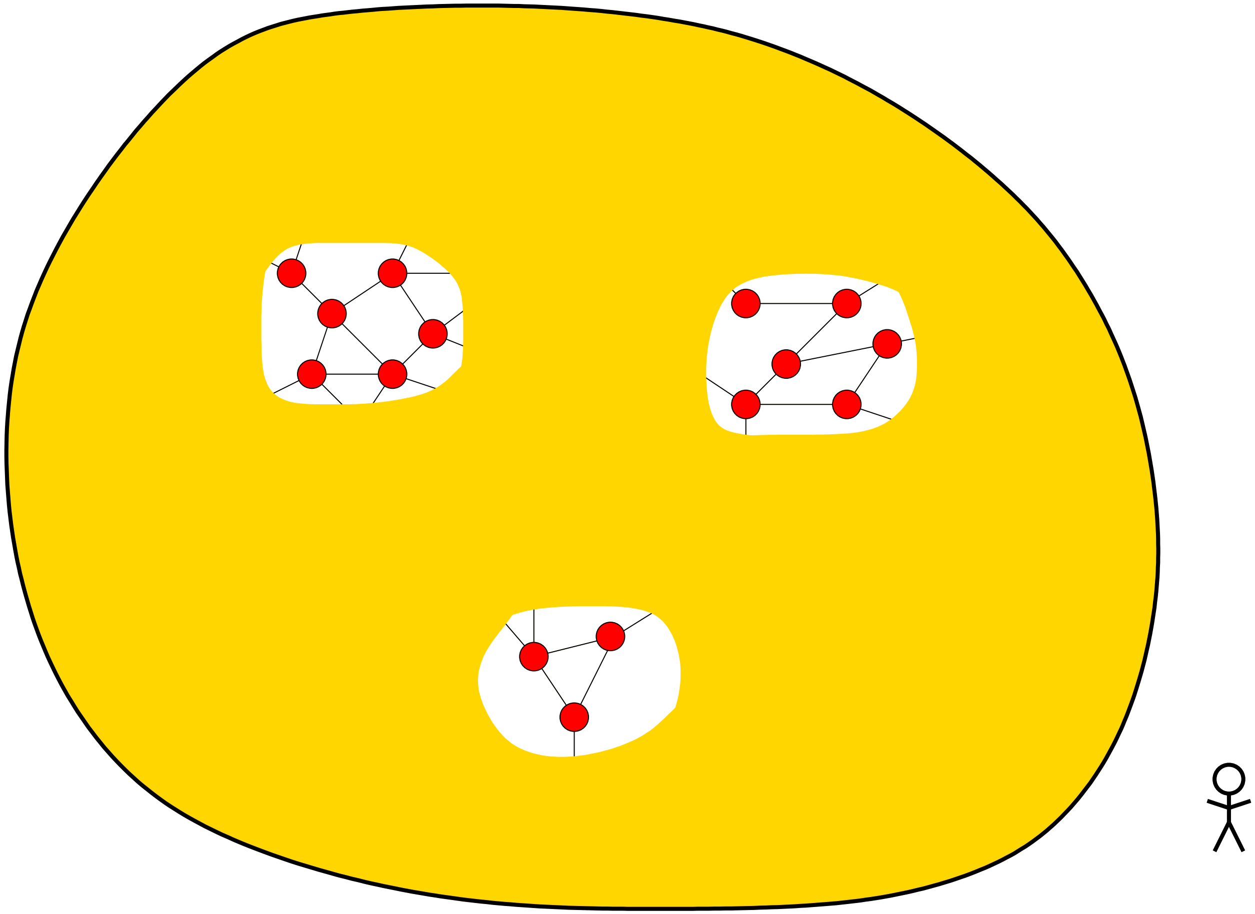
Random Local Exploration Techniques for Sublinear-Time Algorithms

Krzysztof Onak

IBM Research

Sublinear-Time Algorithms

Sublinear-Time Algorithms

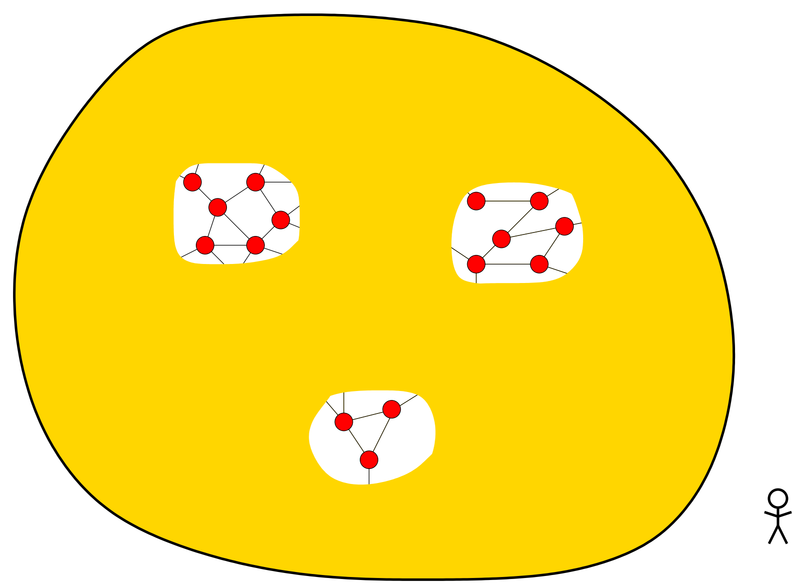


Sublinear-time algorithms:

Fast answer based on inspecting
a tiny fraction of the input

Motivation

- Existing big graph:
  - social network
  - bank transactions
  - network connections
- Goal: quickly learn something about it



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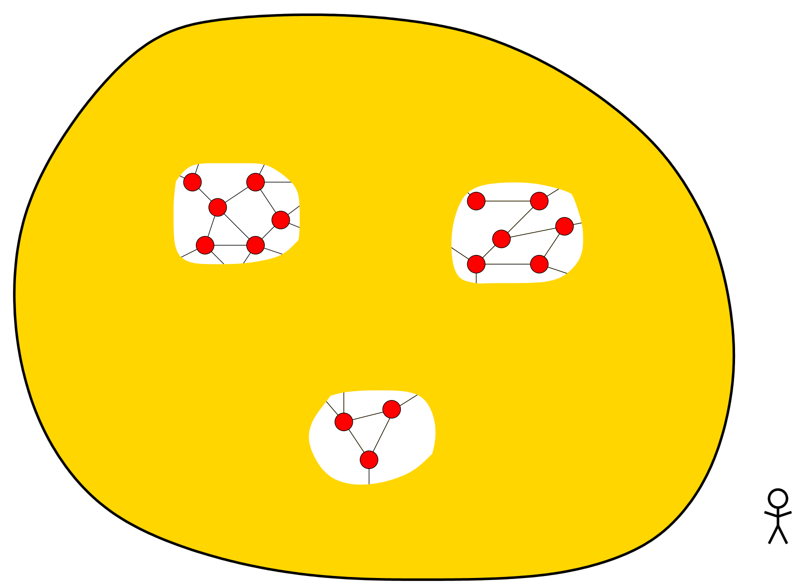
  - bipartite?



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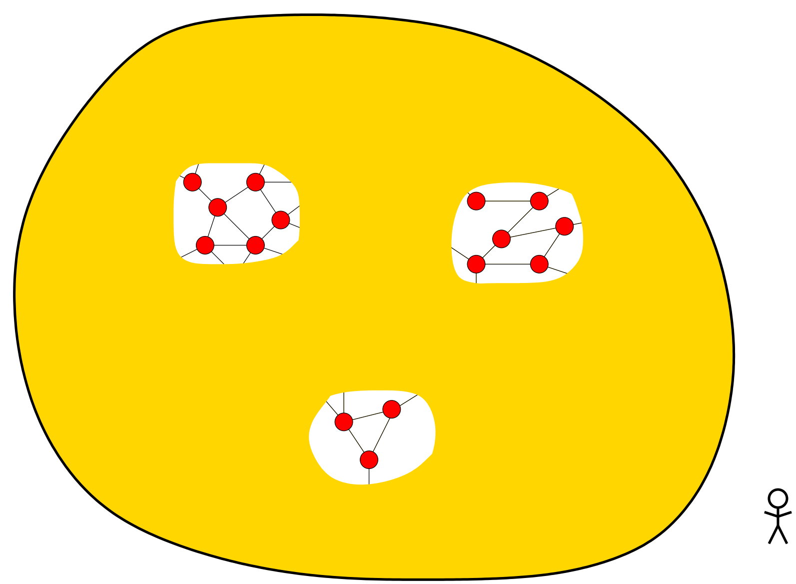
- Estimate a graph parameter:
  - number of triangles
  - dominating set
  - vertex cover



Roadmap

Focus on:

- simple graph problems and properties
- sparse graphs

1. Simulation of greedy algorithms
2. Partitioning oracles
3. Random walks

Graph Access Model

Allowed operations:

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Essentially: query access to adjacency lists

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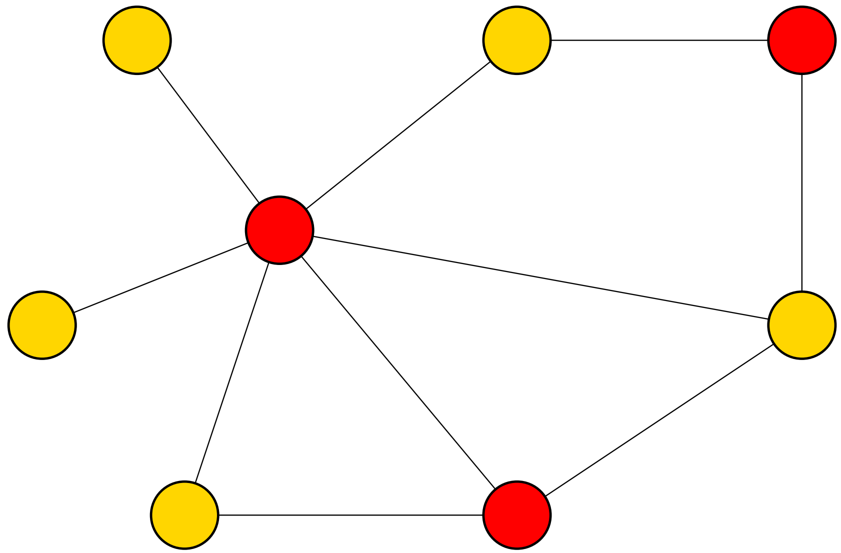
1. Simulation of greedy algorithms

2. Partitioning oracles

3. Random walks

Example: Vertex Cover

Goal: find smallest set S of vertices such that each edge has endpoint in S



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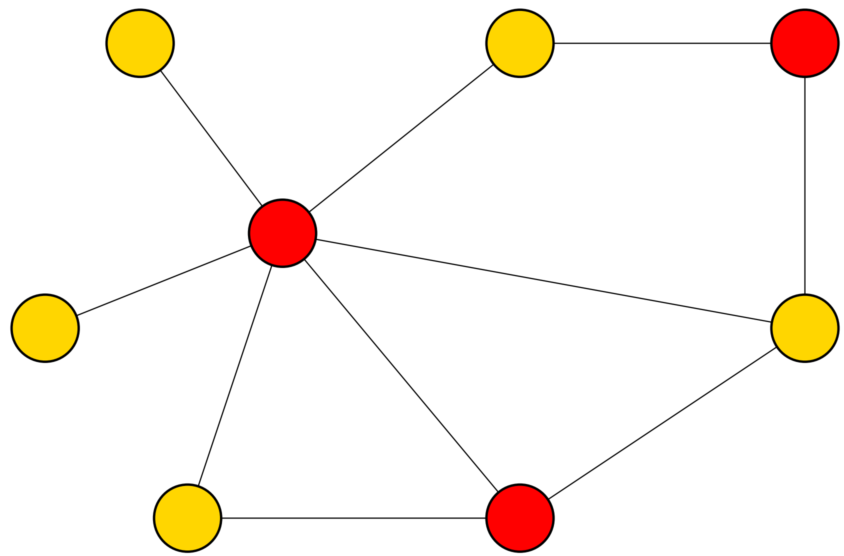

- Best polynomial time algorithm: 2-approximation

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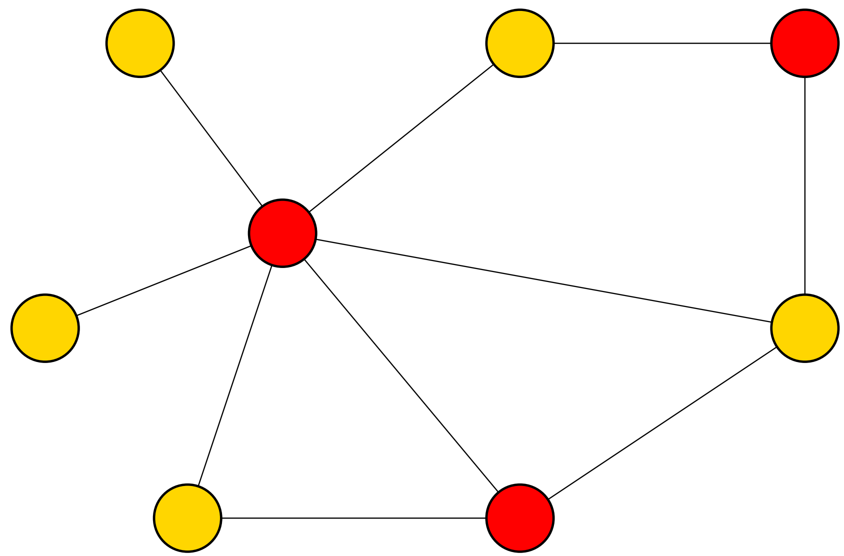

- Best polynomial time algorithm: 2-approximation
- Here:

$$\mathrm{VC} - \epsilon n \leq \text{(computed value)} \leq 2 \cdot \mathrm{VC} + \epsilon n$$

where $\mathrm{VC} =$ minimum vertex cover size
$n =$ number of vertices

Essential Technique

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- Multiple applications:
  - vertex cover approximation
  - maximum matching approximation
  - computing nice partitions of graphs
  - local distributed algorithms
  - approximate planarity verification
  - local computation algorithms

Essential Technique

- We develop a local computation technique
- Multiple applications:
  - vertex cover approximation
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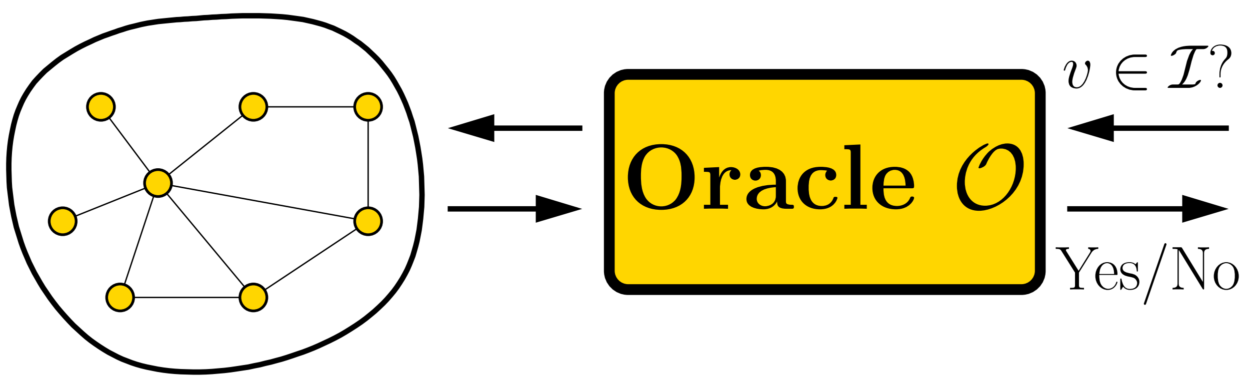
  - computing nice partitions of graphs
  - local distributed algorithms
  - approximate planarity verification
  - local computation algorithms
- Will present and apply a less general version: local computation of maximal independent set

Main Tool:
Constructing a Maximal Independent Set Locally

Oracle for Maximal Independent Set

Want to construct oracle $\mathcal{O}$:

- $\mathcal{O}$ has query access to $G = (V, E)$
- $\mathcal{O}$ provides query access to maximal independent set $\mathcal{I} \subseteq V$
- $\mathcal{I}$ is not a function of queries
  it is a function of G and random bits



Goal: Minimize the query processing time

Challenge of Locality

- Simplest algorithm for Maximal Independent Set $\mathcal{I}$:
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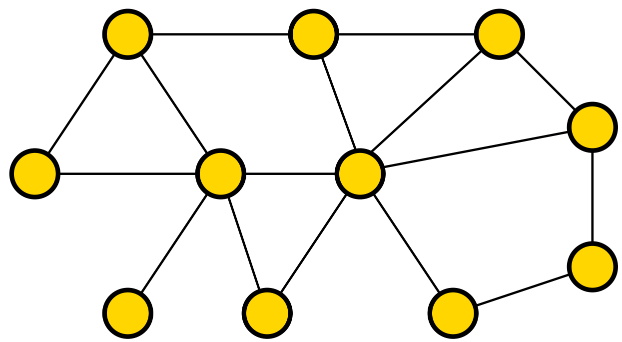

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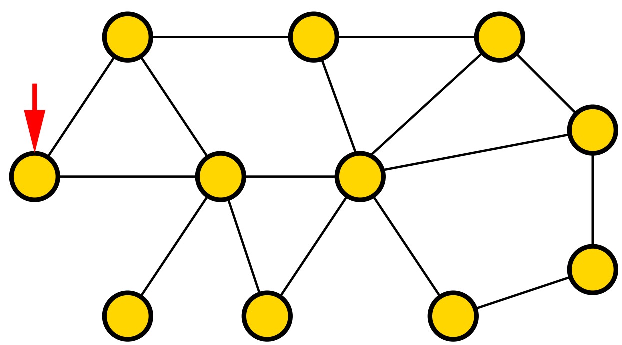

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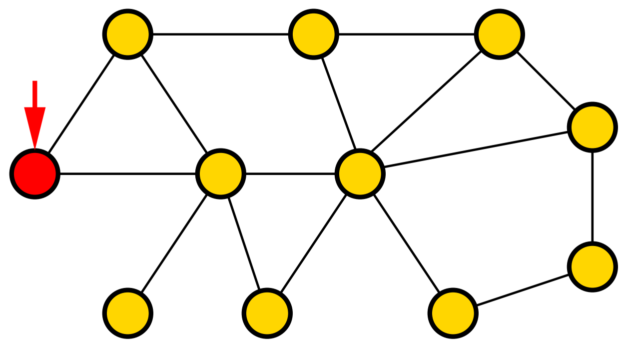

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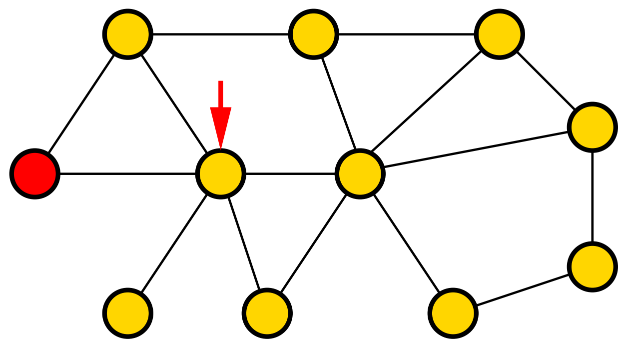

