DISTRIBUTED SIMULATION AND DISTRIBUTED INFERENCE

Last Thing Standing Between You and a Beer

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June 15, 2018

Joint work with Jayadev Acharya (Cornell University) and Himanshu Tyagi (IISc Bangalore)
A STORY
Boaty McBoatface is starting its first mission today!
It’s going to Antarctica to study global warming, not to play.

The world’s oceans are changing, you see. It’s freezing down there, but not as cold as it used to be.
Boaty’s findings will be sent to scientists with care,
By way of a radio link, but with a certain flair.
McBoatfaces are expensive

What is the most ship-efficient protocol to reliably test whether the distribution of temperatures matches the one on record?
DISTRIBUTED INFERENCE
THE SETTING: “SIMULTANEOUS COMMUNICATION PROTOCOL” (SMP)

- an inference task $P$ over $k$-ary distributions
- an unknown $k$-ary distribution $p$
- one centralized “referee” $R$ who needs to solve $P$ on $p$
- $n$ communication-limited players, each can send $\ell$ bits to $R$
- each player independently gets one sample from $p$

Question: As a function of $k$, $\ell$, and all relevant parameters of $P$, how many players $n$ are required?
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Question

As a function of $k$, $\ell$, and all relevant parameters of $\mathcal{P}$, how many players $n$ are required?
Can assume $\ell < \log_2 k$, otherwise trivial.

Inference tasks: density estimation, parameter estimation, functional estimation, hypothesis testing/property testing...

Different flavors: public-coin, pairwise-coin, private-coin...
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Inference tasks: density estimation, parameter estimation, functional estimation, hypothesis testing/property testing...

Different flavors: public-coin, pairwise-coin, private-coin
“SIMULATE-AND-INFER”
One approach to solve it all

Key Observation

If the referee can simulate independent samples from $p$ using the messages from the players, then it can do anything.

Begging the question

Can the referee simulate independent samples from $p$ using the messages from the players?
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Begging the question

Can the referee simulate independent samples from \( p \) using the messages from the players?
Theorem

For every $k \geq 1$ and $\ell < \log k$, there exists no SMP with $\ell$ bits of communication per player for distributed simulation over $[k]$ with any finite number of players. (Even allowing public-coin and interactive protocols.)

Proof.

By contradiction, [...].
Theorem

For every $k \geq 1$ and $\ell < \log k$, there exists no SMP with $\ell$ bits of communication per player for distributed simulation over $[k]$ with any finite number of players. (Even allowing public-coin and interactive protocols.)
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Proof.

By contradiction, [...] pigeonhole principle [...].
Theorem
For every \( k; \ell \geq 1 \), there exists a private-coin protocol with \( \ell \) bits of communication per player for distributed simulation over \( \left[ k \right] \), with expected number of players \( O\left( \frac{k}{2^\ell} \right) \). Moreover, this is optimal even allowing public-coin and interactive protocols.

Proof.
Case \( \ell = 1 \). Player \( 2i \) and \( 2i + 1 \) both send 1 if their sample "hits" \( i \); the referee outputs \( i \) if (i) player \( 2i + 1 \) is the only odd player sending 1, and player \( 2i \) sends 0. Then, conditioned on \( R \) not outputting \( ? \), \( i \) is outputted with probability \( p_i \).

And the probability to output \( ? \) is

\[
1 - \left( 1 - \frac{p_i}{\phi\left( \|p\|_2 \right)} \right)^k
\]

(and some complications to bound this away from 1).
Theorem

For every $k, \ell \geq 1$, there exists a private-coin protocol with $\ell$ bits of communication per player for distributed simulation over $[k]$, with expected number of players $O(k/2^\ell \lor 1)$. Moreover, this is optimal even allowing public-coin and interactive protocols.
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Case $\ell = 1$. 
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Proof.

Case $\ell = 1$. Player $2i - 1$ and $2i$ both send 1 if their sample “hits” $i$;
Theorem

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Theorem

For every \( k, \ell \geq 1 \), there exists a private-coin protocol with \( \ell \) bits of communication per player for distributed simulation over \([k]\), with expected number of players \( O(k/2^\ell \lor 1) \). Moreover, this is optimal even allowing public-coin and interactive protocols.

