d)$ is true for the sink z, at which point we can download the sink axioms and derive a contradiction. The intuition behind this translation is that a black pebble on v means that we know v, which in resolution translates into truth of v. In the pebble game, having a white pebble on v instead means that we need to assume v. By duality, we let this correspond to falsity of v in resolution. Focusing on the pyramid Π_2 and pebbling contradiction $Peb_{\Pi_2}[\vee_2]$ in Figure 1, our intuitive understanding then becomes that white pebbles on x and y and a black pebble on z should correspond to the set of clauses

$$\{\overline{x}_i \vee \overline{y}_j \vee z_1 \vee z_2 \mid i, j = 1, 2\} \tag{1}$$

which indeed encode that assuming $x_1 \vee x_2$ and $y_1 \vee y_2$, we can deduce $z_1 \vee z_2$. See Figure 2(a) for an illustration of this.

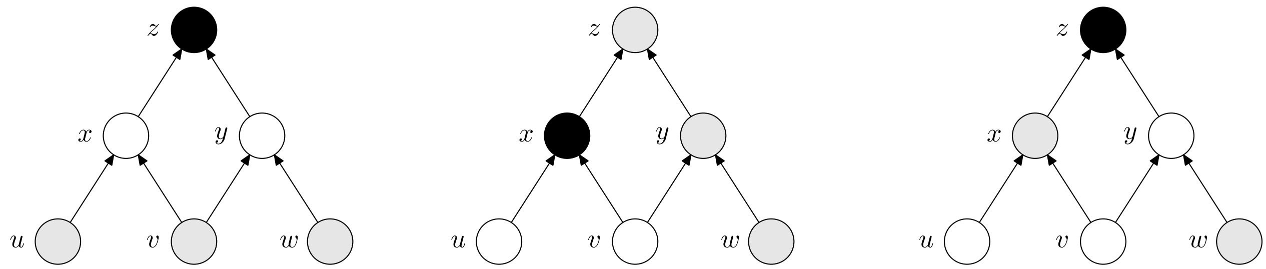
If we now place white pebbles on u and v, this allows us to remove the white pebble from x. Rephrasing this in terms of resolution, we can say that x follows if we assume u and v, which is encoded as the set of clauses

$$\{\overline{u}_i \vee \overline{v}_j \vee x_1 \vee x_2 \mid i, j = 1, 2\} \tag{2}$$

(see Figure 2(b)), and indeed, from the clauses in (1) and (2) we can derive in resolution that z is black-pebbled and u, v and y are white pebbled, i.e., the set of clauses

$$\{\overline{u}_i \vee \overline{v}_j \vee \overline{y}_k \vee z_1 \vee z_2 \mid i, j, k = 1, 2\} \tag{3}$$

(see Figure 2(c)). This toy example indicates one of the problems one runs into when one tries to simulate black-white pebbling in resolution: as the number of white pebbles grows, there is an exponential blow-up in the number of clauses. The clause set in (3) is twice the size of those in (1) and (2), although it corresponds to only one more white



(a) $\{\overline{x}_i \vee \overline{y}_j \vee z_1 \vee z_2 \mid i,j = 1,2\}$. (b) $\{\overline{u}_i \vee \overline{v}_j \vee x_1 \vee x_2 \mid i,j = 1,2\}$. (c) $\{\overline{u}_i \vee \overline{v}_j \vee \overline{y}_k \vee z_1 \vee z_2 \mid i,j,k = 1,2\}$.

Figure 2. Black and white pebbles and (intuitively) corresponding sets of clauses.

pebble. This suggests that as a pebbling starts to make heavy use of white pebbles, a resolution refutation will not be able to mimic such a pebbling in a length- and space-preserving manner.

This leads to the thought that perhaps black pebbling provides not only upper but also lower bounds on resolution refutations of pebbling contradictions. This would be consistent with what has been known so far. For all pebbling contradictions with proven space lower bounds, the underlying graphs have asymptotically the same black and black-white pebbling price, and hence all known lower bounds can be expressed in terms of black pebbling. There have been no examples of pebbling contradictions where resolution can do strictly better than black pebbling and tightly match smaller bounds on space in terms of black-white pebbling.

II. Our Results

Our first set of results is that resolution can in fact be strictly better than black-only pebbling, both for time-space trade-offs and with respect to space in absolute terms. We prove this by designing a limited version of black-white pebbling, where we explicitly restrict the amount of nondeterminism, i.e., white pebbles, a pebbling strategy can use. Such restricted pebbling use "few white pebbles per black pebble" (in a sense that will be made formal below), and can therefore be simulated in a time- and space-preserving manner by resolution, avoiding the exponential blow-up just discussed. We then show that for all known separation results in the pebbling literature where black-white pebbling does asymptotically better than black-only pebbling, there are graphs exhibiting these separations for which optimal black-white pebblings can be carried out in our limited version of the game. This means that resolution refutations of pebbling contradictions over such DAGs can do strictly asymptotically better than what is suggested by black-only pebbling, matching the lower bounds in terms of (general) black-white pebbling.

More precisely, we obtain such results for three families of graphs.[2] The first family are the *bit reversal graphs* studied by Lengauer and Tarjan [30], for which black-white pebbling has quadratically better trade-offs than black pebbling.