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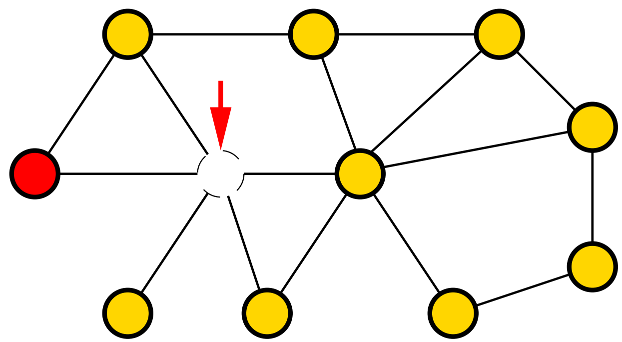

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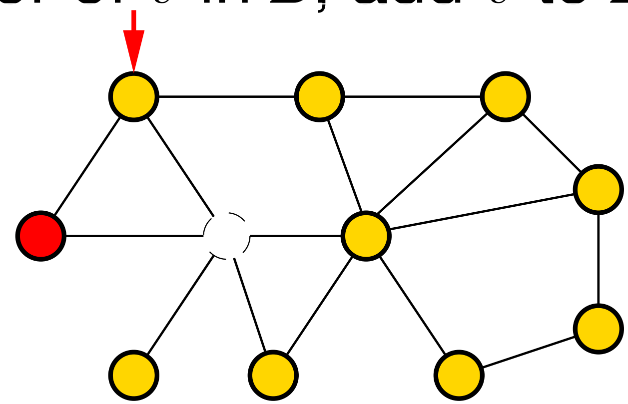

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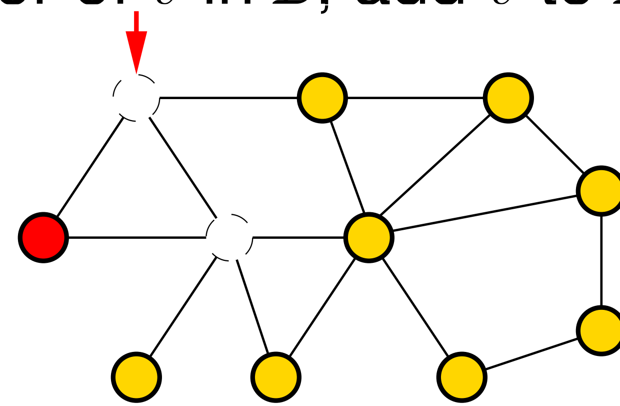

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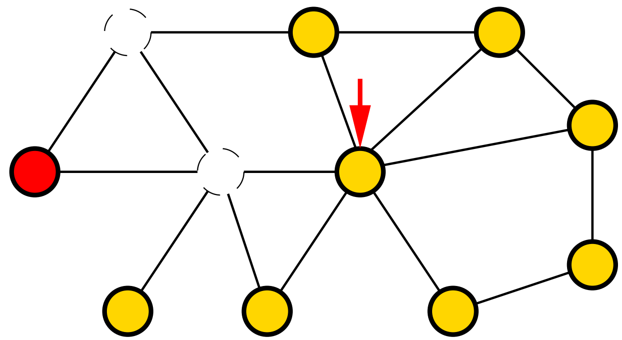

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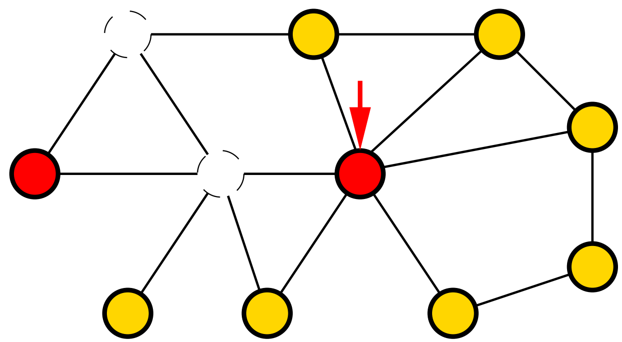

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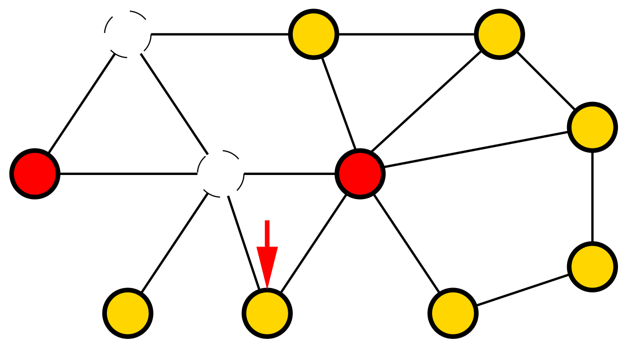

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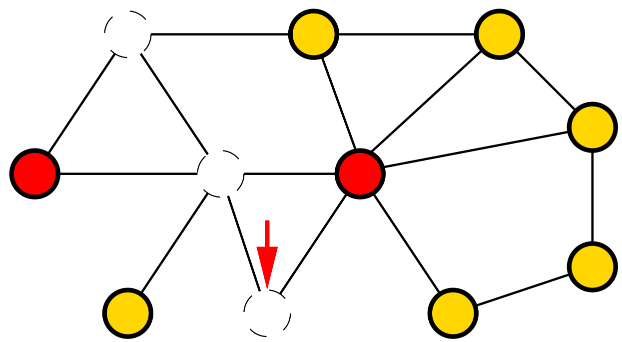

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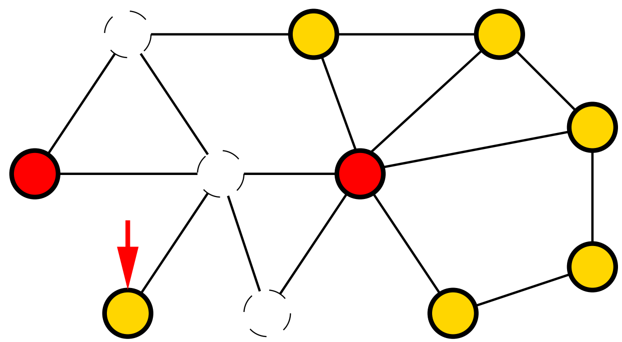

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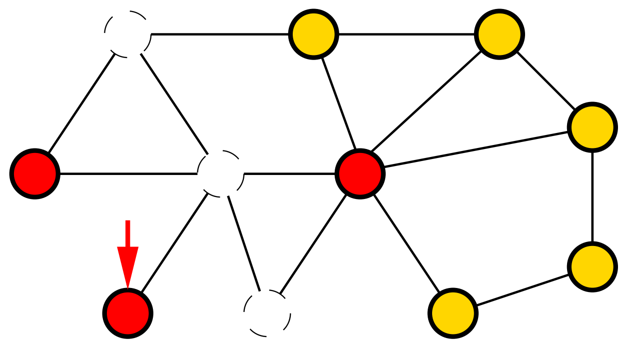

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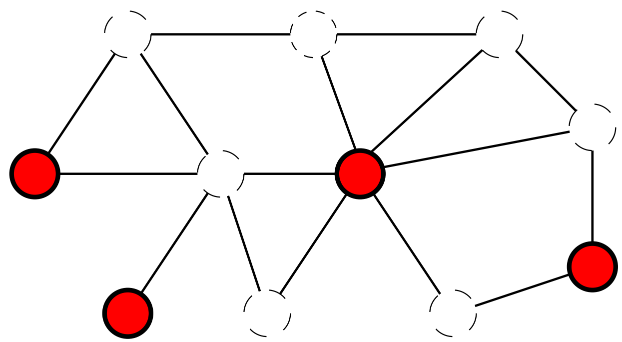

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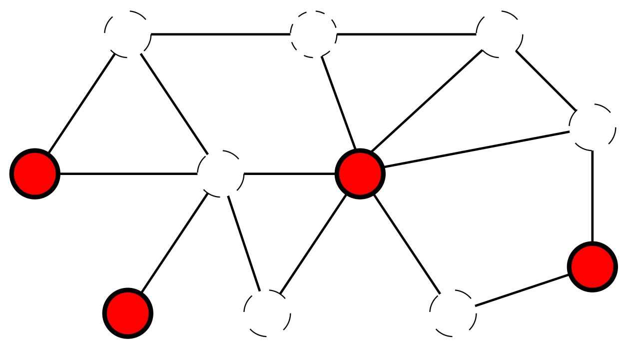

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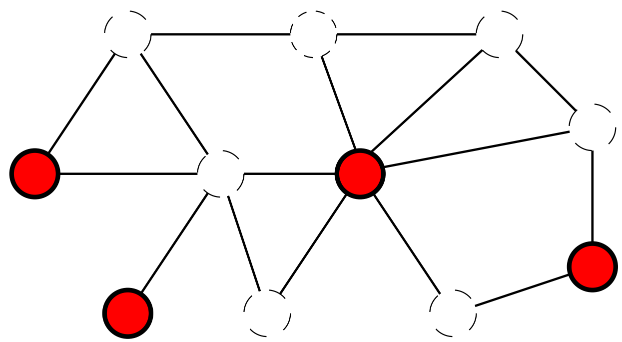

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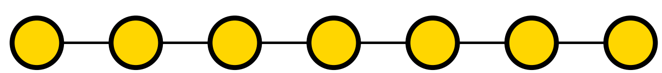

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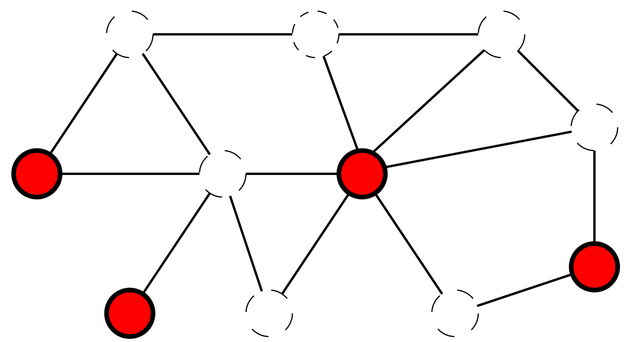

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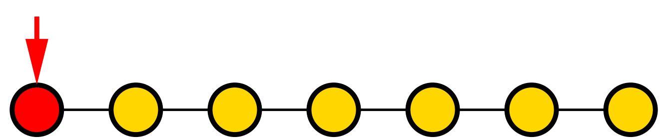

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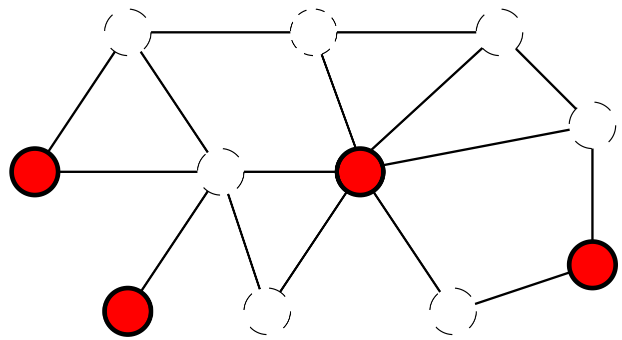

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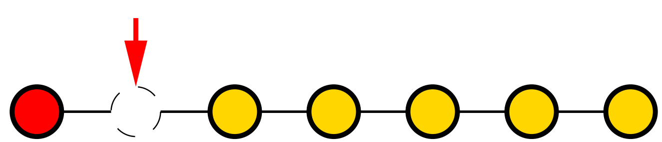

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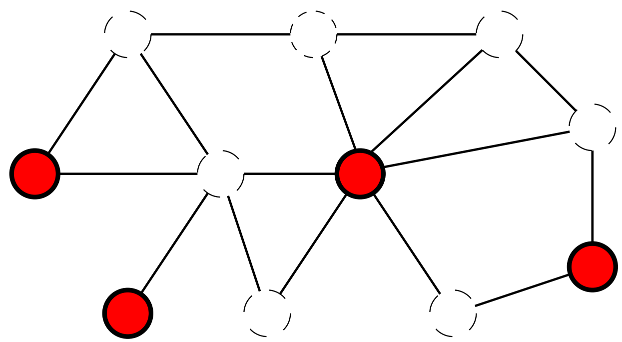

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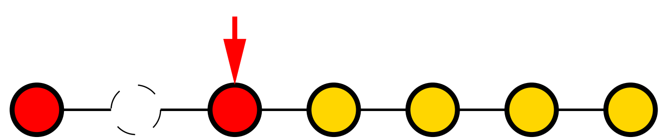

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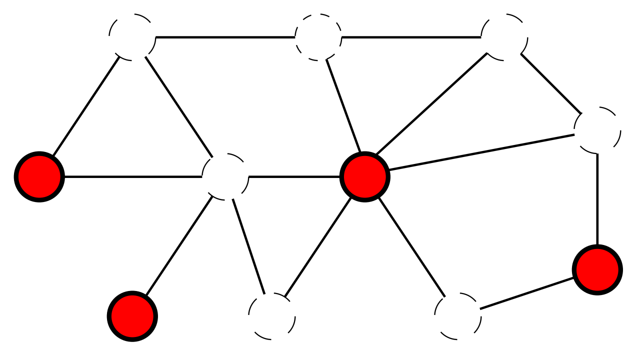

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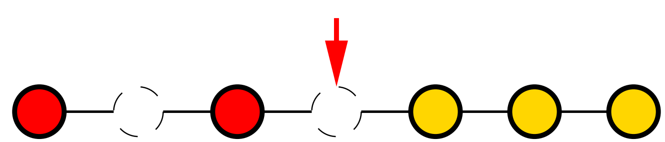

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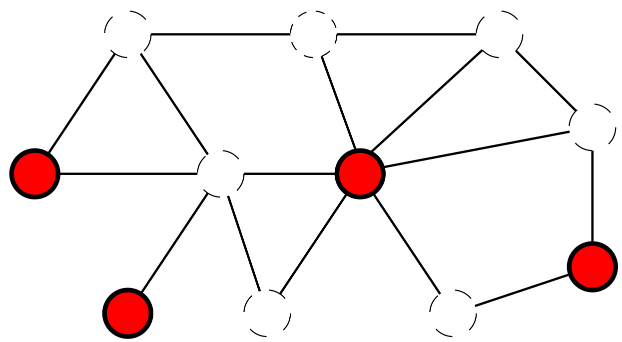

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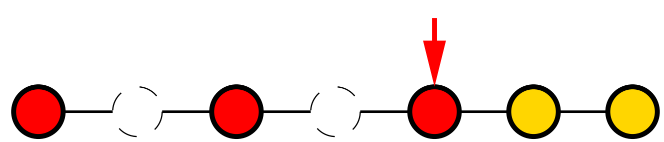

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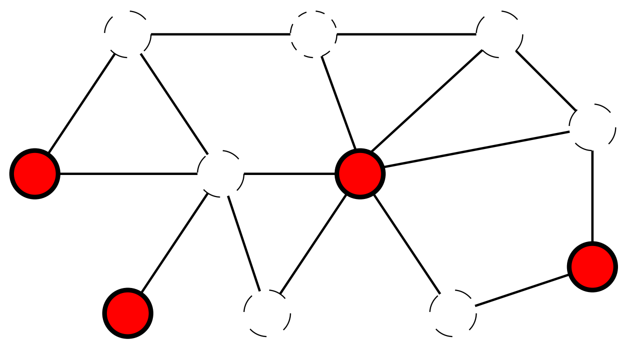

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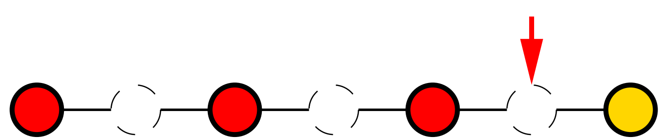

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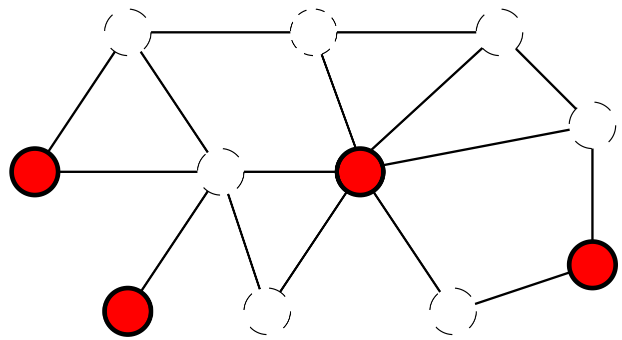

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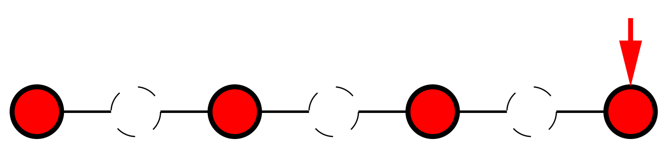

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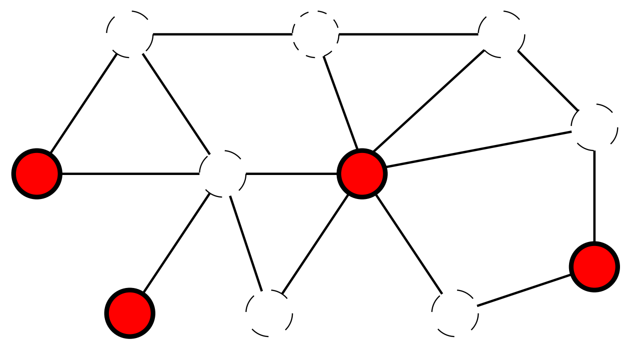

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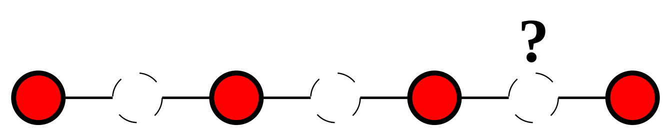

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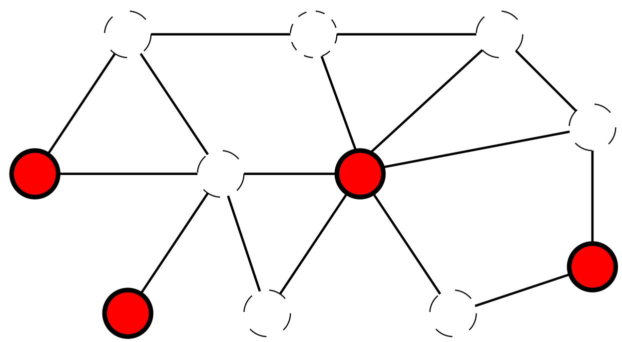

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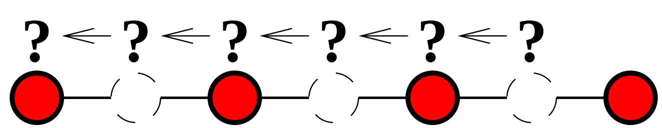