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\[
1 - \prod_{i=1}^{k} (1 - p_i) \leq 1 - \phi(\|p\|_2)
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Theorem

For every $k, \ell \geq 1$, there exists a private-coin protocol with $\ell$ bits of communication per player for distributed simulation over $[k]$, with expected number of players $O(k/2^\ell \lor 1)$. Moreover, this is optimal even allowing public-coin and interactive protocols.

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$$1 - \prod_{i=1}^{k} (1 - p_i) \leq 1 - \phi(\|p\|_2)$$

(and some complications to bound this away from 1).
Corollary (Informal)
For any inference task $P$ over $k$-ary distributions with sample complexity $s$ in the non-distributed model, there is a private-coin protocol for $P$, with $\ell$ bits of communication per player, and $n = O(s^{k/2})$ players.
Corollary (Informal)

For any inference task $\mathcal{P}$ over $k$-ary distributions with sample complexity $s$ in the non-distributed model, there is a private-coin protocol for $\mathcal{P}$, with $\ell$ bits of communication per player, and $n = O(s \cdot k/2^\ell)$ players.
Corollary (Learning in Total Variation)
For every \( k; \ell \), there is a private-coin protocol for learning \( k \)-ary distributions with \( \ell \) bits per player, and \( n = O(k^2 2^{\ell/2}) \) players. (And this is optimal, even for public-coin and interactive protocols.)

Corollary (Testing Uniformity)
For every \( k; \ell \), there is a private-coin protocol for testing uniformity over \( \{k\} \) with \( \ell \) bits per player, and \( n = O(k^3 2^{\ell/2}) \) players.
Corollary (Learning in Total Variation)

For every $k, \ell \leq \log_2 k$, there is a private-coin protocol for learning $k$-ary distributions with $\ell$ bits per player, and $n = O\left(\frac{k^2}{2^{\ell} \varepsilon^2}\right)$ players.
(And this is optimal, even for public-coin and interactive protocols.)
Corollary (Learning in Total Variation)

For every $k, \ell \leq \log_2 k$, there is a private-coin protocol for learning $k$-ary distributions with $\ell$ bits per player, and $n = O\left(\frac{k^2}{2^\ell \varepsilon^2}\right)$ players. (And this is optimal, even for public-coin and interactive protocols.)

Corollary (Testing Uniformity)

For every $k, \ell \leq \log_2 k$, there is a private-coin protocol for testing uniformity over $[k]$ with $\ell$ bits per player, and $n = O\left(\frac{k^{3/2}}{2^\ell \varepsilon^2}\right)$ players.
ONE APPROACH TO REALLY, REALLY SOLVE IT ALL?

Is this "simulate-and-infer" approach optimal?

Conjecture (The Flying Pony Question)
Does the simulate-and-infer scheme that simulates independent samples compressed to the size $*$ of the problem using private-coin protocols, and sends them to the referee who then infers from them, always require the lowest number of players?
Natural Question

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Conjecture (The Flying Pony Question)

Does the simulate-and-infer scheme that simulates independent samples *compressed to the size* of the problem using private-coin protocols, and sends them to the referee who then infers from them, always require the lowest number of players?
NO FLYING PONY
The answer is no:

**Theorem**

There exists an inference task \( \mathcal{P} \) over \( k \)-ary distributions with 
\[ 2^{\text{size}(\mathcal{P})} \cdot \text{samplecomplexity}(\mathcal{P}) = \Omega(k^{3/2}) \], yet for which there is a 1-bit private-coin protocol with \( n = O(k) \) players.
The answer is no:

**Theorem**

There exists an inference task $\mathcal{P}$ over $k$-ary distributions with $2^{\text{size}(\mathcal{P})} \cdot \text{samplecomplexity}(\mathcal{P}) = \Omega(k^{3/2})$, yet for which there is a 1-bit private-coin protocol with $n = O(k)$ players.