Lemma 5 ([30]). *There are DAGs $\{G_n\}_{n=1}^{\infty}$ of size $\Theta(n)$ with black pebbling price $\mathsf{Peb}(G_n) = 3$ such that any optimal black pebbling $\mathcal{P}_n$ of G_n exhibits a trade-off $\mathsf{time}(\mathcal{P}_n) = \Theta\bigl(n^2/\mathsf{space}(\mathcal{P}_n) + n\bigr)$ but optimal black-white pebblings $\mathcal{P}_n$ of G_n achieve a trade-off $\mathsf{time}(\mathcal{P}_n) = \Theta\bigl((n/\mathsf{space}(\mathcal{P}_n))^2 + n\bigr)$.*

Theorem 6. *Fix any non-constant Boolean function f and let $Peb_{G_n}[f]$ be pebbling contradictions over the graphs in Lemma 5. Then for any monotonically nondecreasing function $s(n) = \mathrm{O}(\sqrt{n})$ there are resolution refutations π_n of $Peb_{G_n}[f]$ in total space $\mathrm{O}(s(n))$ and length $\mathrm{O}\bigl((n/s(n))^2\bigr)$, beating the lower bound $\Omega\bigl(n^2/s(n)\bigr)$ for black-only pebblings of G_n.*

Focusing next on absolute bounds on space rather than time-space trade-offs, the best known separation between black and black-white pebbling for polynomial-size graphs is the one shown by Wilber [46].

Lemma 7 ([46]). *There are DAGs $\{G(s)\}_{s=1}^{\infty}$ having size polynomial in s such that the black-white pebbling price of $G(s)$ is $\mathsf{BW\text{-}Peb}(G(s)) = \mathrm{O}(s)$ but the black pebbling price is $\mathsf{Peb}(G(s)) = \Omega(s \log s / \log\log s)$.*

For pebbling formulas over these graphs we do *not* know how to beat the black pebbling space bound—we return to this somewhat intriguing problem in Section IV—but using instead graphs in [26] exhibiting the same pebbling properties, we can obtain the desired result.

Theorem 8. *Fix any non-constant Boolean function f and let $Peb_{G(s)}[f]$ be pebbling contradictions over the graphs $G(s)$ in [26] with pebbling properties as in Lemma 7. Then there are resolution refutations π_n of $Peb_{G(s)}[f]$ in total*

[2]All graphs discussed in this paper are explicitly constructible and have bounded vertex indegree. Also, unless otherwise stated they have a single, unique sink. We do not repeat this in the formal statements here in order not to clutter the text unnecessarily.

space $\mathrm{O}(s)$*, beating the lower bound* $\Omega(s \log s / \log \log s)$ *for black-only pebbling.*

If we remove all restriction on graph size, there is a quadratic separation of black and black-white pebbling established by Kalyanasundaram and Schnitger [26].

Lemma 9 ([26]). *There are DAGs* $\{G(s)\}_{s=1}^{\infty}$ *of size* $\exp(\Theta(s \log s))$ *such that* $\textsf{BW-Peb}(G(s)) \leq 3s + 1$ *but* $\textsf{Peb}(G(s)) \geq s^2$.

For pebbling formulas over these graphs, resolution again matches the black-white pebbling bounds.

Theorem 10. *Fix any non-constant Boolean function* f *and let* $Peb_{G(s)}[f]$ *be pebbling contradictions over the graphs* $G(s)$ *in Lemma 9. Then there are resolution refutations* π_n *of* $Peb_{G(s)}[f]$ *in total space* $\mathrm{O}(s)$*, beating the lower bound* $\Omega(s^2)$ *for black-only pebbling.*

In particular, Theorems 8 and 10 show that the lower bound on proof space for pebbling contradictions in terms of black-white pebbling price in [9] is tight (up to constant factors).

Turning to our second set of results, we first note that in spite of the theorems above, for general pebbling formulas we still do not know of any way of simulating black-white pebbling in resolution. Instead, we are limited to deriving upper bounds from black-only pebblings while lower bounds have to be obtained in terms of black-white pebblings. At first sight, this might not look too bad since the space gap between the two can be at most quadratic, as shown by Meyer auf der Heide [31]. However, the translation given in [31] of a black-white pebbling in space s to a black pebbling in space $\mathrm{O}(s^2)$ incurs an exponential blow-up in pebbling time, destroying all hope of obtaining nontrivial time-space trade-off results for resolution in this way. Hence, to get meaningful trade-offs for pebbling formulas we need graph families with strong *dual* trade-offs for black and black-white pebbling simultaneously. In this paper, we present such a family of graphs, building on and strengthening previous work by Carlson and Savage [14], [15].

Theorem 11. *There is an explicitly constructible two-parameter graph family* $\Gamma(c, r)$*, for* $c, r \in \mathbb{N}^+$*, having unique sink, vertex indegree* 2*, and size* $\Theta(cr^3 + c^3r^2)$*, and satisfying the following properties:*

1) *The graph* $\Gamma(c, r)$ *has black-white pebbling price* $\textsf{BW-Peb}(\Gamma(c, r)) = r + \mathrm{O}(1)$ *and black pebbling price* $\textsf{Peb}(\Gamma(c, r)) = 2r + \mathrm{O}(1)$.
2) *There is a black-only pebbling of* $\Gamma(c, r)$ *in time linear in the graph size and in space* $\mathrm{O}(c + r)$.
3) *Suppose that* $\mathcal{P}$ *is a black-white pebbling of* $\Gamma(c, r)$ *with* $\textsf{space}(\mathcal{P}) \leq r + s$ *for* $0 < s \leq c/8$*. Then* $\textsf{time}(\mathcal{P}) \geq \left(\frac{c-2s}{4s+4}\right)^r \cdot r!$.