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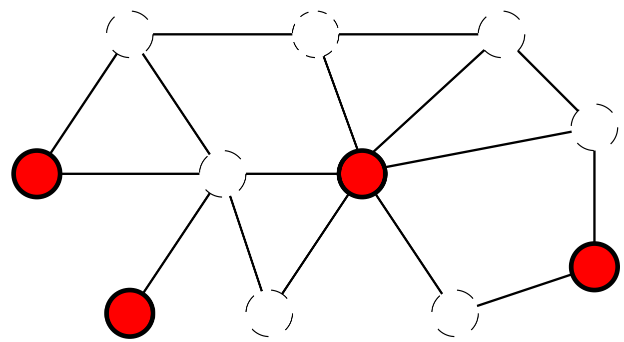

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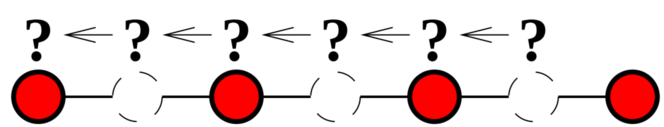


- Solution: consider vertices in random order

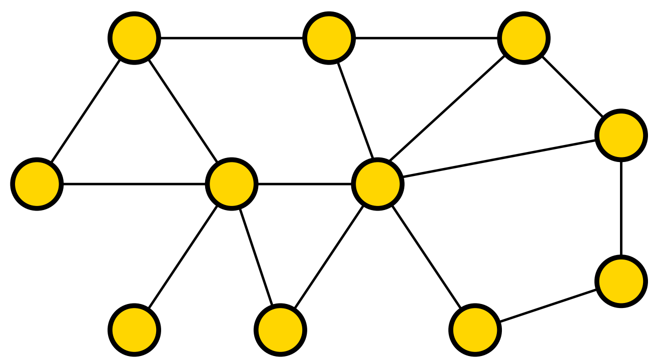
Local MIS Simulation

Nguyen, O. (2008)

Main idea:

- select maximal independent set greedily
- consider vertices in random order

Random order $\equiv$ random numbers $r(v)$ assigned to each vertex



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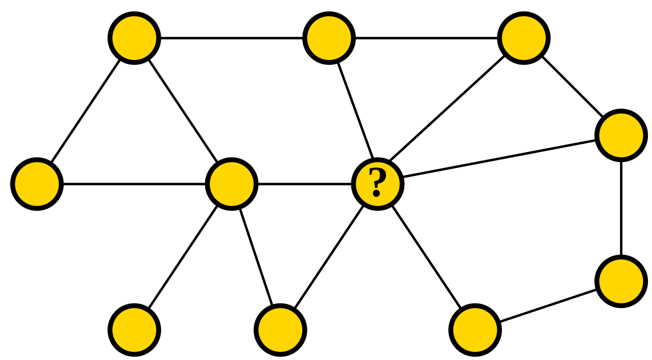

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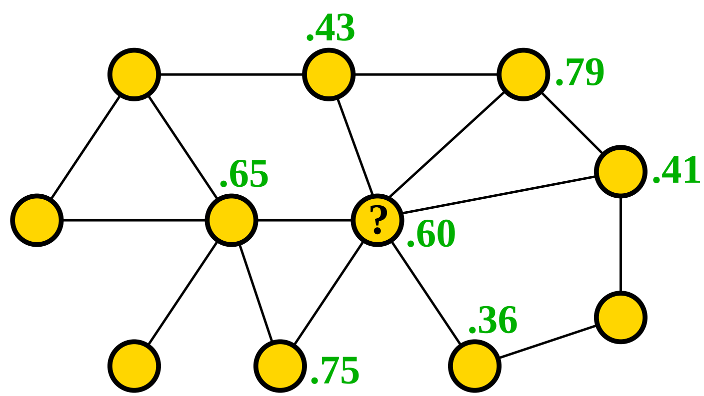

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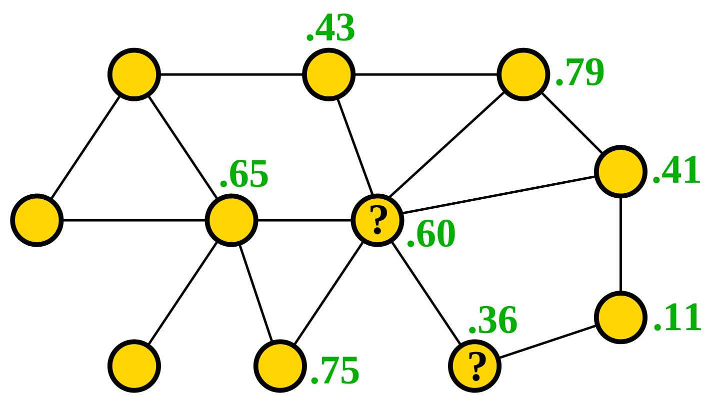

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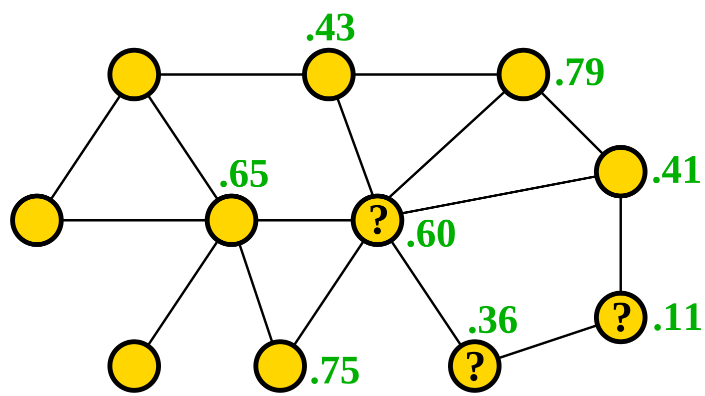

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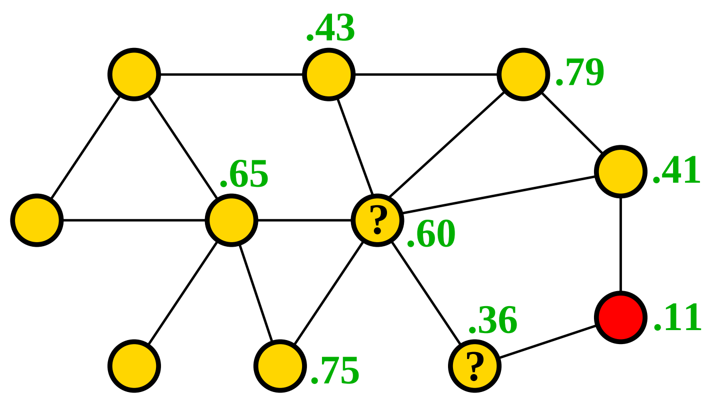

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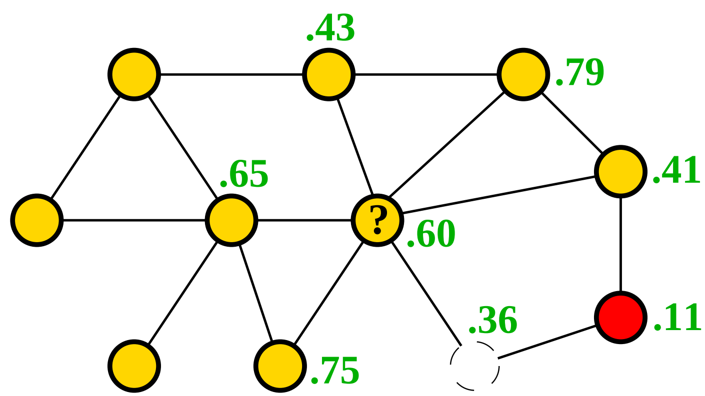

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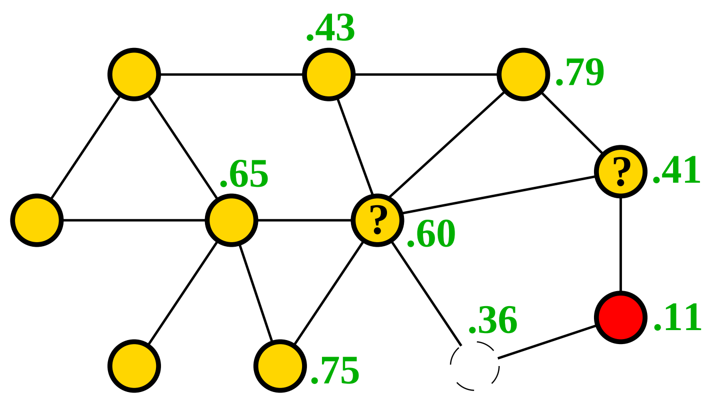

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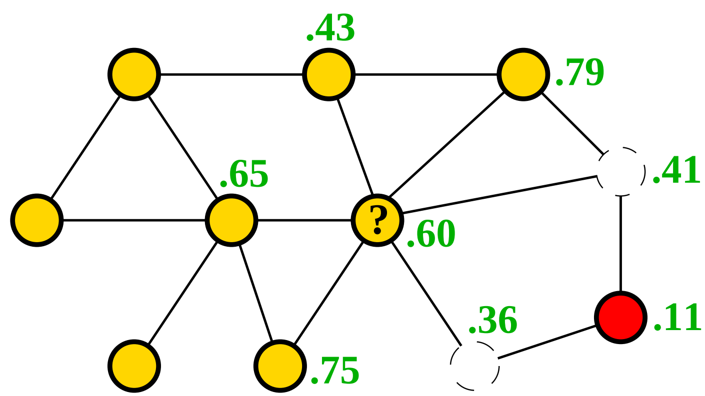

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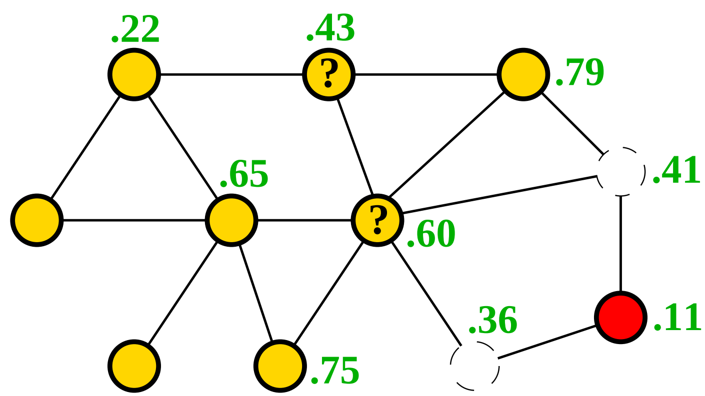

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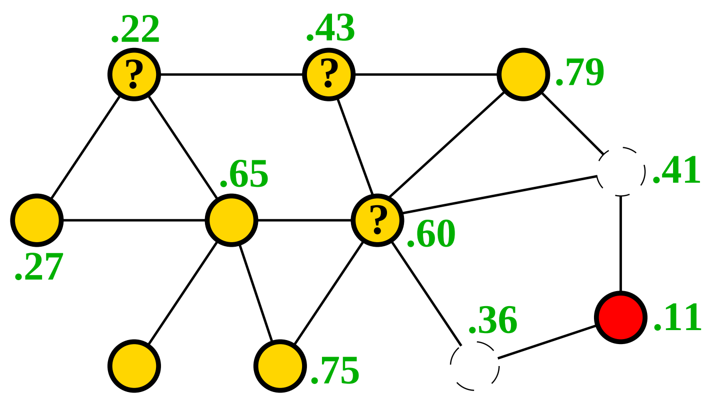

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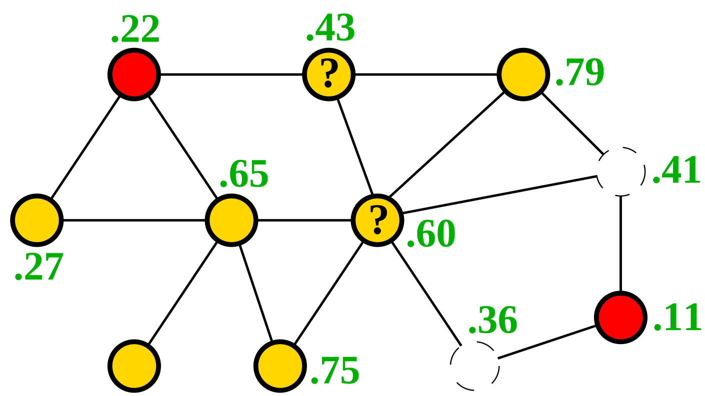

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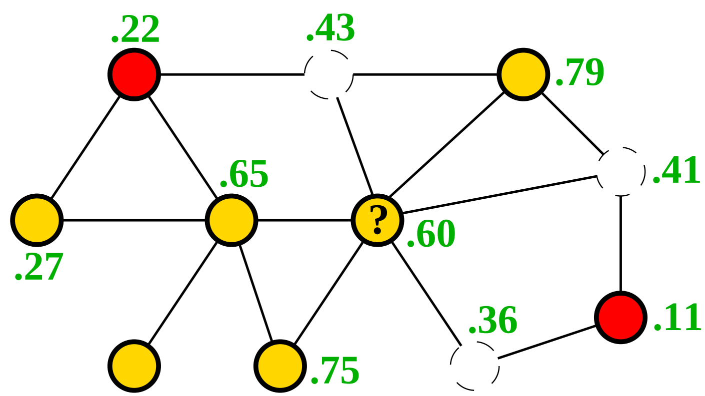

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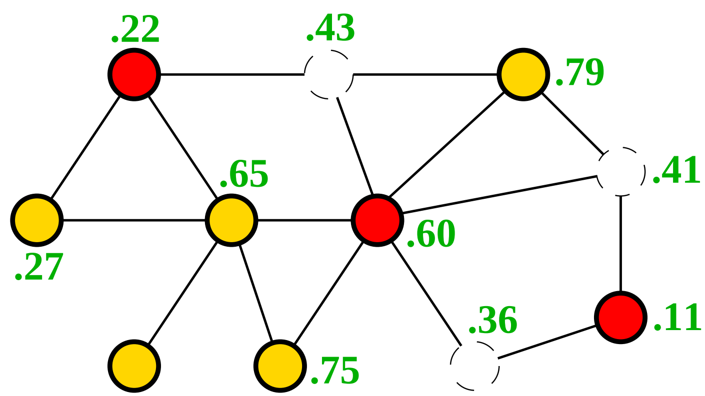

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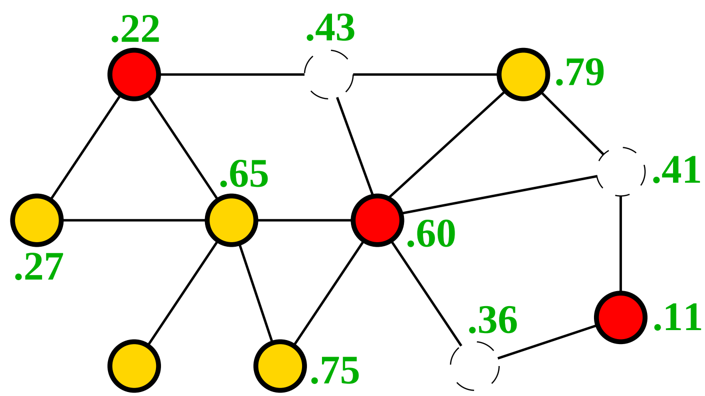

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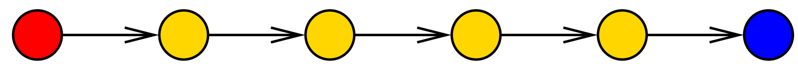
$\mathrm{E}[$#visited vertices$]$ and query complexity of order $2^{O(d)}$

Bounding Expected Query Complexity

- $\Pr[$a given path of length k is explored$] = 1/(k+1)!$



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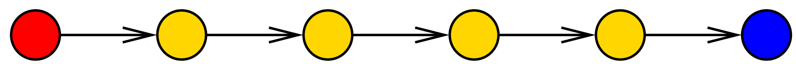
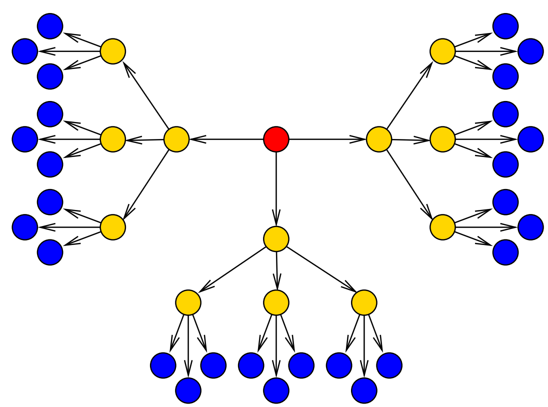


- $(\text{number of vertices at distance } k) \leq d^k$



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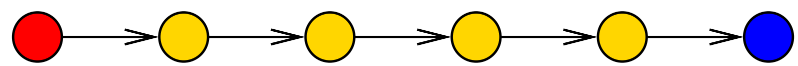

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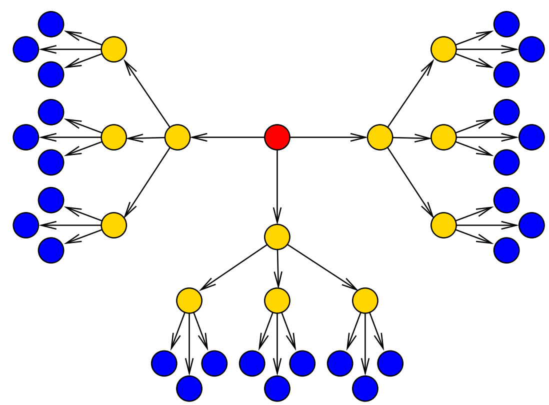

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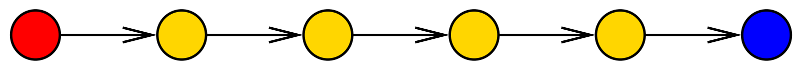

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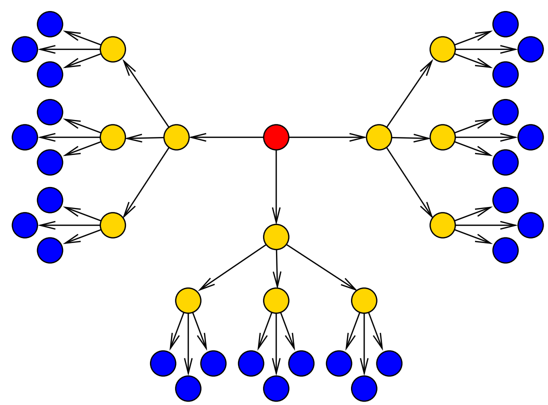

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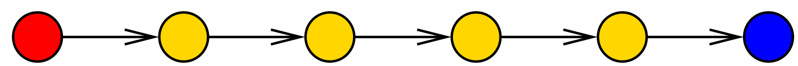

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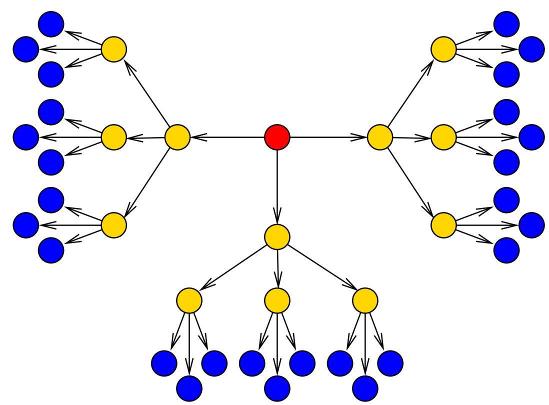

