LOCAL COMPUTATION ALGORITHMS
(a tale of two computation models)

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joint work with:
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Want to solve graph problems

**Labeled Graphs**

* vertices have unique ID's $\in \{1, \ldots, n\}$
* edges have port numbers $\in \{1, \ldots, \Delta\}$

- $v$ is 2'nd neigh. of $u$
- $u$ is 4'th neigh. of $v$
Access Model to Graph

\[
\text{probe}(v) : \quad \text{return list of all neighbors of } v
\]

\[
\Delta \times \text{more expensive than } \text{probe}(v,i) = \text{return } i\text{-th neighbor of } v
\]
Labeled graph $G$ & fixed MIS in $G$

MIS = maximal ind. set
Local Computation Alg (for MIS)

query
V \in \text{MIS}?

yes/no

LCA

 probes

neighbors

G

* LCA generates a seq. of probes
* LCA always answers correctly

(stateless det version of RTVX11)
query \(\text{veMIS?}\) \(\xrightarrow{\text{LCA}}\) G

probes \(\xrightarrow{\text{neighbors}}\)

**GOAL:** \(\text{MIN} \# \text{probes.} \quad (o(n) \text{ probes per query})\)

**Rules:**

- **consistent:** \(\exists \text{MIS} \forall q \quad \text{LCA}(q) = I_{\text{MIS}}(q)\)

- **stateless:** \(\text{LCA can\text{'}t store previous queries, probes, neighbors, answers, MIS...}\)
LCA for MIS?!

* answer yes unless conflicts prev. answer.
  requires storing answers.
* can't store MIS because did not read G.
LCA - application

- uncoord. servers
- no communication between servers
- all consist. wrt to same MIS

[NO-08, YYI-12]

use "LCA" to apx VC
LCA Variations

1) Randomized (use a random seed)  
\[ \text{NO-08, RTVX-11} \]

Never wrong, but

\[
\Pr \left[ \text{too many probes} \right] < \frac{1}{\text{poly}(n)}
\]

2) With States (to generate a random graph)  
\[ \text{ELMR, BRY17} \]
(LCA vs.) Distributed LOCAL

* distributed model in which output of $v$ is determined by Ball ($v$, radius = r)

* equiv. 1 rounds of communication with unbounded msg lengths.

* running time is unbounded.

focus on local info.

goal: MIN # rounds.
Easy Reductions

\[ \text{LCA with probe radius} \leq r \implies \text{LOCAL with } r \text{ rnds.} \]
Easy Reductions

Det. distributed LOCAL with \( r \) rounds

\[ \text{LCA with } \Delta^r \text{ probes} \]

But, \( \Delta = 2 \):

\[ \text{# probes} \leq 2r + 1 \]

2r + 1
Classic Parallel Challenges

MIS, Vertex Cover, Maximal Matching, $\Delta+1$ color (2-approx)

Greedy Seq. Alg.

1) Fix a vertex ordering $v_1, v_2, \ldots, v_n$

2) For $i = 1$ to $n$:
   - add $v_i$ to MIS if $MIS \cap N(v_i) = \emptyset$

Q: Simulate by LCA, LOCAL, PRAM...
PLAN: LCA Simulation of Seq. Greedy

1) compute acyclic orientation of G

2) Sim. Greedy by "DFS".

query: $v_3 \in MLS$?

probe all vertices reachable from $v_3$

THM: DFS version agrees with greedy wrt topological ordering
PLAN: LCA Simulation of Seq. Greedy

1) compute acyclic orientation of G

2) Sim. Greedy by "DFS".

#probes = #vertices reachable from $V_3$.

GOAL! Find orientation such that

$$\max_v |\text{reachable}(v)|$$

is small.
Acyclic Orientation

1) use ID's: \( \text{id}(u) > \text{id}(v) \)

\[
\begin{array}{c}
\text{u} \\
\downarrow \\
\text{v}
\end{array}
\]

may induce long paths:

\[
\begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n
\end{array}
\]

\( \left| \text{reach}(1) \right| = n \)

means \( n \) probes.
2) Random Permutations \([NO-08, YYI-12, ARVX-12, MRVX-12, RV-16]\)

\[ \{r(v)\}_v \text{ i.i.d. unif } [0,1] \]

\[ r(u) > r(v) \]

\[ u \rightarrow v \]

THM:

\[ \mathbb{E}_{r,v} \left[ |\text{reach}(v)| \right] \leq e^\Delta \]

\[ \text{[NO-08]} \]

\[ \Pr_r \left[ \max_v |\text{reach}(v)| > 2^\Delta \cdot \log n \right] < \frac{1}{n^2} \]

\[ \text{[RV16]} \]

\[ \Rightarrow \text{Store } & \# \text{ probes } \geq \Omega(\log n) \]
3) **Vertex Coloring**

\[
\text{color}(u) > \text{color}(v) \Rightarrow \quad \rightarrow
\]

reach radius \( \leq \) # colors
\[
|\text{reach. set}| \leq \Delta \#\text{colors}
\]

lots of distributed vertex coloring alg

[\text{L92, BE09, ...}] achieve

\( \text{poly}(\Delta) \) colors

\( \text{poly}(\Delta) + \log^* n \) rnds
Recap: LCA sim. of seq. Greedy

1) acyclic orientation (via vertex color)
2) run DFS version.
   colors computed on-the-fly during DFS.

probe radius \leq \# colors + \text{rnds}(_{\text{LOCAL}}) proto rad.