The graph family in Theorem 11 turns out to be surprisingly versatile. For instance, we can use it to prove among other things the rather striking statement that for any *arbitrarily slowly growing* non-constant function, there are explicit graphs of such (arbitrarily small) pebbling space complexity that nevertheless exhibit *superpolynomial* time-space trade-offs for black and black-white pebbling simultaneously.

Theorem 12. *Let* $g(n)$ *be any arbitrarily slowly growing*[3] *monotone function* $\omega(1) = g(n) = \mathrm{O}(n^{1/7})$*, and let* $\epsilon > 0$ *be an arbitrarily small positive constant. Then there is a family of explicitly constructible single-sink DAGs* $\{G_n\}_{n=1}^{\infty}$ *of size* $\Theta(n)$ *such that the following holds:*

1) G_n *has black-white pebbling price* $\textsf{BW-Peb}(G) = g(n) + \mathrm{O}(1)$ *and black pebbling price* $\textsf{Peb}(G) = 2 \cdot g(n) + \mathrm{O}(1)$.
2) *There is a complete black pebbling* $\mathcal{P}$ *of* G_n *with* $\textsf{time}(\mathcal{P}) = \mathrm{O}(n)$ *and* $\textsf{space}(\mathcal{P}) = \mathrm{O}\left(\sqrt[3]{n/g^2(n)}\right)$
3) *Any complete black-white pebbling of* G_n *in space at most* $\left(n/g^2(n)\right)^{1/3-\epsilon}$ *requires pebbling time superpolynomial in* n.

More examples of interesting trade-offs that can be obtained from the graphs in Theorem 11 are given in the full-length version [34] of this paper.

In the rest of this paper, we outline the main ideas in the proofs of our results in Section III and then discuss some remaining open problems in Section IV. We refer to [34] for missing formal definitions and full proofs.

III. Outline of Constructions and Proofs

We need to set up a fair amount of technical machinery before we can give the full, formal proofs of our results. In order not to obscure unnecessarily what are in essence reasonably straightforward arguments, in this extended abstract we focus on giving an overview of the main ideas, eschewing unnecessary technicalities.

A. Labelled Black-White Pebbling and Resolution

Let us start by discussing the tools used to establish Theorems 6, 8, and 10. The idea is to design a version of the black-white pebble game that is tailor-made for resolution. This game is essentially just a formalization of the naive resolution simulation sketched in Section I, but before stating the formal definitions, let us try to provide some intuition why the rules of this new game look the way they do.

First, if we want a game that can be mimicked by resolution, then placements of isolated white vertices do not

[3]Note that we also assume $g(n) = \mathrm{O}(n^{1/7})$, i.e., that $g(n)$ does not grow to fast. This is just a simplifying technical assumption. If we allow the minimal space to grow as fast as n^{ϵ} for some $\epsilon > 0$, then it is easy to use our graph family with other parameter settings to obtain even stronger results. Hence, the interesting aspect here is that $g(n)$ is allowed to grow arbitrarily slowly.

make much sense. What a resolution derivation can do is to download axiom clauses, and intuitively this corresponds to placing a black pebble on a vertex together with white pebbles on its immediate predecessors, if it has any. Therefore, we adopt such aggregate moves as the only admissible way of placing new pebbles. For instance, looking at the graph Π_2 and pebbling contradiction $Peb_{\Pi_2}[\vee_2]$ in Figure 1 again, placing a black pebble on z and white pebbles on x and y corresponds to downloading the axiom clauses in (1).

Second, note that if we have a black pebble on z with white pebbles on x and y corresponding to the clauses in (1) and a black pebble on x with white pebbles on u and v corresponding to the clauses in (2), we can derive the clauses in (3) corresponding to z black-pebbled and u, v and y white-pebbled but no pebble on x. This suggests that a natural rule for white pebble removal is that a white pebble can be removed from a vertex if a black pebble is placed *on that same vertex* (and not on its immediate predecessors).

Third, if we then just erase all clauses in (3), this corresponds to all pebbles disappearing. On the face of it, this is very much unlike the rule for white pebble removal in the standard pebble game, where it is absolutely crucial that a white pebble can only be removed when its predecessors are pebbled. However, the important point here is that not only do the white pebbles disappear—the black pebble that has been placed on z with the help of these white pebbles disappears as well. What this means is that we cannot treat black and white pebbles in isolation, but we have to keep track of for each black pebble which white pebbles it depends on, and make sure that the black pebble also is erased if any of the white pebbles supporting it is erased. The way we do this is to label each black pebble v with its supporting white pebbles W, and define the pebble game in terms of moves of such labelled *pebble subconfigurations* $v\langle W\rangle$.

Definition 13 (Pebble subconfiguration). For v a vertex and W a set of vertices, we say that $v\langle W\rangle$ is a *pebble subconfiguration* with a black pebble on v supported by white pebbles on W. The black pebble on v is said to be *dependent* on the white pebbles in its *support* W. We refer to $v\langle\emptyset\rangle$ as an *independent black pebble*.

Our next definition now formalizes the informal description of our new pebble game. We remark that this definition is quite similar to the pebble game defined in [32], and that we have borrowed freely from notation and terminology there.

Definition 14 (Labelled pebbling). For G any DAG with unique sink z, a (complete) *labelled pebbling* of G is a sequence $\mathcal{L} = \{\mathbb{L}_0, \ldots, \mathbb{L}_\tau\}$ of labelled pebble configurations such that $\mathbb{L}_0 = \emptyset$, $\mathbb{L}_\tau = \{z\langle\emptyset\rangle\}$, and for all $t \in [\tau]$ it holds that $\mathbb{L}_t$ can be obtained from $\mathbb{L}_{t-1}$ by one of the following rules:

Introduction $\mathbb{L}_t = \mathbb{L}_{t-1} \cup \{v\langle pred(v)\rangle\}$, where $pred(v)$ is the set of immediate predecessors of v.