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- Expected query complexity $= O(d) \cdot e^d/d = O(e^d)$

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Heuristic:

- Consider neighbors w of v in ascending order of $r(w)$
- Once you find $w \in \mathcal{I}$, $v \notin \mathcal{I}$
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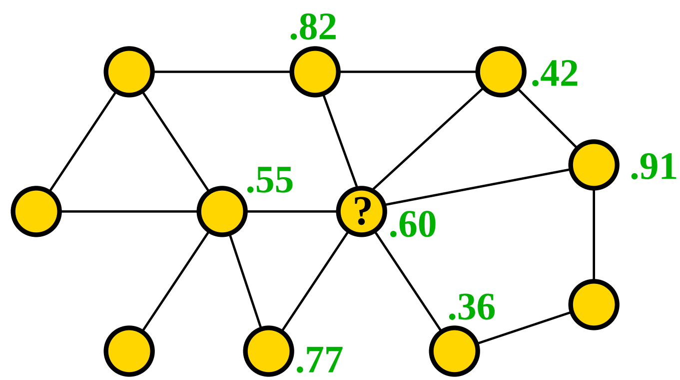

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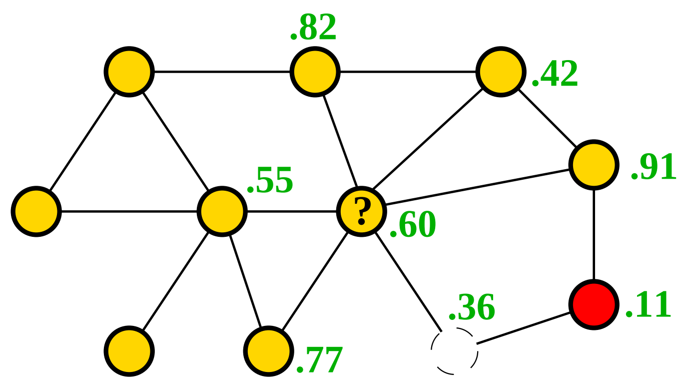

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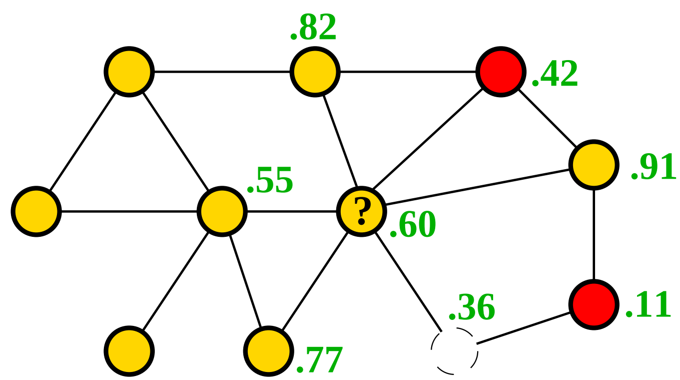

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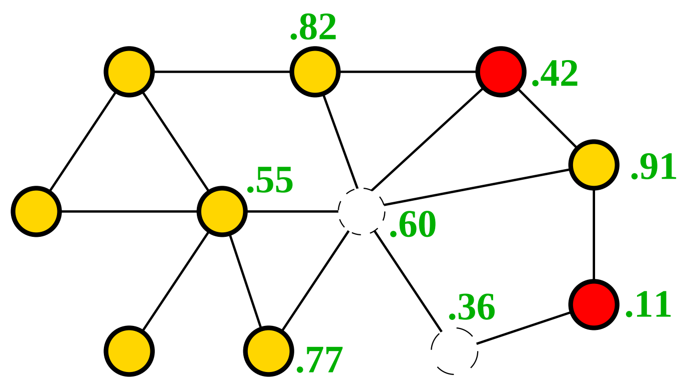

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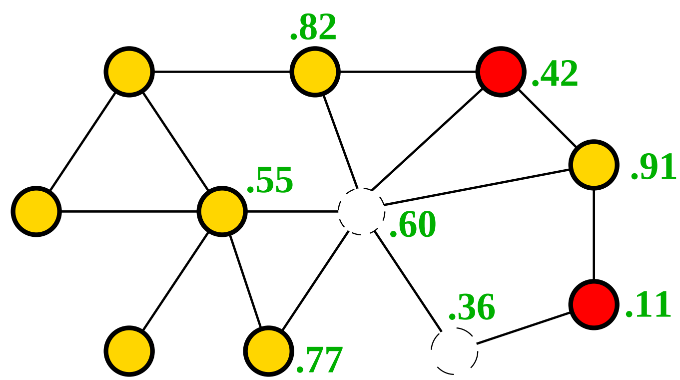

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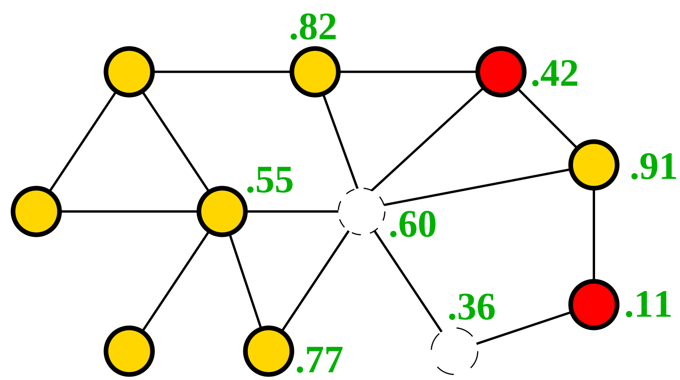

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Which gives:

expected query complexity for random vertex = $O(d^2)$

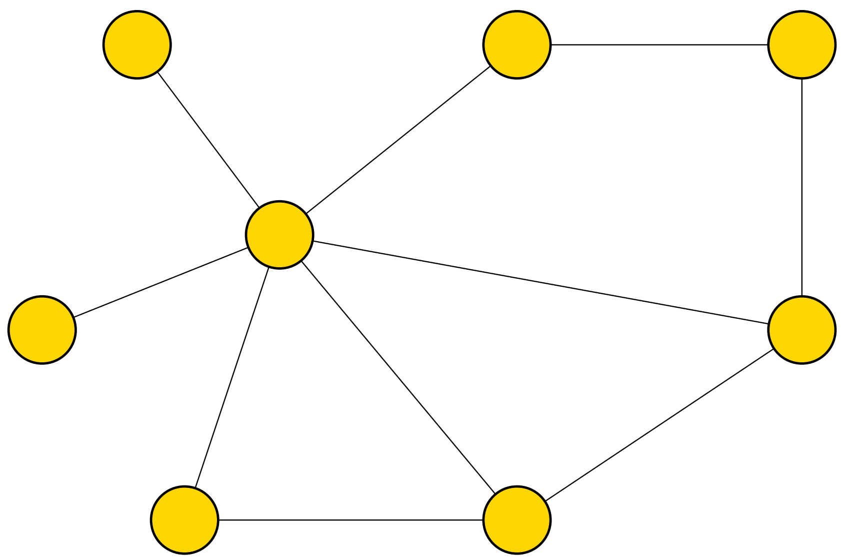
Algorithm for Vertex Cover

Vertex Cover

Goal: find smallest set S of nodes such that each edge has endpoint in S

Classical 2-approximation algorithm [Gavril & Yannakakis]:

- Greedily find a maximal matching M
- Output the set of nodes matched in M



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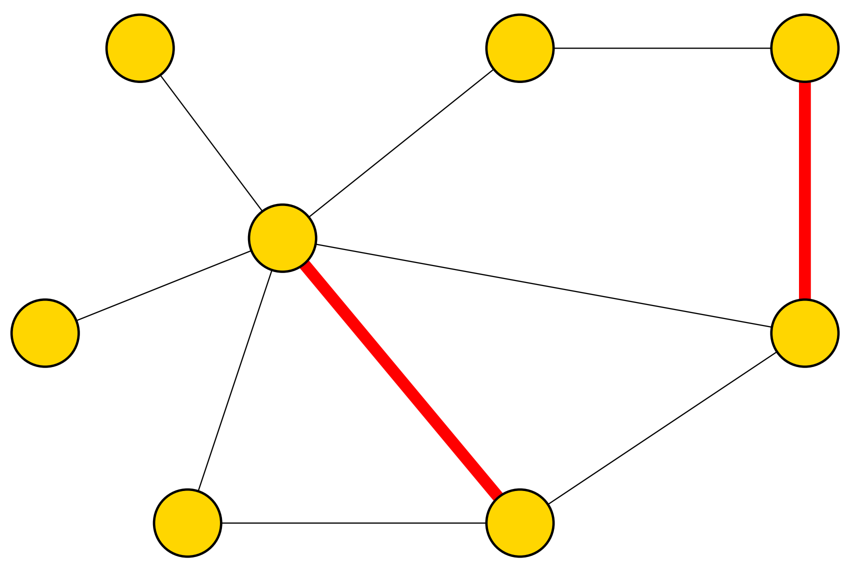

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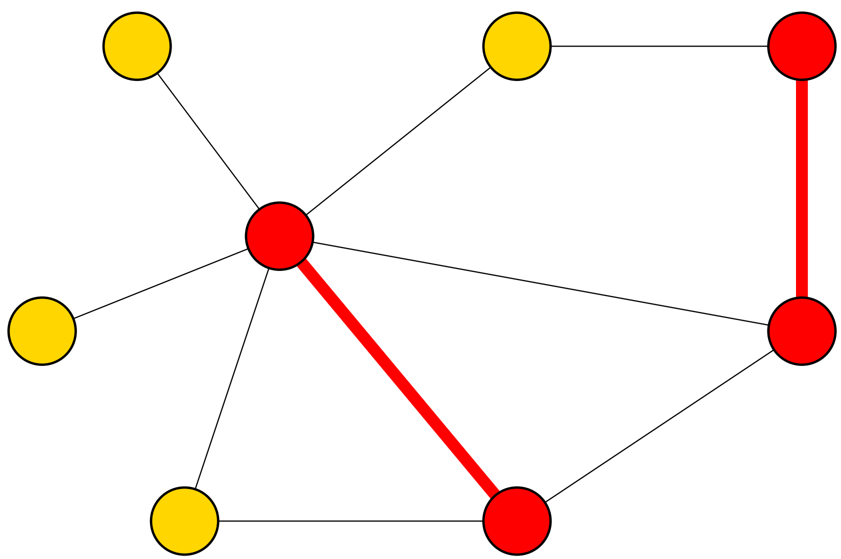

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General Idea:

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This gives:

$$\mathrm{VC} - \epsilon n \leq (\text{computed value}) \leq 2 \cdot \mathrm{VC} + \epsilon n$$

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Running time: $2^{O(d)}/\epsilon^2$

$\mathrm{VC} - \epsilon n \leq \textbf{output} \leq 2 \cdot \mathrm{VC} + \epsilon n$

Parnas, Ron (2007): $d^{O(\log(d)/\epsilon^3)}$ queries

- via simulation of local distributed algorithms

Marko, Ron (2007): $d^{O(\log(d/\epsilon))}$ queries

- via Luby's algorithm

Nguyen, O. (2008): $2^{O(d)}/\epsilon^2$ queries

- the algorithm and proof presented here

Yoshida, Yamamoto, Ito (2009): $O(d^4/\epsilon^2)$ queries

- the Nguyen, O. algorithm + analysis of the heuristic

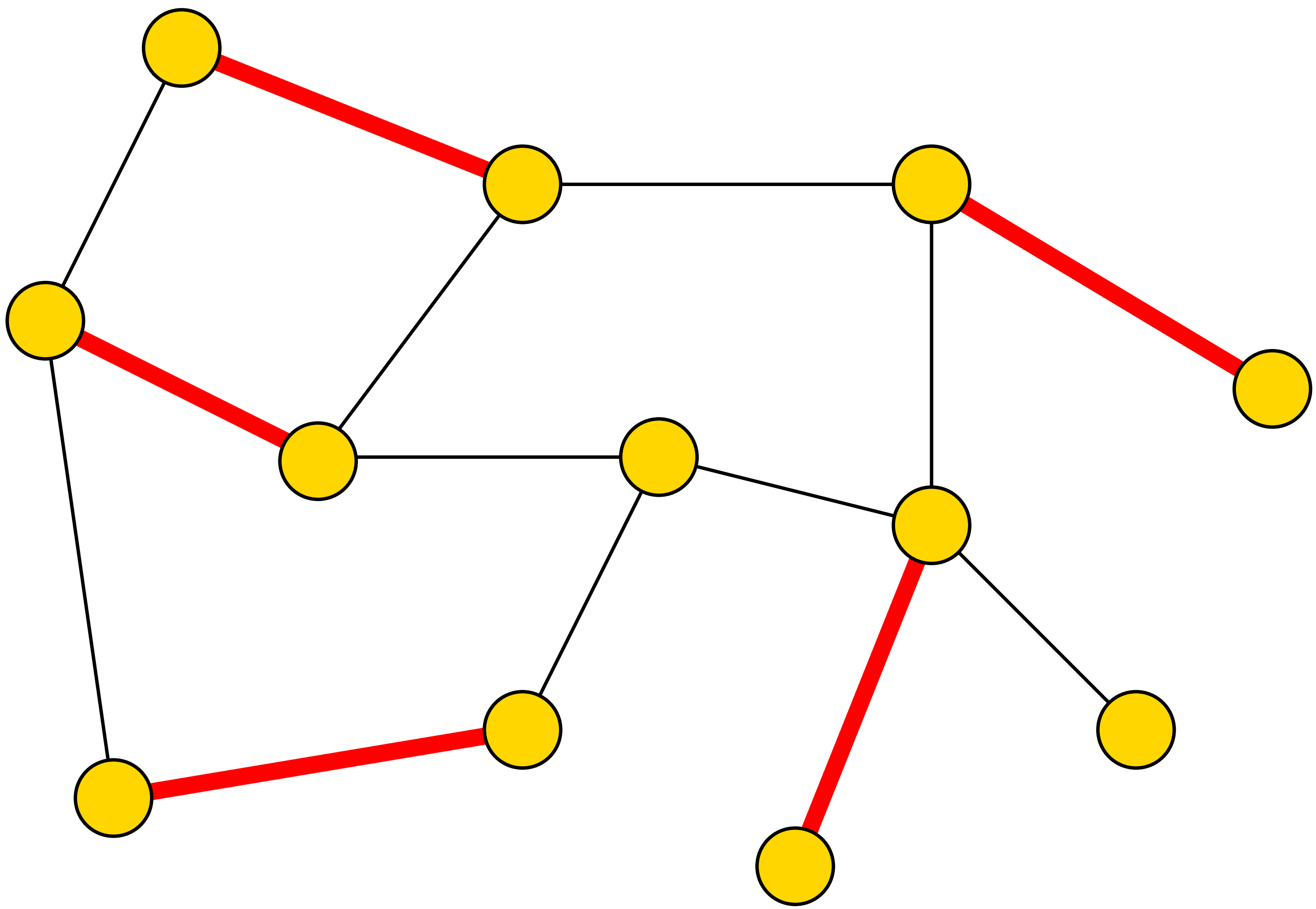
O., Ron, Rosen, Rubinfeld (2012): $\tilde{O}(d/\epsilon^3)$ queries

- further refinements of NO and YYI
- sampling from the neighbor sets
- near optimal: $\Omega(d)$ lower bound due to Parnas, Ron (2007)

Better Approximation for Maximum Matching

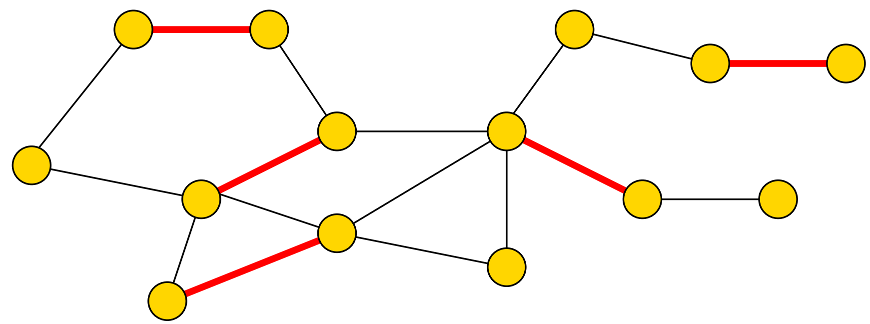
Maximum Matching

Goal: find a set of disjoint edges of maximum cardinality



Review of Properties

Augmenting Path: a path that improves matching



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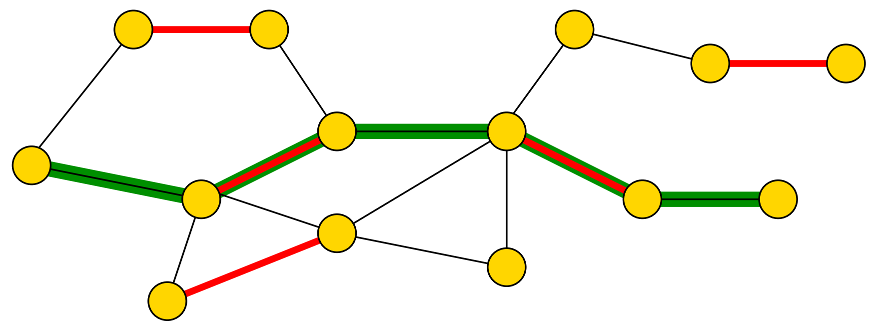

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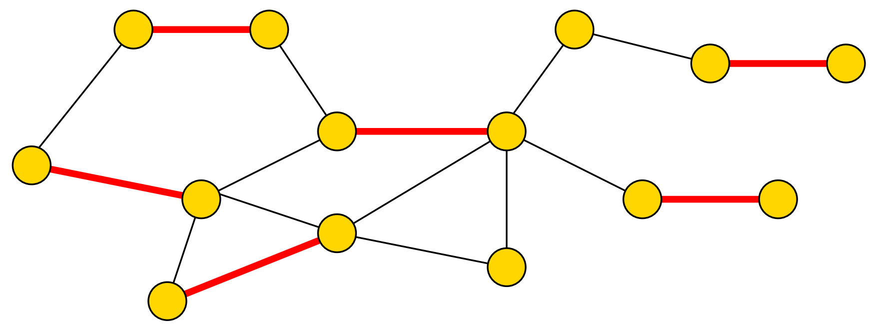

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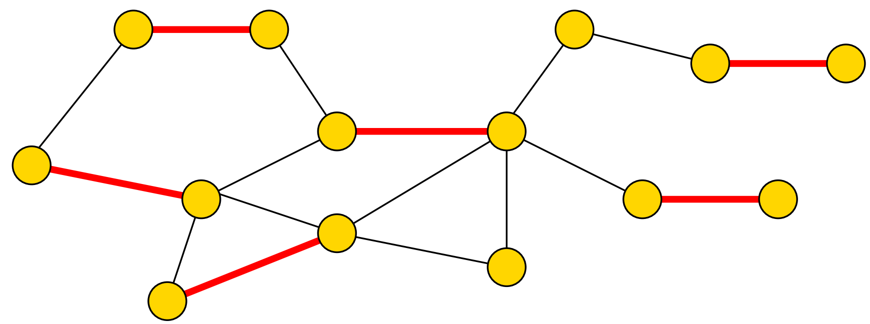