**Proof.**

Promise problem: $p$ is either uniform, or uniform on an arbitrary subset of $k/2$ elements.
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**Proof.**

Promise problem: $p$ is either uniform, or uniform on an arbitrary subset of $k/2$ elements. $\text{samplecomplexity}(\mathcal{P}) = \sqrt{k}$ (folklore); $2^\text{size}(\mathcal{P}) = \Omega(k)$ (from other theorems);
The answer is no:

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PUBLIC-COIN UNIFORMITY TESTING
Must decide:

\[ p = u_k \text{ (uniform)} \]
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(and be correct on any \( p \) with probability at least 2/3)

Fundamental property of distributions, building block for testing many others. [BKR04, Gol16, CDGR17]
UNIFORMITY TESTING, RECAP

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- completely understood in the non-distributed setting: \( n = \Theta(\sqrt{k}/\varepsilon^2) \) samples [GR00, BFR+00, Pan08, DGPP17]
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- general “simulate-and-infer” scheme gives private-coin protocol with \( n = O(k^{3/2}/\varepsilon^2) \) players
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- general “simulate-and-infer” scheme gives private-coin protocol with \( n = O(k^{3/2}/\varepsilon^2) \) players (optimal?)

- what if we allow public coins?
Theorem (Upper Bound)

For every $k, \ell \leq \log_2 k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell$ bits per player, and $n = O\left(\frac{k}{2^{\ell/2} \varepsilon^2}\right)$ players.
Theorem (Upper Bound)

For every $k, \ell \leq \log_2 k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell$ bits per player, and $n = O\left(\frac{k}{2^{\ell/2} \varepsilon^2}\right)$ players.

Theorem (Lower Bound)

This is optimal.
Theorem (Warm Up)

For every $k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell = 1$ bit per player, and $n = O\left(\frac{k}{\varepsilon^3 \log \frac{1}{\varepsilon}}\right)$ players.
Theorem (Warm Up)

For every \( k \), there is a public-coin protocol for testing uniformity over \([k]\) with \( \ell = 1 \) bit per player, and \( n = O\left(\frac{k}{\varepsilon^3} \log \frac{1}{\varepsilon}\right) \) players.

Proof.

Starting point: if \( p \) is \( \varepsilon \)-far from uniform, by definition,

\[
\mathbb{E}_{x \sim u}[|p(x) - 1/k|] > \varepsilon/k .
\]
Theorem (Warm Up)

For every $k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell = 1$ bit per player, and $n = O\left(\frac{k}{\varepsilon^3} \log \frac{1}{\varepsilon}\right)$ players.

Proof.

Starting point: if $p$ is $\varepsilon$-far from uniform, by definition,

$$\mathbb{E}_{x \sim u}[|p(x) - 1/k|] > \varepsilon/k.$$ 

Now, by an averaging argument (Markov),

$$\Pr_{x \sim u}[p(x) < (1 - \varepsilon/2)/k] > \varepsilon/2$$
Theorem (Warm Up)

For every $k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell = 1$ bit per player, and $n = O\left(\frac{k}{\epsilon^3 \log \frac{1}{\epsilon}}\right)$ players.

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and therefore [...]
Theorem

For every $k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell = 1$ bit per player, and $n = O(k/\varepsilon^2)$ players.
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Proof.

Same starting point. Now, by a better averaging argument (Levin’s work investment strategy), there exists $1 \leq j \leq L := \log_2(1/\varepsilon)$ s.t.

$$\Pr_{x \sim U}[p(x) < (1 - 2^{-j})/k] > \varepsilon \cdot 2^j/(L + 1 - j)^2$$
Theorem

For every $k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell = 1$ bit per player, and $n = O(k/\varepsilon^2)$ players.

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and therefore [...] (also, don’t pay for the union bound!)
Theorem

For every $k, \ell \leq \log_2 k$, there is a public-coin protocol for testing uniformity over $[k]$ with $\ell$ bits per player, and $n = O\left(k/(2^{\ell/2}\varepsilon^2)\right)$ players.
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Proof.

Starting point: for a set $S \subseteq [k]$ of $2^\ell - 1$ elements with $p(S) \approx 2^\ell / k$, testing uniformity of the conditional distribution $p_S$ would cost

$$(k/2^\ell) \cdot \sqrt{2^\ell / \varepsilon^2} = k/(2^{\ell/2} \varepsilon^2)$$

samples, by rejection sampling.
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samples, by rejection sampling. Now, if $p$ is $\varepsilon$-far from uniform then, on a u.a.r. set $S \subseteq [k]$ of $2^\ell - 1$ elements, $p_S$ is $\varepsilon$-far from uniform on expectation.
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samples, by rejection sampling. Now, if $p$ is $\varepsilon$-far from uniform then, on a u.a.r. set $S \subseteq [k]$ of $2^\ell - 1$ elements, $p_S$ is $\varepsilon$-far from uniform on expectation. Then, same ideas as before: Levin’s strategy+careful allocation of the failure probabilities.
Theorem