Erasure $\mathbb{L}_t = \mathbb{L}_{t-1} \setminus \{v\langle V\rangle\}$ for $v\langle V\rangle \in \mathbb{L}_{t-1}$.

Merger $\mathbb{L}_t = \mathbb{L}_{t-1} \cup \{v\langle (V \cup W) \setminus \{w\}\rangle\}$ for $v\langle V\rangle$, $w\langle W\rangle \in \mathbb{L}_{t-1}$ with $w \in V$. We denote this subconfiguration $\mathsf{merge}(v\langle V\rangle, w\langle W\rangle)$, and refer to it as a *merger on* w.

Let the set of all black-pebbled vertices in $\mathbb{L}_t$ be denoted $Bl(\mathbb{L}_t) = \bigcup\{v \mid v\langle W\rangle \in \mathbb{L}_t\}$ and let $Wh(\mathbb{L}_t) = \bigcup\{W \mid v\langle W\rangle \in \mathbb{L}_t\}$ be the set of all white-pebbled vertices. Then the space of an labelled pebbling $\mathcal{L} = \{\mathbb{L}_0, \ldots, \mathbb{L}_\tau\}$ is $\max_{\mathbb{L}\in\mathcal{L}}\{|Bl(\mathbb{L}) \cup Wh(\mathbb{L})|\}$ and the time is $\mathit{time}(\mathcal{L}) = \tau$.

Figures 2(a) and 2(b) are both examples of subconfigurations resulting from introduction moves, and if we merge the two we get the subconfiguration in Figure 2(c).

The game in Definition 14 might look quite different from the standard black-white pebble game, but it is not hard to show that labelled pebblings are essentially just a restricted form of black-white pebblings. (See [34] for the proof.)

Lemma 15. *If G is a single-sink DAG and $\mathcal{L}$ is a complete labelled pebbling of G, then there is a complete black-white pebbling $\mathcal{P}_\mathcal{L}$ of G with $\mathit{time}(\mathcal{P}_\mathcal{L}) \leq \frac{4}{3}\mathit{time}(\mathcal{L})$ and $\mathit{space}(\mathcal{P}_\mathcal{L}) \leq \mathit{space}(\mathcal{L})$.*

However, the definition of space of labelled pebblings does not seem quite right from the point of view of resolution. Not only does the space measure fail to capture the exponential blow-up in the number of white pebbles discussed above. We also have the problem that if one white pebble is used to support many different black pebbles, then in a resolution refutation simulating such a pebbling we have to pay multiple times for this single white pebble, once for every black pebble supported by it. To get something that can be simulated by resolution, we therefore need to restrict the labelled pebble game even further.

Definition 16 (Bounded labelled pebblings). An (m, S)-*bounded labelled pebbling* is a labelled pebbling $\mathcal{L} = \{\mathbb{L}_0, \ldots, \mathbb{L}_\tau\}$ such that every $\mathbb{L}_t$ contains at most m pebble subconfigurations $v\langle W\rangle$ and every $v\langle W\rangle$ has white support size $|W| \leq S$.

It is easy to see that boundedness automatically implies low space complexity, since an (m, S)-bounded pebbling $\mathcal{L}$ clearly satisfies $\mathit{space}(\mathcal{L}) \leq m(S + 1)$. And using the concept of bounded labelled pebblings, we can show that if there is such a pebbling of a graph G, then this pebbling can be used as a template for a resolution refutation of any pebbling contradiction $Peb_G[f]$. (We again refer to [34] for the proof.)

Lemma 17. *Suppose that $\mathcal{L}$ is any complete (m, S)-bounded pebbling of a graph G and that f is any nonconstant Boolean function of arity d. Then there is a resolution refutation $\pi_\mathcal{L}$ of the formula $Peb_G[f]$ in simultaneous*

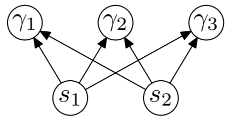


Figure 3. Base case for Carlson-Savage graph with 3 spines and sinks.

length $L(\pi_{\mathcal{L}}) = \textsf{time}(\mathcal{L}) \cdot \exp\bigl(\mathrm{O}(dS)\bigr)$ *and total space* $TotSp(\pi_{\mathcal{L}}) = m \cdot \exp\bigl(\mathrm{O}(dS)\bigr)$*. In particular, fixing* f *it holds that resolution can simulate* $(m, \mathrm{O}(1))$*-bounded pebblings in a time- and space-preserving manner.*

The whole problem thus boils down to the question whether there are graphs with (a) asymptotically different properties for black and black-white pebbling for which (b) optimal black-white pebblings can be carried out in the bounded labelled pebbling framework. The answer to this question turns out to be yes, and the space upper bounds for the pebbling contradictions in Theorems 6, 8, and 10 are all proven by exhibiting bounded labelled pebblings for the corresponding graphs. The details concerning how these graphs are constructed, as well as how they are pebbled, are somewhat intricate, however, and cannot be presented here due to space constraints. We instead refer the reader to the full-length version [34] of this paper.

B. A Graph Family with Tight Dual Trade-off Properties

Let us next outline the proof of our graph pebbling trade-off results in Theorem 11. We remark that in what follows, we will discuss a slightly different setting where graphs may have multiple sinks, not just one, and where we only require that a pebbling *visits* every sink once, touching it with a black or white pebble, instead of leaving a black pebble on the sink until the end of the pebbling. It is straightforward to translate results for such pebblings back to the setting in Theorem 11. (See [34] for the technical details.)