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Fact: There are $|M^*| - |M|$ disjoint augmenting paths for M

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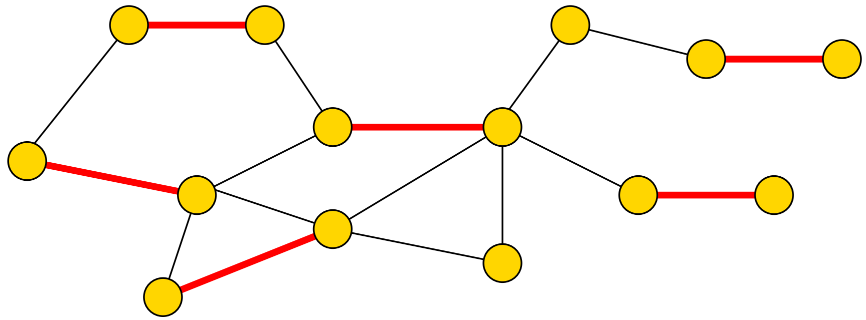

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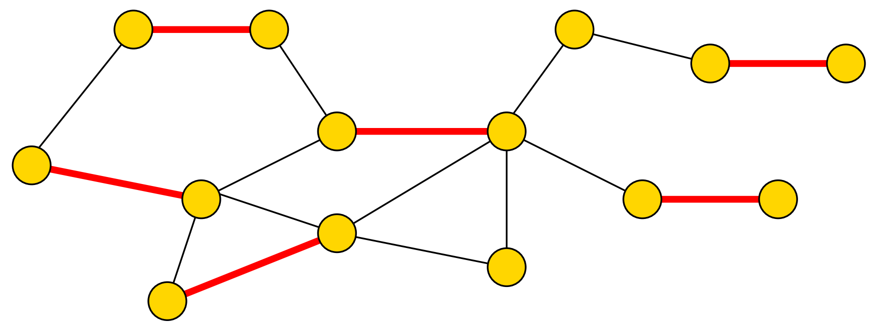

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To get $(1+\epsilon)$-approximation, set $k = \lceil 1/\epsilon \rceil$

Standard Algorithm

Lemma [Hopcroft, Karp 1973]:

M = matching with no augmenting paths of length $< t$

P = maximal set of vertex-disjoint augmenting paths of length t for M

M' = M with all paths in P applied

Claim: M' has only augmenting paths of length $> t$

Standard Algorithm

Lemma [Hopcroft, Karp 1973]:

M = matching with no augmenting paths of length $< t$

P = maximal set of vertex-disjoint augmenting paths of length t for M

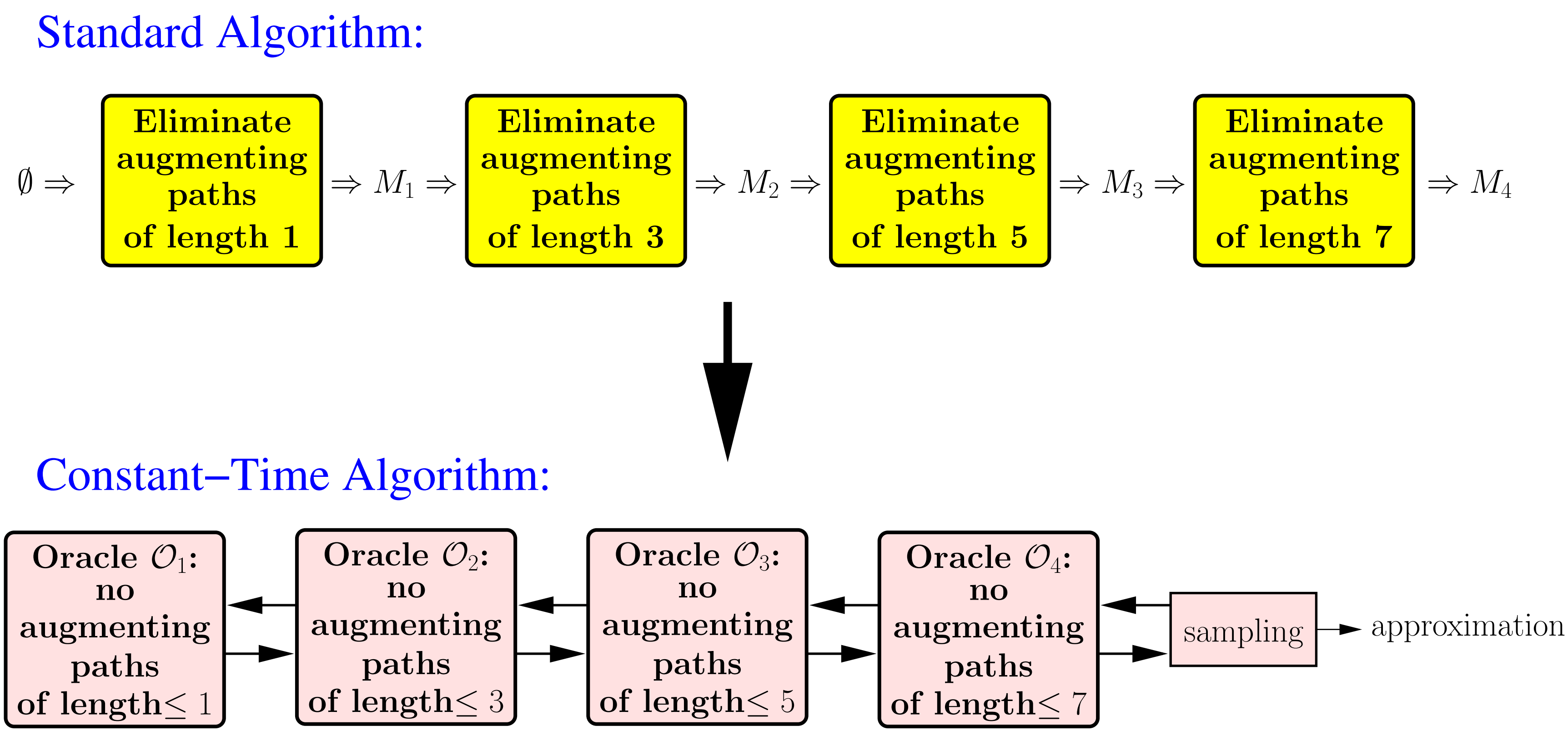
M' = M with all paths in P applied

Claim: M' has only augmenting paths of length $> t$

Algorithm:

$M :=$ empty matching
for $i = 1$ to k:
  find maximal set of disjoint augmenting paths of length $2i - 1$
  apply all paths to M
return M

Transformation

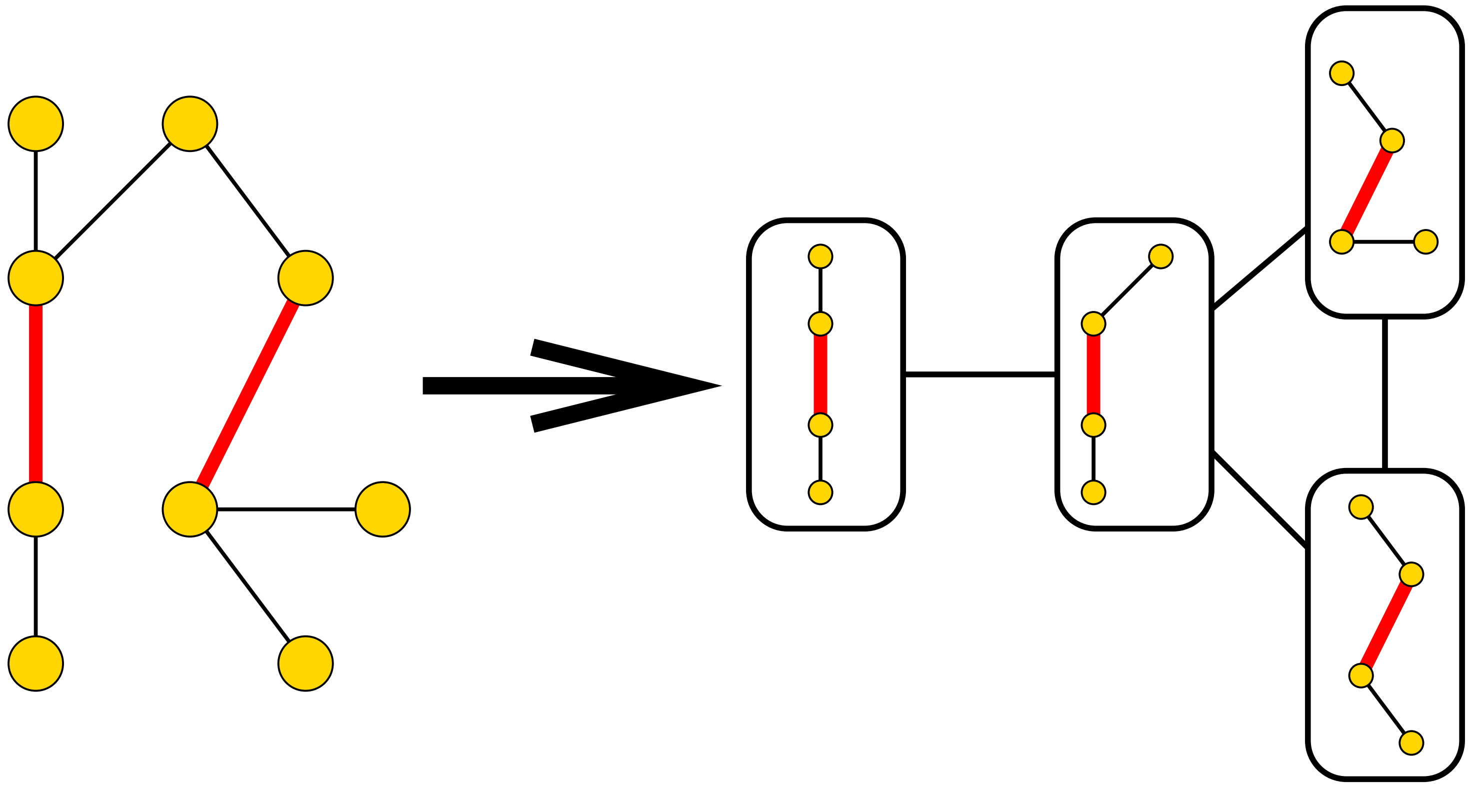


Oracle $\mathcal{O}_i$:

- provides query access to M_i
- simulates applying to M_{i-1} a maximal set of disjoint augmenting paths of length $2i-1$

Transformation

Sample graph considered by $\mathcal{O}_2$:



$\mathcal{O}_i$'s graph has degree $d^{O(i)}$

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Can’t apply the previous approach!

- every query may disclose some information about the random numbers
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Yoshida, Yamamoto, Ito (2009)

- Query complexity: $d^{O(1/\epsilon^2)}$
- uniform on higher level $\Rightarrow$ close to uniform on lower

Roadmap

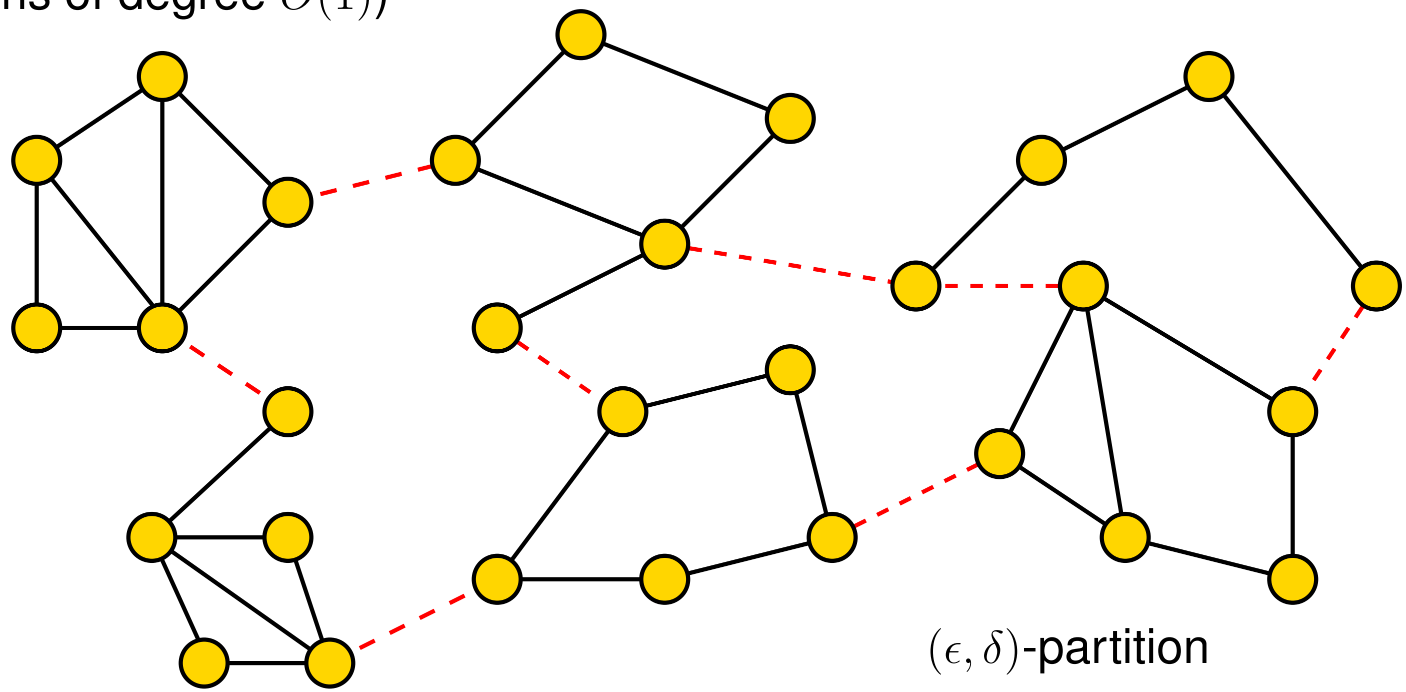
1. Simulation of greedy algorithms

2. Partitioning oracles

3. Random walks

Hyperfinite Graphs

(All graphs of degree $O(1)$)



(ϵ, δ)-partition

- (ϵ, δ)-hyperfinite graphs: can remove $\epsilon|V|$ edges and get components of size at most δ

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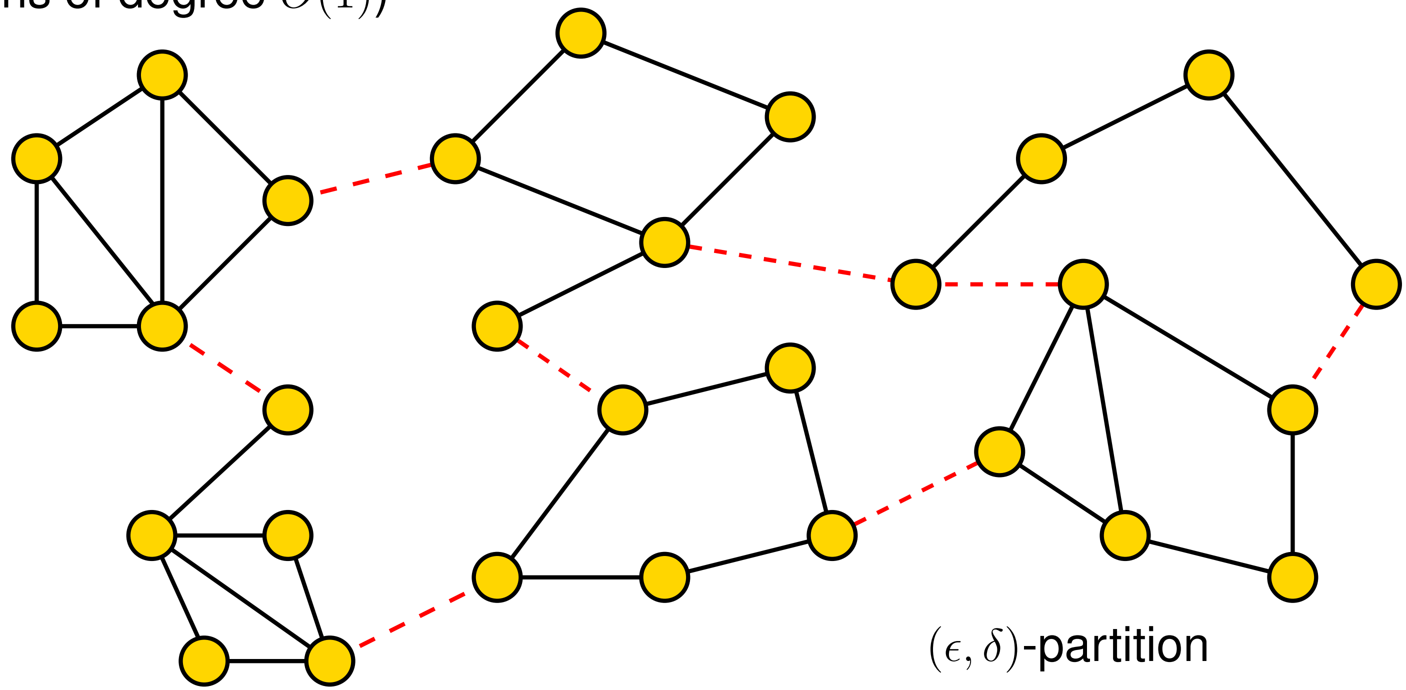
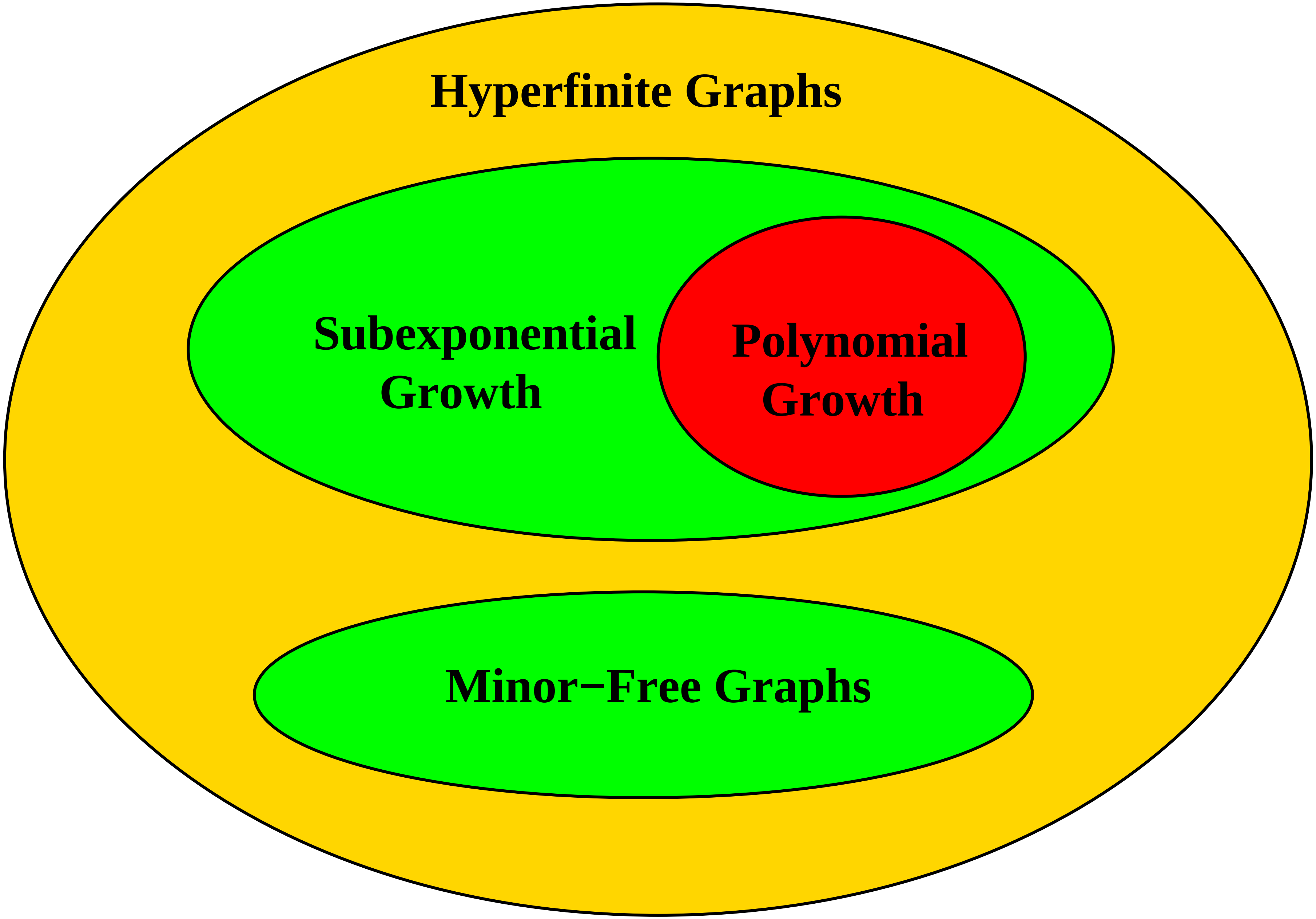