\Rightarrow \# probes \leq \Delta

E.G. \# probes \leq \Delta^{\text{poly}(\Delta) + \lg^* n}
LCA Simulation of Seq. Greedy (MIS)

THM: \#probes ≤ Δ^2, Δ, log* n

probe radius ≤ Δ + log* n

holds: MIS, (Δ+1)-coloring, maximal Match., 2 apx VC

need to describe how to compute vertex coloring in LCA
(Slightly) Better LCA vertex coloring

THM: \( O(\Delta^2) \) colors with
\[
\# \text{probes} \leq \Delta^4 \cdot \log^* n
\]
\[
\text{probe rad} \leq \log^* n
\]

proof:
0) First design dist. LOCAL alg.
1) Partition \( E \) to \( \Delta^2 \) disjoint paths/cycles.

\[
E_{i,j} = \{ (v,u) \mid v = \text{neighbor}(u,j) \} \]
\[
(\deg_{E_{i,j}} \leq 2) \]

zero reads [Kuhn 09]
(Slightly) Better LCA vertex coloring

THM: \( O(\Delta^2) \) colors with
\[
\#\text{probes} \leq \Delta^4 \cdot \log^* n
\]
\[
\text{probe rad} \leq \log^* n
\]

proof:

1) partition \( E = \bigcup_{i,j} E_{i,j} \) (paths/cycles)

2) 3-color every \( E_{i,j} \) \( O(\log^* n) \) runs \([PR]\)

\[\Rightarrow \] 3\(\Delta^2\) colors for \( G \)

\[
\text{color}(v) = \{ c_{i,j}(v) \}_{i,j}, \quad c_{i,j}(v) \in \{1,2,3\}
\]
Better LCA vertex coloring

THM: $O(\Delta^2)$ colors with

# probes $\leq \Delta^4 \cdot \log^* n$

probe rad $\leq \log^* n$

proof:

2) $3^{\Delta^2}$ - coloring

3) reduce # colors using Linial 1-rnd alg.

$3^{\Delta^2} \xrightarrow{1\text{-rnd}} 5\Delta^2 \log(3^{\Delta^2}) \xrightarrow{1\text{-rnd}} O(\Delta^3) \xrightarrow{1\text{-rnd}} O(\Delta^2)$
(Slightly) Better LCA vertex coloring

THM: $O(\Delta^2)$ colors with

$\#\text{probes} \leq \Delta^4 \cdot \log^* n$

$\text{probe rad} \leq \log^* n$

proof:
Sim. LOCAL coloring alg by LCA:

path/cycle: $\log^* n$ rnds $\mapsto 2 \cdot \log^* n$ probes.

$\forall$ vertex incident to $\leq \Delta$ paths/cycles

$\Rightarrow 2\Delta \cdot \log^* n$ probes.

3 distributed rnds $\Rightarrow \Delta^3 \cdot 2\Delta \cdot \log^* n$ probes!
LCA Simulation of Seq. Greedy (MIS)

THM: \#probes \leq \Delta^2 \cdot \Delta \cdot \log^* n

probe radius \leq \Delta + \log^* n

holds: MIS, (\Delta+1)-coloring, maximal Match., 2 aox VC

proof:

\[
\begin{align*}
\text{query}(v) & \leq (\Delta^2) \\
|\text{DFS}| & \leq \Delta \\
\text{LCA color}(x) & \leq \Delta^4 \cdot \log^* n \\
\end{align*}
\]
$(1-\varepsilon)$-apx max. match. in distributed LOCAL

CLA-MIS via acyclic orient via dist. vertex color

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CLA

global alg

recursive improvement [HK, NO-08]

$\#$ probes $= 2^{\Delta^{1/2}} \cdot (\log^* n)^{1/2}$

trivial reduction

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distributed LOCAL alg

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$\#$ rnds $= \Delta^{(1/2)} + \varepsilon^{-2} \cdot \log^* n$
1) random graph generation
   * pref. attach. [ELMR]
   * \( G(n,p) \) with random/next neigh...

2) \((2+\varepsilon)\)-apx max. matching with
   \( \Delta \cdot \log^* n \) probes [Fischer 17]

3) SLOCAL: generalize greedy seq
to Balls of radius \( r \) [GKM 16]