Our graph family is built on a construction by Carlson and Savage [14], [15]. Carlson and Savage only prove their trade-off for black pebbling, however, and the extension of their results to black-white pebbling requires changing the construction and doing a nontrivial amount of extra work (as is usually the case when one wants to lift a black pebbling result to black-white pebbling). The formal definition of the family of graphs, which we will refer to as *Carlson-Savage graphs*, is probably easier to parse if the reader first studies the illustrations in Figures 3 and 4.

Definition 18 (Carlson-Savage graphs). The two-parameter graph family $\Gamma(c, r)$, for $c, r \in \mathbb{N}^+$, is defined by induction over r. The base case $\Gamma(c, 1)$ is a DAG consisting of two sources s_1, s_2 and c sinks $\gamma_1, \ldots, \gamma_c$ with directed edges (s_i, γ_j), for $i = 1, 2$ and $j = 1, \ldots, c$, i.e., edges from both sources to all sinks. The graph $\Gamma(c, r+1)$ has c sinks and is built from the following components:

- c disjoint copies $\Pi^{(1)}_{2r}, \ldots, \Pi^{(c)}_{2r}$ of a pyramid graph[4] of height $2r$ with sinks $z_1, \ldots, z_c$.
- one copy of $\Gamma(c, r)$, for which we denote the sinks by $\gamma_1, \ldots, \gamma_c$.
- c disjoint and identical *spines*, where each spine is composed of cr *sections*, and every section contains $2c$ vertices. We let the vertices in the ith section of a spine be denoted $v[i]_1, \ldots, v[i]_{2c}$.

The edges in $\Gamma(c, r+1)$ are as follows:

- All "internal edges" in $\Pi^{(1)}_{2r}, \ldots, \Pi^{(c)}_{2r}$ and $\Gamma(c, r)$ are present also in $\Gamma(c, r+1)$.
- For each spine, there are edges $\bigl(v[i]_j, v[i]_{j+1}\bigr)$ for all $j = 1, \ldots, 2c-1$ within each section i and edges $\bigl(v[i]_{2c}, v[i+1]_1\bigr)$ from the end of a section to the beginning of next for $i = 1, \ldots, cr-1$, i.e., for all sections but the final one, where $v[cr]_{2c}$ is a sink.
- For each section i in each spine, there are edges $\bigl(z_j, v[i]_j\bigr)$ from the jth pyramid sink to the jth vertex in the section for $j = 1, \ldots, c$, as well as edges $\bigl(\gamma_j, v[i]_{c+j}\bigr)$ from the jth sink in $\Gamma(c, r)$ to the $(c+j)$th vertex in the section for $j = 1, \ldots, c$.

Let us focus on the trade-off lower bound in part 3 of Theorem 11, which is the hard part to prove, and let us start by trying to provide some intuition why this bound should hold. For simplicity, consider first black-only pebblings. Assume inductively that part 3 of Theorem 11 has been proven for $\Gamma(c, r-1)$ and consider $\Gamma(c, r)$. Any pebbling strategy for this DAG will have to pebble through all sections in all spines. Consider the first section anywhere, let us say on spine j, that has been completely pebbled, i.e., there have been pebbles placed on and removed from all vertices in the section. Let us say that this happens at time τ_1. But this means that $\Gamma(c, r-1)$ and all pyramids $\Pi^{(1)}_{2(r-1)}, \ldots, \Pi^{(c)}_{2(r-1)}$ must have been completely pebbled during this part of the pebbling as well. Fix any pyramid and consider some point in time $\sigma_1 < \tau_1$ when there are at least $r+1$ pebbles on its vertices, which must happen because of known pebbling lower bounds for pyramids [17], [27]. At this point, the rest of the graph must contain very few pebbles (think of s here as being very small). In particular, there are very few pebbles on the subgraph $\Gamma(c, r-1)$ at time σ_1, so for all practical purposes we can think of $\Gamma(c, r-1)$ as being essentially empty of pebbles.

Consider now the next section in the spine j that is completed, say, at time $\tau_2 > \tau_1$. Again, we can argue that some pyramid is completely pebbled in the time interval $[\tau_1, \tau_2]$, and thus has $r+1$ pebbles on it at some time $\sigma_2 > \tau_1 > \sigma_1$. This means that $\Gamma(c, r-1)$ is essentially empty of pebbles at time σ_2 as well. But note that all sinks in the subgraph $\Gamma(c, r-1)$ must have been pebbled in the

[4]We omit the formal definition here, but as an example the graph in Figure 1(a) is a pyramid of height 2.