(ϵ, δ)-partition

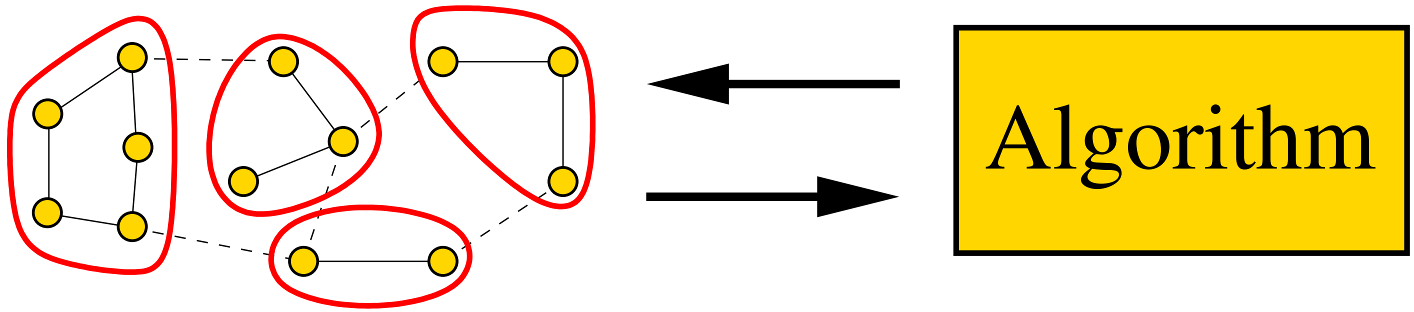
- (ϵ, δ)-hyperfinite graphs: can remove $\epsilon|V|$ edges and get components of size at most δ
- hyperfinite family of graphs: there is ρ such that all graphs are $(\epsilon, \rho(\epsilon))$-hyperfinite for all $\epsilon > 0$

Taxonomy


Hyperfinite Graphs
Subexponential Growth
Polynomial Growth
Minor−Free Graphs

Using a Partition

If someone gave us a $(\epsilon/2, \delta)$-partition:



- Sample $O(1/\epsilon^2)$ vertices
- Compute minimum vertex cover for the sampled components
- Return the fraction of the sampled vertices in the covers

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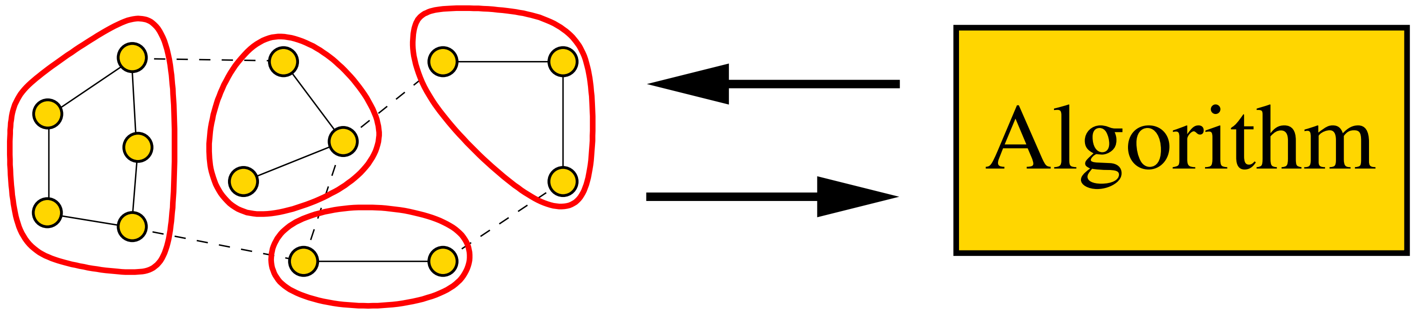

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This gives $\pm\epsilon$ approximation to $\mathrm{VC}(G)/n$ in constant time:

- Cut edges change $\mathrm{VC}(G)$ by at most $\epsilon n/2$
- Can compute vertex cover separately for each component

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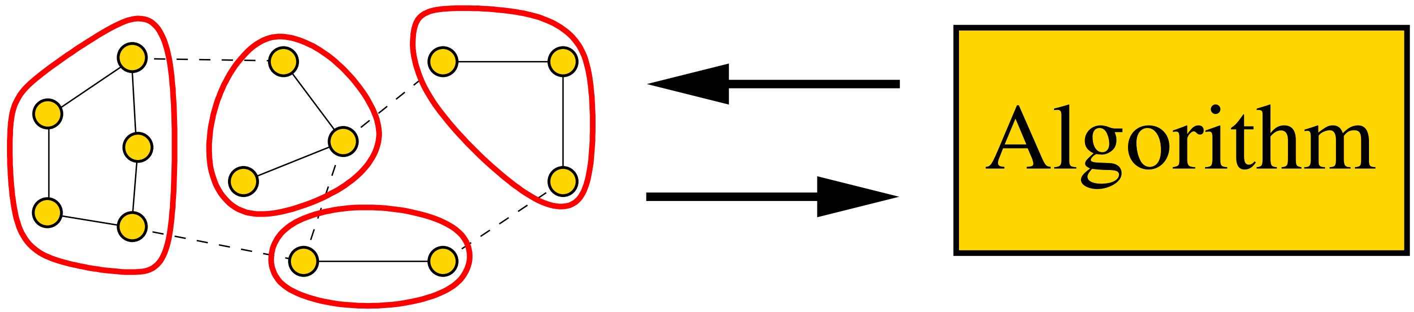

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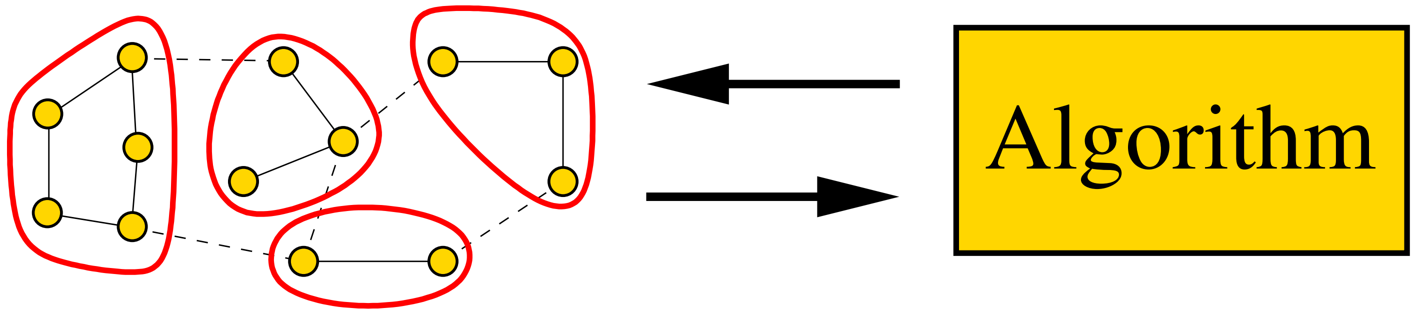

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New Tool: Partitioning Oracles

Partitioning Oracle

Hassidim, Kelner, Nguyen, O. (2009)

$\mathcal{C} =$ fixed hyperfinite class

- oracle has query access to $G = (V, E)$
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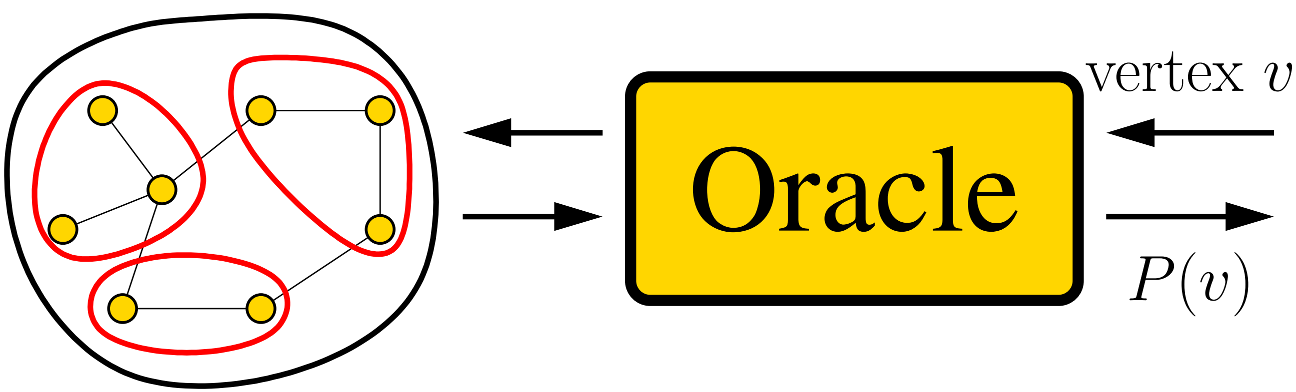

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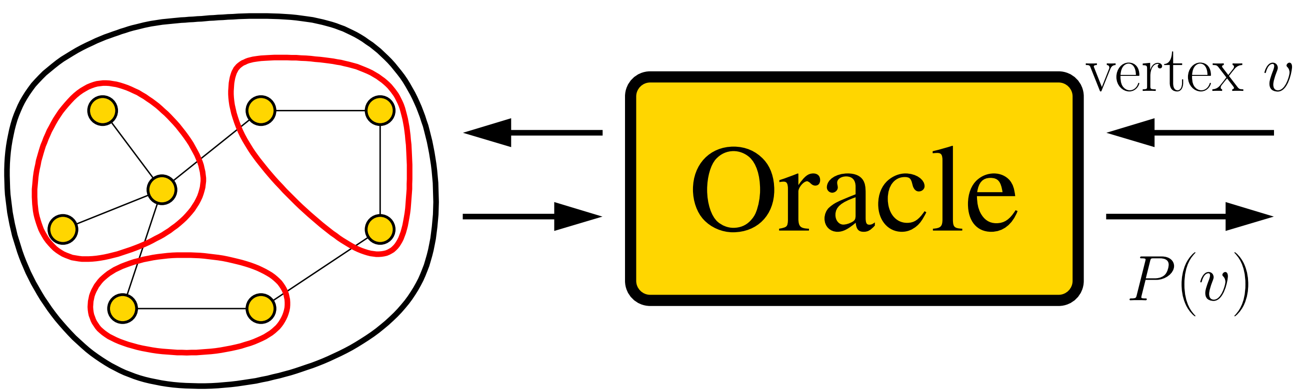

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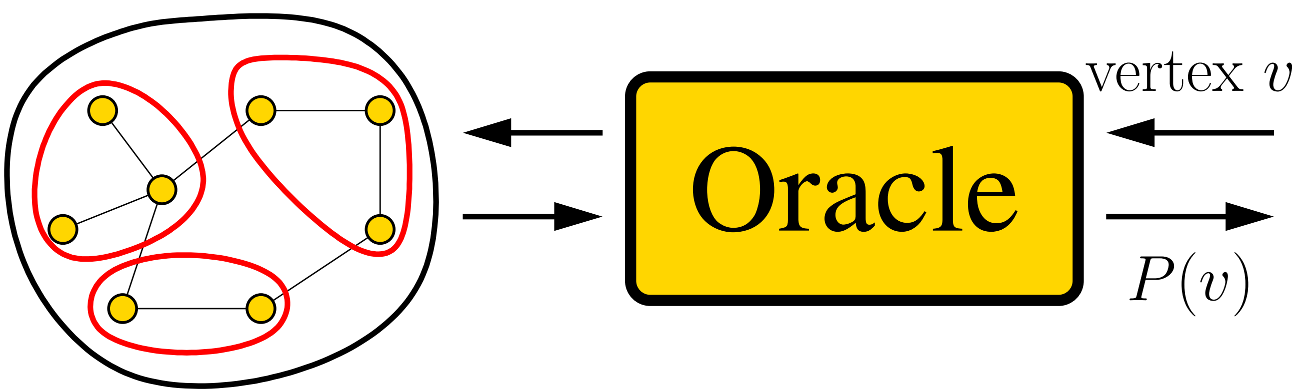

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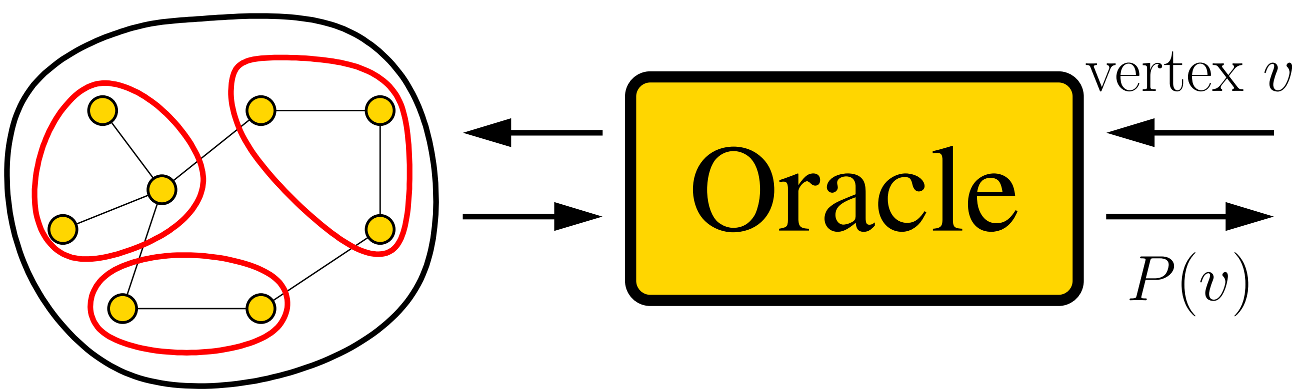

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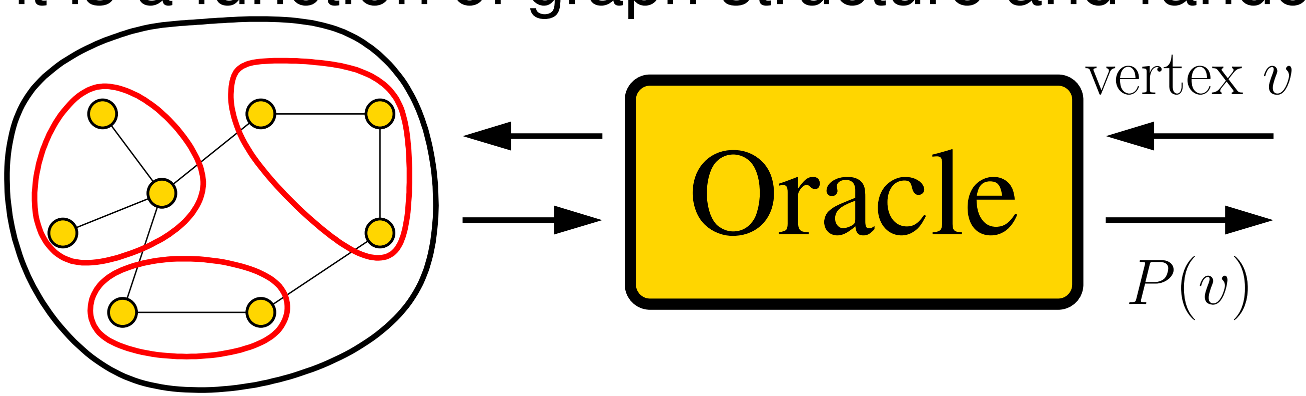
  - partition $P(\cdot)$ is not a function of queries,
    it is a function of graph structure and random bits



Oracle Implementations

- Generic oracle for any hyperfinite class of graphs
  - Query complexity: $2^{d^{O(\rho(\epsilon^3/C))}}$ for some constant C
  - Via local simulation of a greedy partitioning procedure (uses [Nguyen, O. 2008])

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- Constant Treewidth:
  - Query complexity: $\mathrm{poly}(d/\epsilon)$
  - Edelman, Hassidim, Nguyen, O. (2011)

Two Applications

1. Approximately learning hyperfinite graphs
   - Then solve an arbitrary problem on almost the same graph

Two Applications

1. Approximately learning hyperfinite graphs
   - Then solve an arbitrary problem on almost the same graph
2. Testing minor-closed properties
   - Simpler proof of the result due to Benjamini, Schramm, and Shapira (2008)
   - Much faster tester

Application 1: Learning

- Input graphs can be decomposed into constant size components by cutting few edges
- Algorithm:
  - sample large constant number of vertices
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- Application: solve any testing or approximation problem on almost the same graph
- First proof: Newman and Sohler (2011)

Application 2: Testing

Testing H-minor-freeness in the sparse graph model of Goldreich and Ron (1997)

- Input: query access to constant degree graph G
  & parameter $\epsilon > 0$
- Goal: w.p. $2/3$
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- Via partitioning oracles: $2^{\mathrm{polylog}(1/\epsilon)}$ and simpler proof

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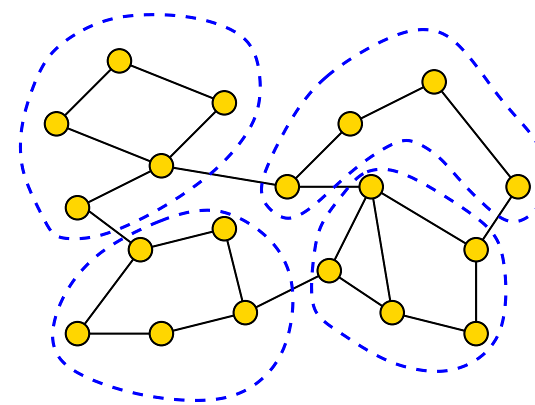