For every \( k, \ell \leq \log_2 k \), every public-coin protocol for testing uniformity over \([k]\) with \( \ell \) bits per player, must have \( n = \Omega(k/(2^{\ell/2}\varepsilon^2)) \) players.
Theorem

For every $k, \ell \leq \log_2 k$, every public-coin protocol for testing uniformity over $[k]$ with $\ell$ bits per player, must have $n = \Omega(k/(2^{\ell/2} \varepsilon^2))$ players.

Proof.

By Le Cam’s two-point method, consider a distribution over “hard instances”:

$$\forall 1 \leq i \leq k/2, \quad p(2i - 1), p(2i) = \left( \frac{1 \pm \varepsilon}{k}, \frac{1 \mp \varepsilon}{k} \right)$$

uniformly and independently at random. (Paninski’s construction [Pan08]).
Theorem

For every $k, \ell \leq \log_2 k$, every public-coin protocol for testing uniformity over $[k]$ with $\ell$ bits per player, must have $n = \Omega(k/(2^{\ell/2}\varepsilon^2))$ players.

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uniformly and independently at random. (Paninski’s construction [Pan08]). But needs to upper bound the TV distance between (i) distribution of $n$ messages sent to the referee when $p = u_k$, and (ii) distribution of $n$ messages under average hard instance.
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For every $k, \ell \leq \log_2 k$, every public-coin protocol for testing uniformity over $[k]$ with $\ell$ bits per player, must have $n = \Omega(k/(2^{\ell/2}\varepsilon^2))$ players.

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uniformly and independently at random. (Paninski’s construction [Pan08]). But needs to upper bound the TV distance between (i) distribution of $n$ messages sent to the referee when $p = u_k$, and (ii) distribution of $n$ messages under average hard instance. The latter is not a product distribution...
General framework for distributed inference problems over discrete distributions, in the communication-starved regime.
CONCLUSION

- General framework for distributed inference problems over discrete distributions, in the communication-starved regime
- Tight bounds for distributed simulation (and distributed learning [DGL+17, HMÖW18, HÖW18])
- First work on distributed testing
- Optimal protocols for public-coin uniformity testing
- Many questions and directions to explore
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First work on distributed testing

Optimal protocols for public-coin uniformity testing

Many questions and directions to explore*
THANK YOU

ILLUSTRATION ©DAMI LEE
Tuğkan Batu, Lance Fortnow, Ronitt Rubinfeld, Warren D. Smith, and Patrick White.
Testing that distributions are close.

Tuğkan Batu, Ravi Kumar, and Ronitt Rubinfeld.
Sublinear algorithms for testing monotone and unimodal distributions.

Clément L. Canonne, Ilias Diakonikolas, Themis Gouleakis, and Ronitt Rubinfeld.
Testing shape restrictions of discrete distributions.

Ilias Diakonikolas, Elena Grigorescu, Jerry Li, Abhiram Natarajan, Krzysztof Onak, and Ludwig Schmidt.
Communication-efficient distributed learning of discrete distributions.

Ilias Diakonikolas, Themis Gouleakis, John Peebles, and Eric Price.
Sample-optimal identity testing with high probability.

Oded Goldreich.
The uniform distribution is complete with respect to testing identity to a fixed distribution.
Electronic Colloquium on Computational Complexity (ECCC), 23:15, 2016.

Oded Goldreich and Dana Ron.
On testing expansion in bounded-degree graphs.

Yanjun Han, Pritam Mukherjee, Ayfer Ö zgür, and Tsachy Weissman.
Distributed statistical estimation of high-dimensional and nonparametric distributions with communication constraints, February 2018.
Talk given at ITA 2018.

Yanjun Han, Ayfer Özgür, and Tsachy Weissman.
Geometric Lower Bounds for Distributed Parameter Estimation under Communication Constraints.
abs/1802.08417.

Liam Paninski.
A coincidence-based test for uniformity given very sparsely sampled discrete data.