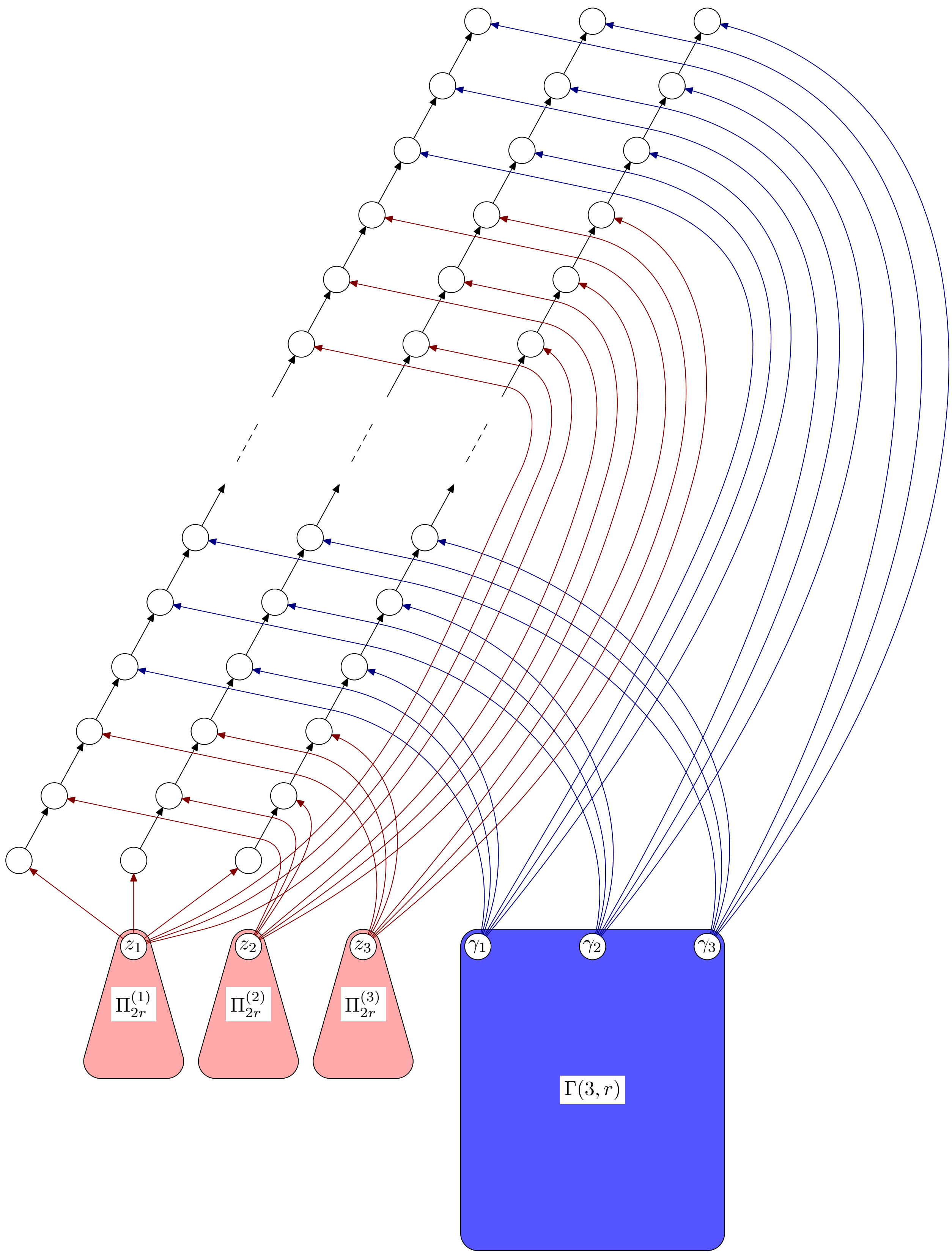


Figure 4. Inductive definition of Carlson-Savage graph $\Gamma(3, r+1)$ with 3 spines and sinks.

time interval $[\sigma_1, \sigma_2]$, and since we know that $\Gamma(c, r-1)$ is (almost) empty at times σ_1 and σ_2, this allows us to apply the induction hypothesis. Since $\mathcal{P}$ has to pebble through a lot of sections in different spines, we will be able to repeat the above argument many times and apply the induction hypothesis on $\Gamma(c, r-1)$ each time. Adding up all the lower bounds obtained in this way, the induction step goes through.

This is the spirit of the proof of the black-only pebbling trade-off in [15]. When we instead want to deal with black-white pebblings, things get much more complicated. Black pebblings must by necessity pebble through a graph in a bottom-up fashion, and it is therefore straightforward to measure "how far" a black pebbling has progressed. A black-white pebbling, however, can place and remove pebbles anywhere in the DAG at any time. Therefore, it is more difficult to control the progress of a black-white pebbling, and one has to use different ideas and work harder in the proof.

We establish part 3 of Theorem 11 by proving a slightly stronger lemma, dealing with *conditional* pebblings that start with some pebbles already present on the graph, and can also leave some pebbles on the graph at the end of the pebbling. A crucial ingredient in the proof is that we assume below (without loss of generality) that all pebblings are *frugal*, meaning that no obviously redundant pebble placements are made, but that all pebbles placed on the graph are used to place other black pebbles on successors or to remove white pebbles from successors. (Again, we refer to [34] for a more thorough discussion of these pebbling technicalities.)

Lemma 19. *Suppose that $\mathcal{P} = \{\mathbb{P}_\sigma, \dots, \mathbb{P}_\tau\}$ is a conditional black-white pebbling on $\Gamma(c, r)$ such that*

1) $\max\{\mathit{space}(\mathbb{P}_\sigma), \mathit{space}(\mathbb{P}_\tau)\} < s$ *for s in the range* $0 < s \le c/8 - 1$.
2) *$\mathcal{P}$ pebbles all sinks in $\Gamma(c, r)$ during the interval $[\sigma, \tau]$.*
3) $\mathit{space}(\mathcal{P}) < r + s + 2$.

Then it holds that $\mathit{time}(\mathcal{P}) = \tau - \sigma \ge \left(\frac{c-2s}{4s+4}\right)^r \cdot r!$.

To establish this result we will need the following four technical lemmas, the proofs of which are omitted due to space constraints but can be found in [34]. The first two technical lemmas are easy, but the second pair of lemmas are somewhat less immediate and provide the key to the proof.

Our first technical lemma gives important information about how black-white pebbling strategies must treat the spines of the Carlson-Savage graphs.