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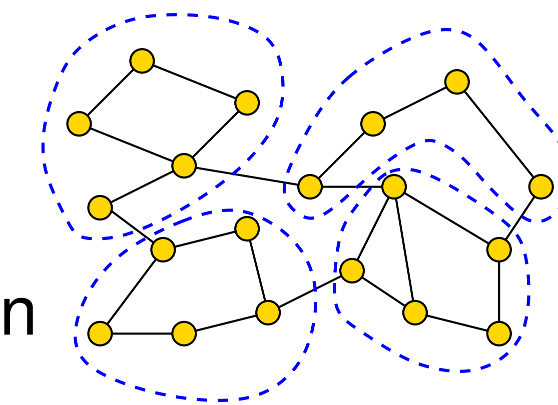
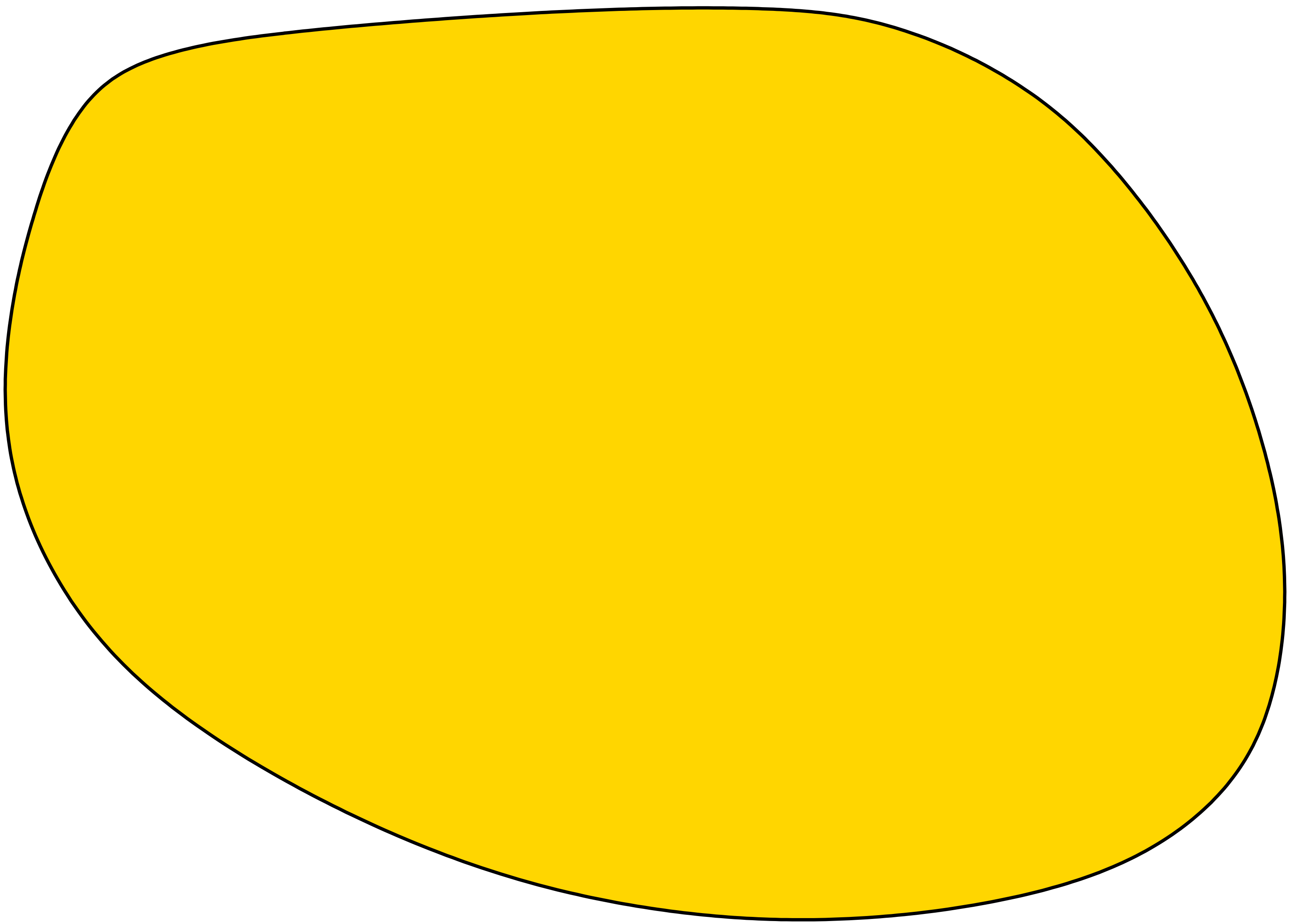
- Why it works:
  - planar: few edges cut in the partition
  - ϵ-far: either many edges cut or many copies of $K_{3,3}$ or K_5



Simplest Oracle

Iterative Procedure

Global procedure:



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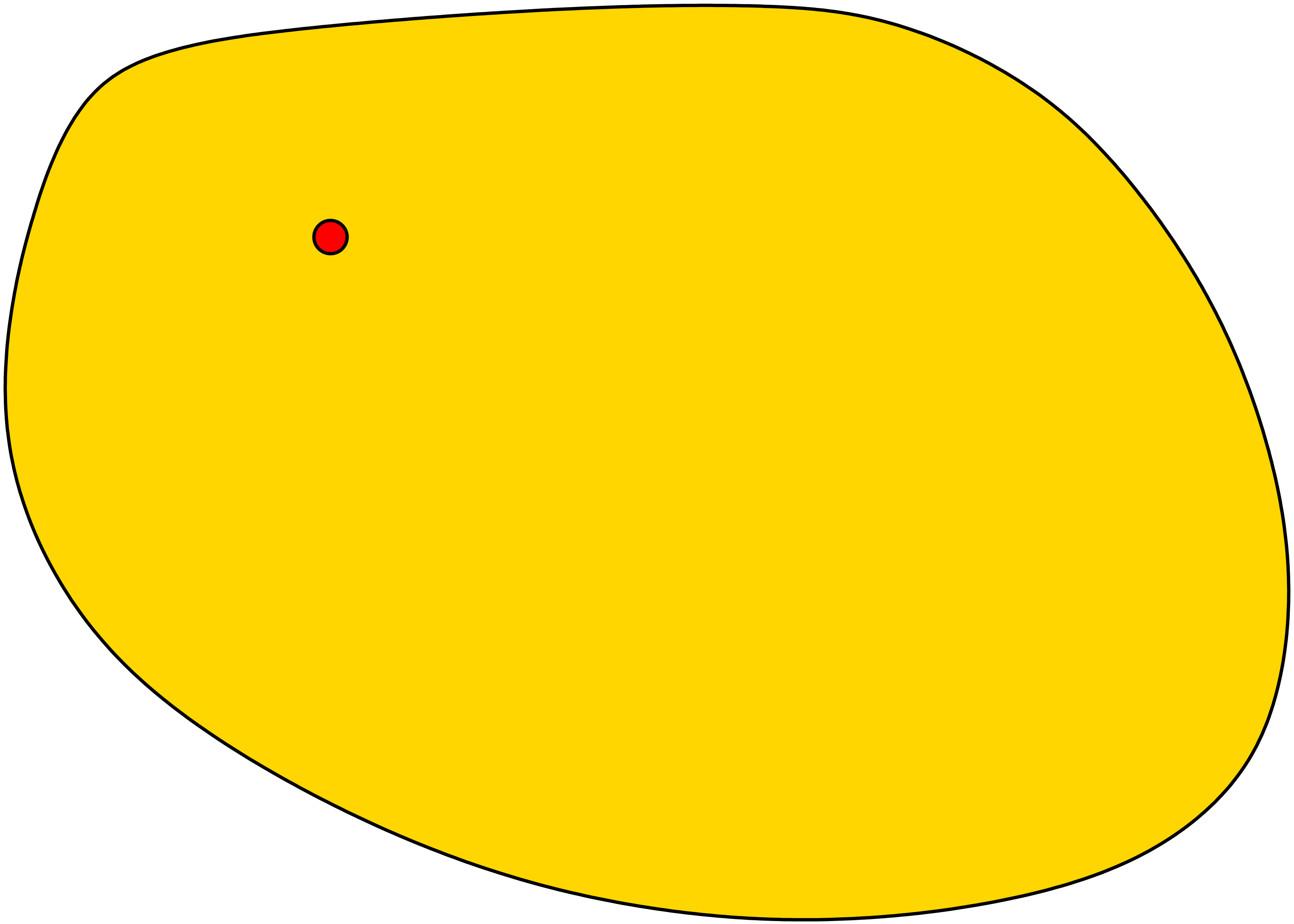

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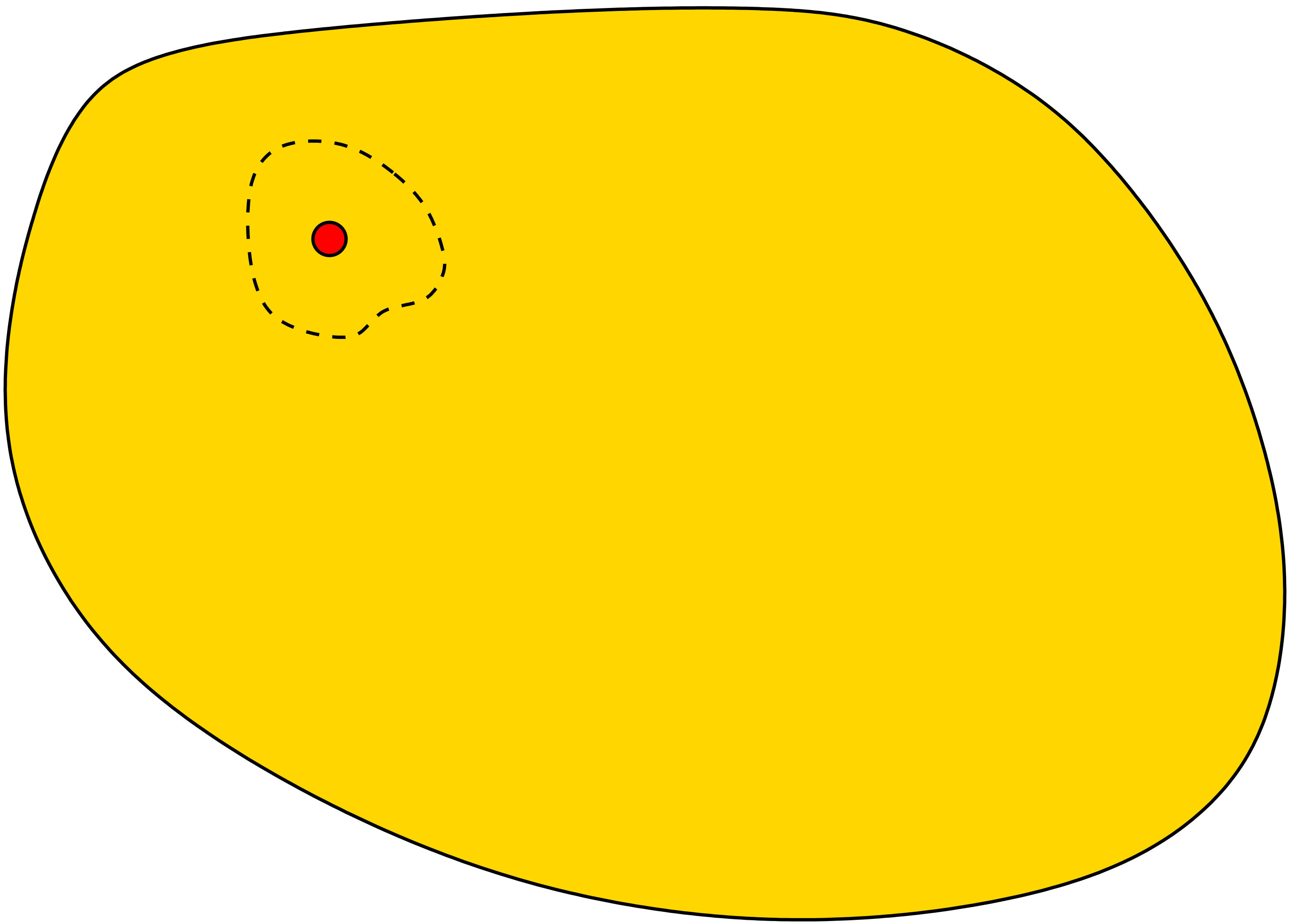

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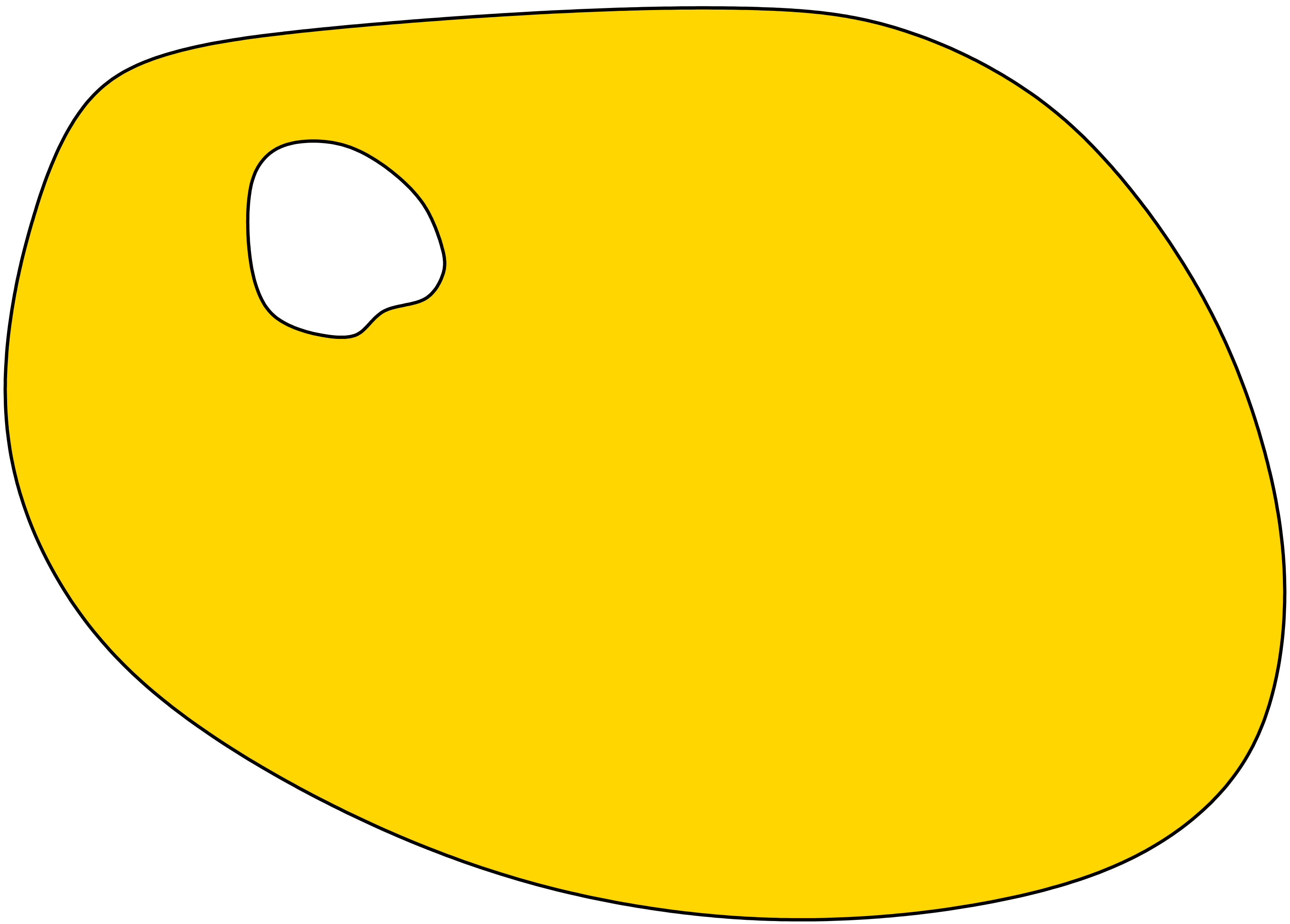

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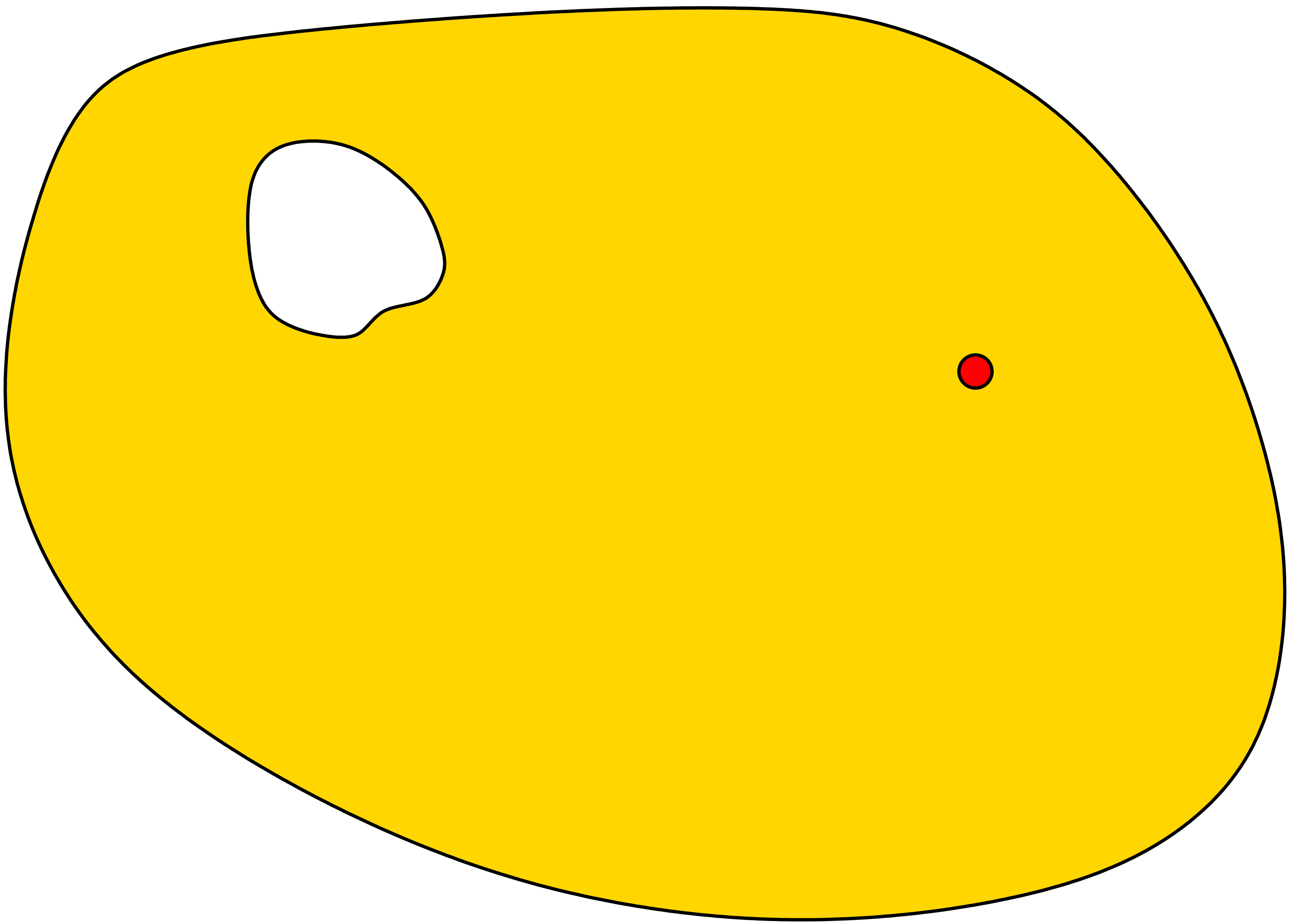

