Performance of Local Algorithms in Random Structures. Power and limitations

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Sub-Linear Algorithms Bootcamp

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Outline

Part I. Challenges

Part II. Solution Space Geometry

Part III. Local Algorithms

Part I. Challenges

(Arguably) Most Embarrassing Algorithmic Problem in Random Graphs

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- Still open. This is embarrassing...

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Algorithmically (G & Li [2017])

$$\text{Ave}_{ALG}(C_{I,J}) = \frac{4(1+o(1))}{3\sqrt{2}}\sqrt{\frac{\log n}{k}}.$$

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Efficiently (convex optimization) only when $n \ge n^*_{\text{Convex}} = \Omega\left(k \log p\right)$.
Donoho, Tanner, Wainwright, Hastie, Tibshirani, Candes, Tao [1996–]

Part II. Geometry of the solution space

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The set of R-optimal solutions is partitioned into separated disconnected components.

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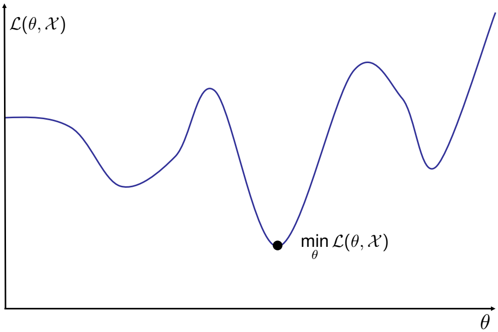

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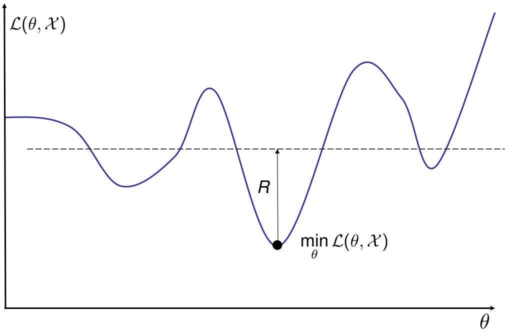

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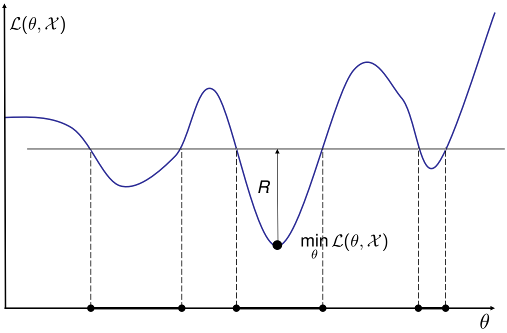

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- connected for $\beta < \frac{1}{2}$, Coja–Oghlan & Efthymiou [2013].
- Extension to multi-overlaps $|I_1 \cap \cdots \cap I_m|$ by Rahman & Virag [2015].

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- disconnected when $d > \frac{2^K \log^2 K \log 2}{K} = \frac{\log^2 K}{K} d^*$, Achlioptas, Coja-Oghlan & Ricci-Tersenghi [2011], G & Sudan [2014] (for NAE-K-SAT).

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- non-monotone when $n < cn^*_{\text{Convex}}$, G & Zadik [2017].

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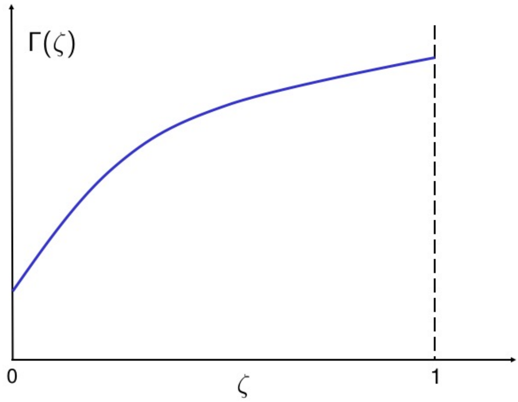
When $n > Cn^*_{\text{Convex}}$



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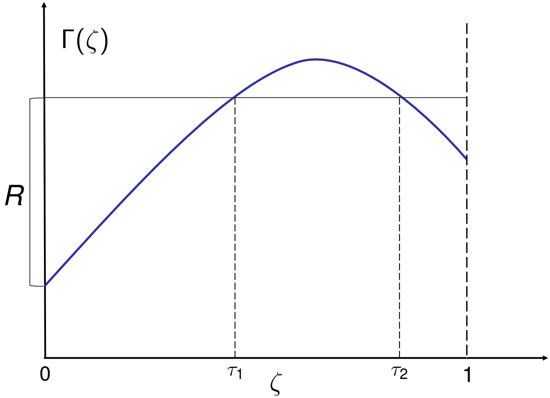

Part III. Local algorithms

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- Gradient Descent (GD): keep improving while you can (MCMC with $\beta = \infty$)

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  - The probability of this is $O\left(\frac{1}{r!}\right)$ – decays faster than the exponential growth rate d^r of the neighborhood of r.

Properties

- Greedy and Local-in-Parallel are special cases of Local-Sequential (Obvious).
- Greedy for ind set can be well approximated by Local-in-Parallel, G & Goldberg [2010]
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    - The probability of this is $O\left(\frac{1}{r!}\right)$ – decays faster than the exponential growth rate d^r of the neighborhood of r.
    - As a result, for most nodes the decision is r-localized.

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- As a result, most of the Gibbs ”mass” is concentrated on ind sets I with $|I| \approx |I^*|$.

Summary

Punchline: For all of the discussed models, below the OGP one of the versions of the local algorithms finds a solution. Above the threshold no algorithms are known and possibly no poly-time algorithms exist.

Examples: Ind set in sparse random graphs

In either $\mathbb{G}(n, \frac{d}{n})$ or $\mathbb{G}_d(n)$, local parallel (FIID) algorithms can construct ind sets with size

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Also MCMC fail finding ind sets of the same size Coja–Oghlan & Efthymiou [2013] (MCMC mixes in exponential time).

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OGP serves as an obstruction to the GD algorithm, G & Zadik [2017]

Conclusion

In conclusion, we have a fairly complete theory: optimization in random graphs is solvable by local algorithms below the OGP threshold. Above the threshold local algorithms provably fails, but no other poly-time algorithms are known, and possibly none exist.

Future Work

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- Planted clique problem $\mathbb{G}(n, \frac{1}{2}, m)$ exhibits non-monotonicity implying OGP ***iff*** the size of the planted clique $m = o(\sqrt{n})$, G & Zadik [201?]

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- Sparse Planted XOR-K-SAT formula $\Phi(n, dn)$ exhibits non-monotonicity implying OGP.

Thank you