Lemma 20. *Suppose that G is a DAG and that v is a vertex in G with a path Q to some sink z_i of G such that all vertices in $Q \setminus \{z_i\}$ have outdegree 1. Then any frugal black-white pebbling strategy pebbles v exactly once, and the path Q contains pebbles during one contiguous time interval.*

The next technical lemma speaks about subgraphs H of a DAG G whose only connection to the rest of the graph $G \setminus H$ are via the sink of H. Note that the pyramids in $\Gamma(c, r)$ satisfy this condition.

Lemma 21. *Let G be a DAG and H a subgraph in G such that H has a unique sink z_h and the only edges between $V(H)$ and $V(G) \setminus V(H)$ emanate from z_h. Suppose that $\mathcal{P}$ is any frugal complete pebbling of G having the property that H is completely empty of pebbles at some given time τ' but at least one vertex of H has been pebbled during the time interval $[0, \tau']$. Then $\mathcal{P}$ pebbles H completely during the interval $[0, \tau']$.*

Our final two technical lemmas say that not too many pyramids can be pebbled simultaneously in a space-efficient pebbling, and that this is true for the spines as well. This means that any space-efficient pebbling will have to alternate back and forth between time intervals when there are a lot of pebbles on some pyramid and time intervals when all sinks in $\Gamma(c, r-1)$ are being pebbled. Assuming inductively that we already know that Lemma 19 holds for $\Gamma(c, r-1)$, these alternations in the pebbling strategies will allow us to apply the induction hypothesis to $\Gamma(c, r-1)$ multiple times, which yields the required lower bound for $\Gamma(c, r)$.

Lemma 22. *At all times during a pebbling of $\Gamma(c, r)$ as in Lemma 19, strictly less than $4(s+1)$ pyramids $\Pi_{2r}^{(j)}$ contain pebbles simultaneously.*

Lemma 23. *At all times during a pebbling of $\Gamma(c, r)$ as in Lemma 19, strictly less than $4(s+1)$ spine sections contain pebbles simultaneously.*

Now we can prove the black-white pebbling time-space trade-offs for the Carlson-Savage graphs.

Proof of Lemma 19: Let $\mathcal{P} = \{\mathbb{P}_\sigma, \dots, \mathbb{P}_\tau\}$ be a pebbling as in the statement of the lemma. We show that $\mathit{time}(\mathcal{P}) \ge T(c, r, s) = \left(\frac{c-2s}{4s+4}\right)^r \cdot r!$ by induction over r.

For $r = 1$, the assumptions in the lemma imply that more than $c - 2s$ sinks are empty at times σ and τ. These sinks must be pebbled, which trivially requires strictly more than $c - 2s > \left(\frac{c-2s}{4s+4}\right) = T(c, 1, s)$ time steps.

Assume that the lemma holds for $\Gamma(c, r-1)$ and consider any pebbling of $\Gamma(c, r)$. Less than $2s$ spines contain pebbles at time σ or time τ. All the other strictly more than $c - 2s$ spines are empty at times σ and τ but must be completely pebbled during $[\sigma, \tau]$ since their sinks are pebbled during this time interval. (This can be more formally argued but we omit the technicalities here.)

Consider the first time σ' when any spine gets a pebble for the first time. Let us denote this spine by Q'. By Lemma 20 we know that Q' contains pebbles during a contiguous time interval until it is completely pebbled and emptied at, say, time τ'. During this whole interval $[\sigma', \tau']$ less than $4s + 4$ sections contain pebbles at any one given time by Lemma 23, so in particular less then $4s + 4$ spines contain pebbles. Moreover, Lemma 20 says that every spine containing pebbles will remain pebbled until completed. What this means is that if we order the spines with respect to the time when they first receive a pebble in groups of size $4s + 4$, no spine in the second group can be pebbled until the at least one spine in the first group has been completed.

We observe that this divides the spines that are empty at the beginning and end of $\mathcal{P}$ into strictly more than $\frac{c-2s}{4s+4}$ groups. Furthermore, we claim that completely pebbling just one empty spine requires at least $r \cdot T(c, r-1, s)$ time steps. Given this claim we are done, since it follows that the total pebbling time must then be lower-bounded by $\frac{c-2s}{4s+4} r \cdot T(c, r-1, s) = T(c, r, s)$. This is so since at least one spine from each group is pebbled in a time interval totally disjoint from the time intervals for all spines in the next group.

It remains to establish the claim. To this end, fix any spine Q^* empty at times σ^* and τ^* but completely pebbled in $[\sigma^*, \tau^*]$. Consider the first time $\tau_1 \in [\sigma^*, \tau^*]$ when any section in Q^*, let us denote it by R_1, has been completely pebbled (i.e., all vertices has been touched by pebbles but are now empty again). During the time interval $[\sigma^*, \tau_1]$ all pyramid sinks $z_1, \ldots, z_c$ must be pebbled (since they are immediate predecessors). Since less than $2 \cdot (4s + 4) < c$ pyramids contain pebbles at times σ^* or τ_1 (Lemma 22), at least one pyramid is pebbled completely (Lemma 21), which requires $r+1$ pebbles. Moreover, there is at least one pebble on the section R_1 during this whole interval. Hence, there must exist a point in time $\sigma_1 \in [\sigma^*, \tau_1]$ when there are strictly less than $(r+2)+s-(r+1)-1 = s$ pebbles on the subgraph $\Gamma(c, r-1)$. Also, at this time σ_1 less than $4s + 4$ sections contain pebbles (Lemma 23), and in particular this means that there are pebbles on less than $4s+3$ other section of our spine Q^*. This puts an upper bound on the number of sections of Q^* that can have been touched by pebbles this far, since every section is completely pebbled during a contiguous time interval before being emptied again, and we chose to focus on the first section R_1 in Q^* that was finished.