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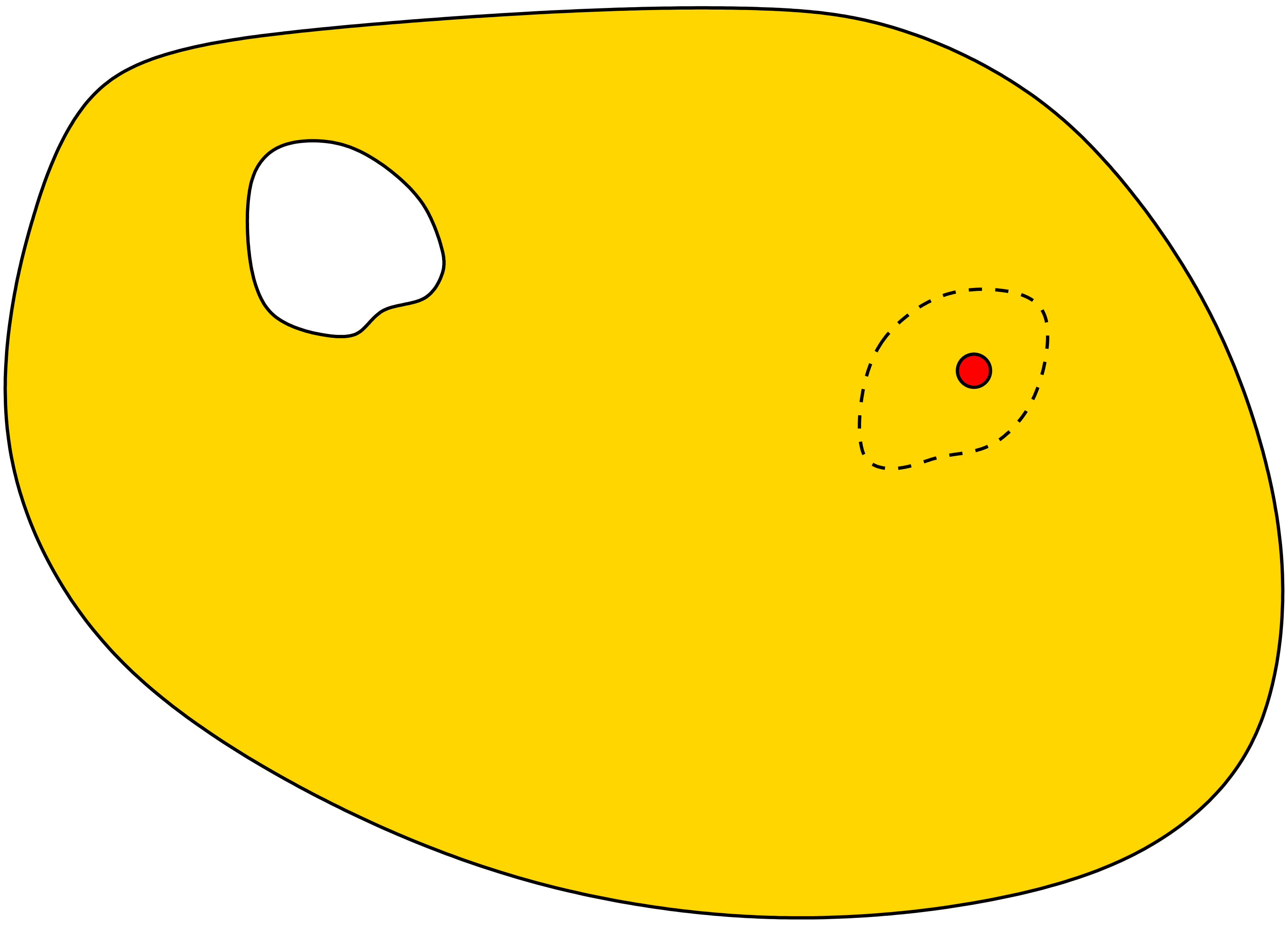

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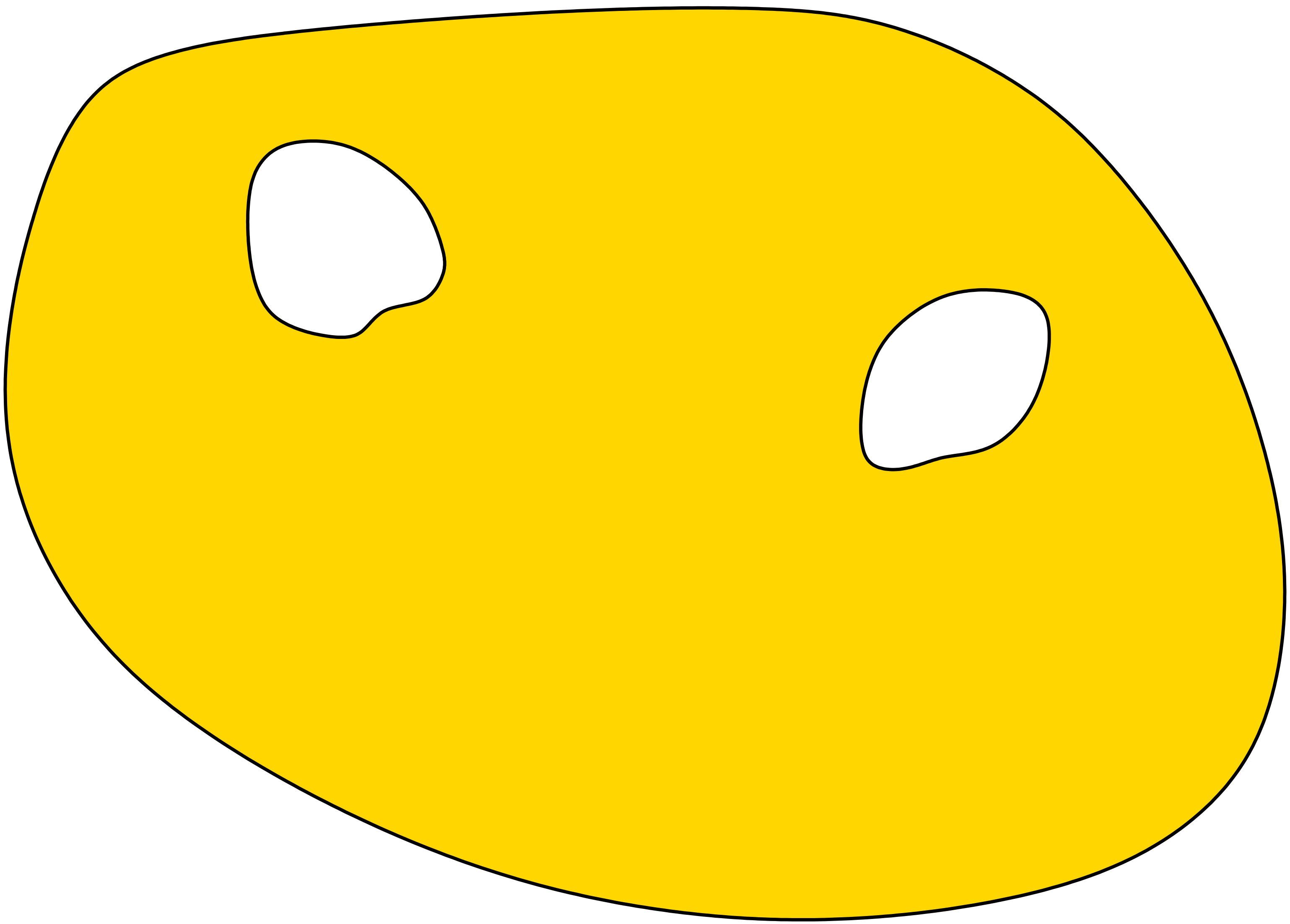

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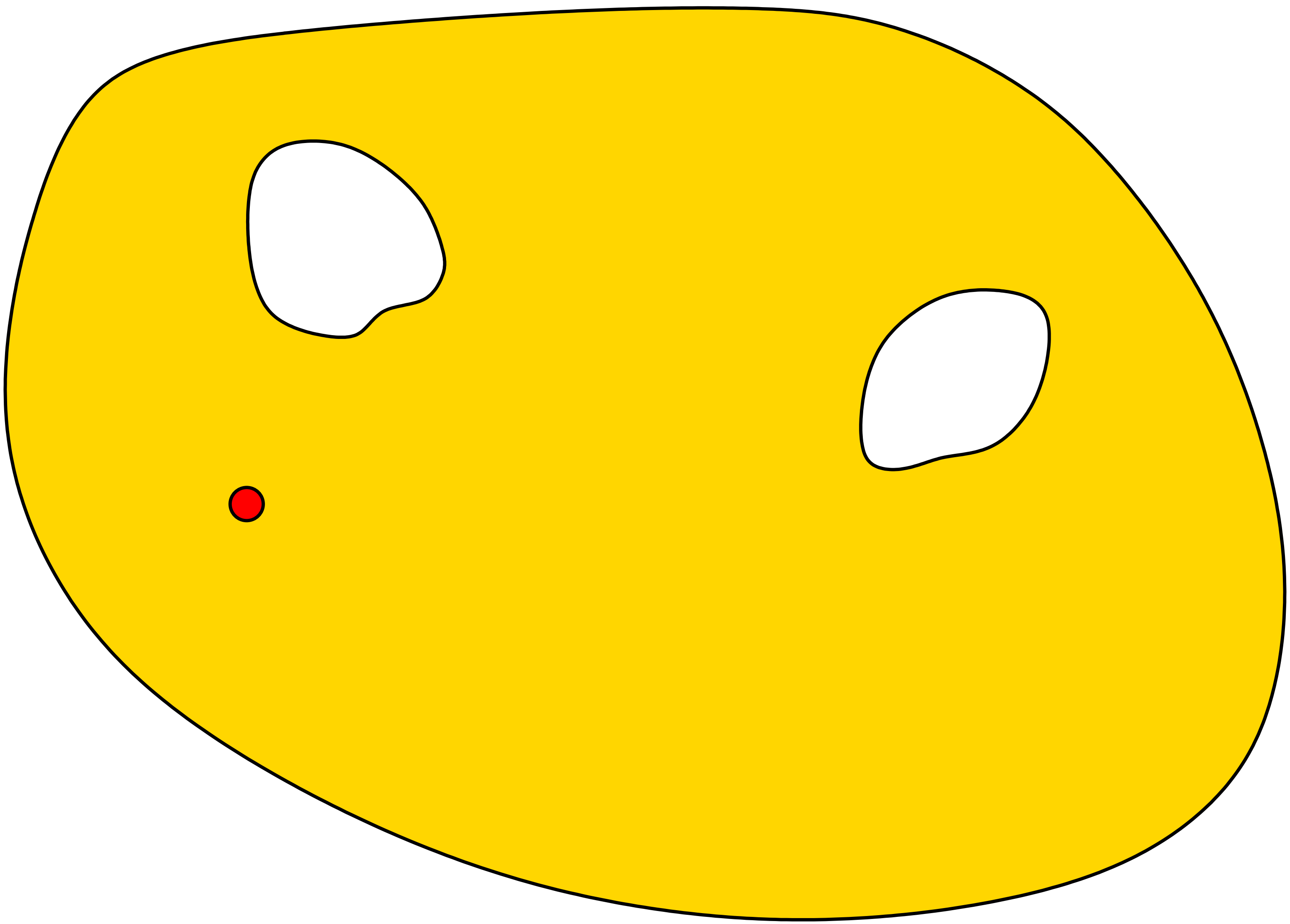

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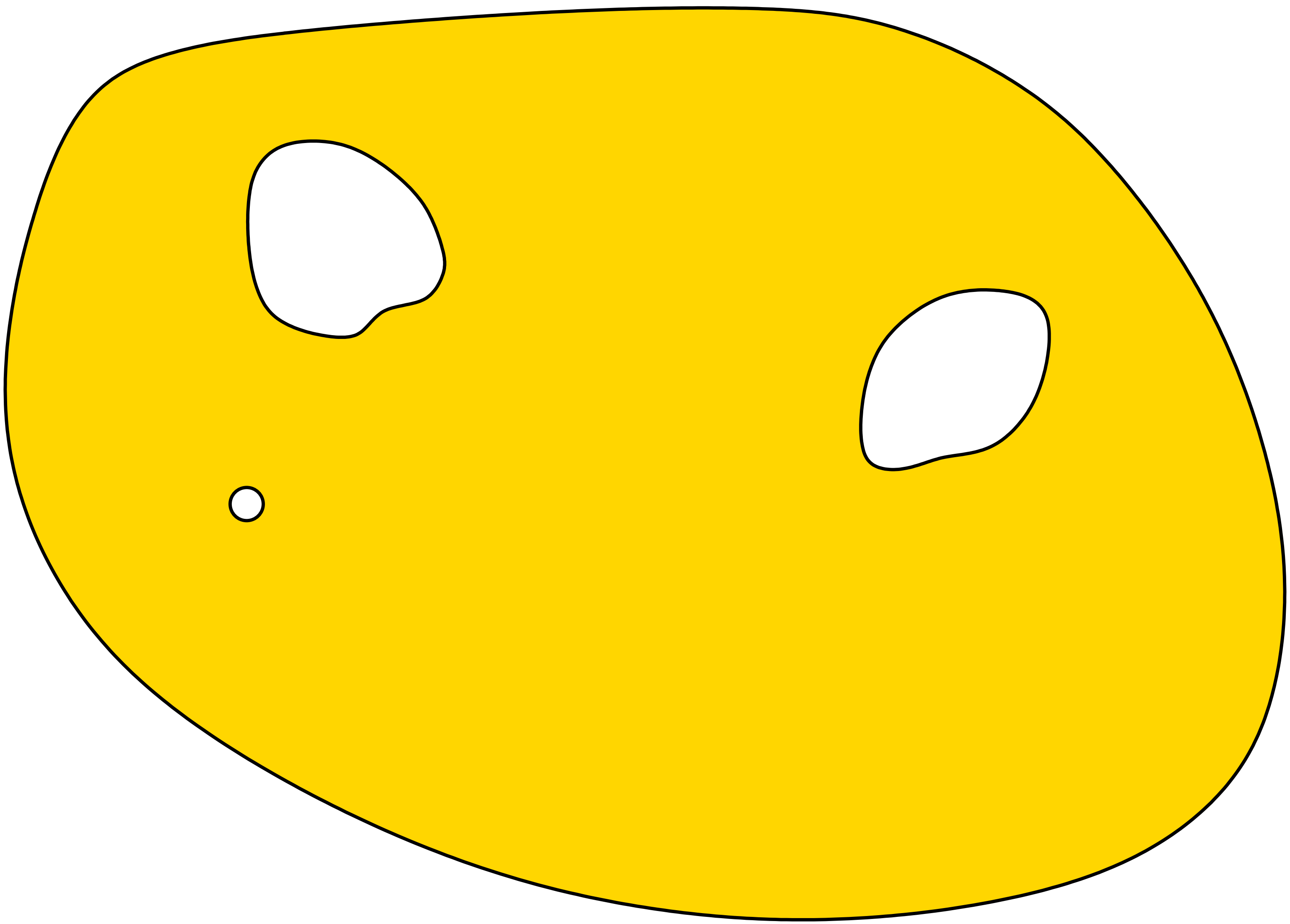

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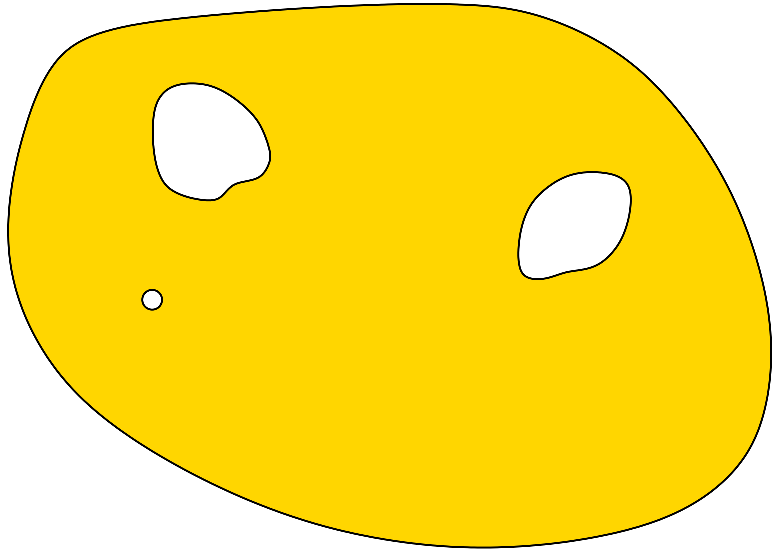

Same technique as for MIS:

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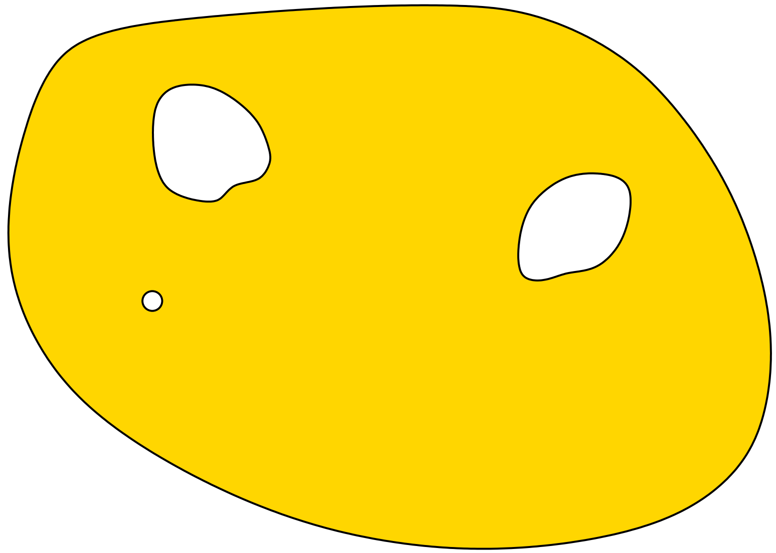

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- More recently Czumaj, Peng, and Sohler (2014) gave tester for k-clusterability

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- Testing bipartiteness: Goldreich, Ron (1998)
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  Czumaj, Goldreich, Ron, Seshadhri, Shapira, Sohler (2010)
  Fichtenberg, Levi, Vasudev, Wötzel (2017)
  Kumar, Seshadhri, Stolman (2018)

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- Later this week: extensions to some other properties
  (Czumaj, Sohler)

Questions?