Look now at the first section R_2 in Q^* *other than the less than* $4s + 4$ *sections containing pebbles at time* σ_1 that is completely pebbled, and let the time when R_2 is finished be denoted τ_2 (clearly, $\tau_2 > \tau_1$). During $[\sigma_1, \tau_2]$ all sinks of $\Gamma(c, r-1)$ must have been pebbled, and at time $\tau_2 - 1$ less than $4s + 3$ other section in Q^* contain pebbles.

Finally, consider the first new section R_3 in our spine Q^* to be completely pebbled among those not yet touched at time $\tau_2 - 1$. Suppose that R_3 is finished at time τ_3. Then during $[\tau_2, \tau_3]$ some pyramid is completely pebbled, and thus there is some time $\sigma_3 \in (\tau_2, \tau_3)$ when there are at least $r+1$ pebbles on this pyramid and at least one pebble on the spine Q^*, leaving less than s pebbles for $\Gamma(c, r-1)$. But this means that we can apply the induction hypothesis on the interval $[\sigma_1, \sigma_3]$ and deduce that $\sigma_3 - \sigma_1 \geq T(c, r-1, s)$. Note also that at time σ_3 less than $8s + 8 < c$ sections in Q^* have been finished.

Continuing in this way, for every group of $8s + 8 < c$ finished sections in the spine Q^* we get one pebbling of $\Gamma(c, r-1)$ in space less than $r + s + 1$ and with less than s pebbles in the start and end configurations, which allows us to apply the induction hypothesis a total number of at least $\frac{cr}{8s+8} > r$ times. (Just to argue that we get the constants right, note that $8s+8 < c$ implies that after the final pebbling of the sinks of $\Gamma(c, r-1)$ has been done, there is still some empty section left in Q^*. When this final section is taken care of, we will again get at least $r + 1$ pebbles on some pyramid while at least one pebble resides on Q^*, so we get the space on $\Gamma(c, r-1)$ down below s as is needed for the induction hypothesis.)

This proves our claim that pebbling one spine takes time at least $r \cdot T(c, r-1, s)$. Lemma 19 now follows. ■

IV. Concluding Remarks

It is known that the black-white pebbling price is always a lower bound on the resolution space of refuting pebbling contradictions $Peb_G[f]$ with respect to the "right" functions f, as proven in [9]. Also, for all graphs studied in this context so far there have been shown to exist refutations of the corresponding pebbling contradictions in space upper-bounded by the black-white pebbling price—trivially for graphs where the black and black-white pebbling prices coincide, and more interestingly for the graphs in the current paper where the black-white pebbling price is asymptotically smaller than the black pebbling price. This naturally raises the question whether it holds in general that the refutation space of pebbling contradictions is asymptotically equal to the black-white pebbling price of the underlying graphs.

Open Question 1. *Is in true for any DAG G with bounded vertex indegree and any (fixed) Boolean function f that the pebbling contradiction $Peb_G[f]$ can be refuted in total space $\mathrm{O}(\textit{BW-Peb}(G))$?*

More specifically, one could ask—as a natural first line of attack if one wants to investigate whether the answer to the above question could be yes—if it holds that bounded labelled pebblings are in fact as powerful as general black-white pebblings. In a sense, this is asking whether only a very limited form of nondeterminism is sufficient to realize the full potential of black-white pebbling.

Open Question 2. *Does it hold that any complete black-white pebbling $\mathcal{P}$ of a single-sink DAG G with bounded vertex indegree can be simulated by a $(\mathrm{O}(\textit{space}(\mathcal{P})), \mathrm{O}(1))$-bounded pebbling $\mathcal{L}$?*

Note that a positive answer to this second question would immediately imply a positive answer to the first question as well by Lemma 17.

We have no strong intuition either way regarding Open Question 1, but as to Open Question 2 it would perhaps be somewhat surprising if bounded labelled pebblings turned out to be as strong as general black-white pebblings. Interestingly, although the optimal black-white pebblings of the graphs in Lemma 9 can be simulated by bounded pebblings, the same approach does *not* work for the original graphs separating black-white from black-only pebbling in [46]. Indeed, these latter graphs might be a candidate graph family for answering Open Question 2 in the negative.

Finally, we are intrigued by the question of whether the properties of the formulas $Peb_G[f]$ shown to hold in [9], [11] for "the right kind" of functions f in fact extend to the simpler formulas $Peb_G[\vee]$ defined in terms of non-exclusive or.

Open Question 3. *Is it true for any DAG G that any resolution refutation π of $Peb_G[\vee]$ can be translated into a black-white pebbling with time and space upper-bounded in terms of the length and space of π?*

Earlier results in [32], [35] can be interpreted as indicating that this should be the case, but the results there only apply to limited classes of graphs and only capture space lower bounds, not time-space trade-offs. And the papers [9], [11] do not shed any light on this question, as the techniques used there inherently cannot work for formulas defined in terms of non-exclusive or.

If the answer to Open Question 3 is yes—which we would cautiously expect it to be—then this could be useful for settling the complexity of decision problems for resolution proof space, i.e., the problem given a CNF formula F and a space bound s to determine whether F has a resolution refutation in space at most s. Reducing from pebbling space by way of formulas $Peb_G[\vee]$ would avoid the blow-up of the gap between upper and lower bounds on pebbling space that cause problems when using, for instance, exclusive or.

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