Relaxed Locally Correctable Codes in Computationally Bounded Channels

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Classical Locally Decodable/Correctable Codes

Encoding: E: $\{0,1\}^k \to \{0,1\}^n$

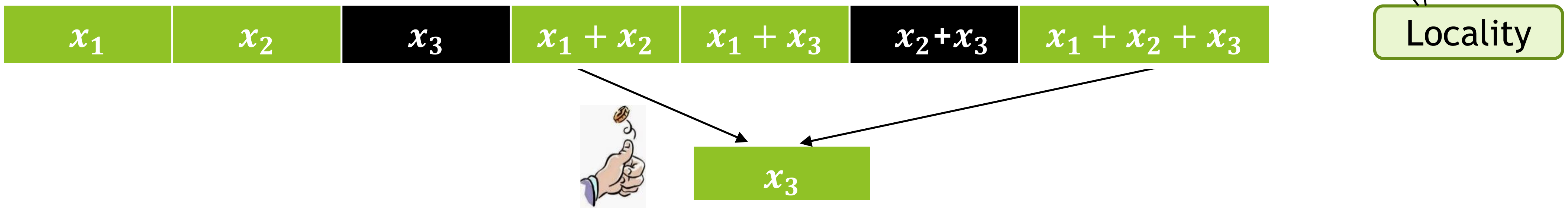
Decoding: D: $\{0,1\}^n \to \{0,1\}^k$ such that given w with $\mathrm{dist}(w, E(m)) < \delta n$ then $D(w) = m$.

Goal: *efficient* encoding/decoding

Parameters: information rate: k/n; minimum distance: $\min \mathrm{dist}(E(m_1), E(m_2))$

Locally decodable/correctable codes (LDCs/LCCs)

- LDC: Given oracle access to input w with $\mathrm{dist}(w, E(m)) < \delta n$, and i, compute m_i with $o(n)$ queries
- LCC: Given oracle access to input w with $\mathrm{dist}(w, E(m)) < \delta n$, and i, compute $(E(m))_i$ with $o(n)$ queries



Status: $q = 2^{O(\sqrt{\log n})}$, any constant rate $0 < R < 1$ [KMRS17]

$q > 2$ (constant), $n = 2^{2^{\sqrt{\log n}}}$ [Yek08, DGY11, Efr12]

Relaxed LDCs/LCCs (RLDCs/RLCCs)

RLDC/RLCCs: Given oracle access to input w with $\mathrm{dist}(w, E(m)) < \delta n$, $\boldsymbol{D}$ makes $q = o(n)$ queries and:

1) $\forall\, i,\ D_i(w) = m_i$ if $w = E(m)$ (RLDC); $D_i(w) = E(m)_i$ (RLCC)

2) $\forall\, i,\ \Pr\left[b \notin \{m_i, \perp\}\right] < 1/3$ (RLDC)

$\Pr\left[b \notin \{E(m)_i, \perp\}\right] < 1/3$ (RLCC)

3) Let Good= $\left\{j \mid \Pr\left[D_j(w) = m_j\right] > \frac{2}{3}\right\}$ (RLDC)

Good= $\left\{j \mid \Pr\left[D_j(w) = (E(m))_j\right] > \frac{2}{3}\right\}$ (RLCC)

Then $|\mathrm{Good}| > \rho n$, for some constant ρ.

Observation: 1) + 2) imply 3) for constant query codes and constant error rate

Status: RLDCs [BGHSV06]: $q = \Theta(1), n = k^{1+\varepsilon}$

RLCCs [GRR18]: $q = \Theta(1), n = \Theta(poly(k))$,

$q = (\log n)^{O(\log\log n)}$, $n = \Theta(k)$

Our results: $q = poly \log n$, $n = \Theta(k)$
for crypto version of definitions

Codes for Computationally Bounded Channels (CBC)

Hamming channel: the channel corrupts any pattern (possibly takes long time to corrupt adversarially)

Shannon channel: the channel introduces independent errors

Lipton channel: Computationally bounded – the channel is a PPT adversary ← This talk

Previous work:

General Codes in CBCs achieve better communication capabilities than in the Hamming model

[Lip94, DGL04, Langberg04, MPSW05, Smith07, GS16, SS16]

Locally Decodable Codes in CBCs: Requires trusted setup/key exchange

- Private-key LDCs [OPS07] - Assumes existence of OWF, shared secret key
  - $\Theta(1)$ info rate and error rate over binary alphabet, $q = \omega(1)$
- Public-key LDCs [HO08, HOSW11] - Crypto assumptions: ϕ-hiding schemes and IND-CPA secure cryptosystems

Computational Relaxed LCCs (CRLCC)

Security parameter λ, **s**=Gen(1^{λ}), **s** is public

$\xleftarrow{m}$ **A**

Sender $\xrightarrow{E(s,m)}$ **A** $\xrightarrow{i,\, w = E(s,m) + e}$ **Receiver** $\xrightarrow{b = D(s,w,i)}$

Defs: $\boldsymbol{p_{A,s}} = \Pr\left[b \notin \{w_i,\, \perp\}\right]$ (Decoder's error probability)

$\text{Good}_{\text{A,s}} = \left\{ i \mid \Pr[D(s,w,i) = (E(m))_i] > \frac{2}{3} \right\}$

Def: (Gen, E, D) is a CRLCC with parameters q queries, τ error rate, $0 < \rho \leq 1$, against PPT adversaries if D makes q queries to input w and

1) For all s, if $w = E(s,m)$ then $D(s,m,i) = \left(E(s,m)\right)_i$
2) For all A in the class, $\Pr[\Pr[b \notin \{w_i,\, \perp\}]>$ negl.$]$ < negl.
3) For all A in the class, $\Pr[\text{Good}_{\text{A,s}} < \rho n]$< negl.

Computational Relaxed LCCs

$(\boldsymbol{Gen}, \boldsymbol{E}, \boldsymbol{D})$ is a CRLCC with parameters $\boldsymbol{q}$ queries, $\boldsymbol{\tau}$ error rate, $\boldsymbol{\rho}$, against a class of adversaries (here PPT) if $\boldsymbol{D}$ makes q queries to input w and

1) For all s, if $w = E(s,m)$ then $D(s,m,i) = \big(E(s,m)\big)_i$
2) For all A in the class, $\Pr[\ \Pr[b \notin \{w_i,\ \perp\}] > \gamma$ =negl.] < μ = negl.
3) For all A in the class, $\Pr[\mathrm{Good}_{A,s} < \rho] < \mu$ = negl.

Weak CRLCC

Strong CRLCC

$p_{A,s} = \Pr[b \notin \{w_i,\ \perp\}]$; $\mathrm{Good}_{A,s} = \left\{i \mid \Pr[D(s,w,i) = (E(m)_i] > \frac{2}{3}\right\}$

Observation: Classical RLCC: for all A (not necessarily PPT) $\forall i$, |Good|>ρ, γ=1/3, μ=0

Our results: Weak and Strong CRLCC for binary alphabet, constant information and error rate, $\mathit{poly}\log(n)$ queries, assuming the existence of collision-resistant hash functions.

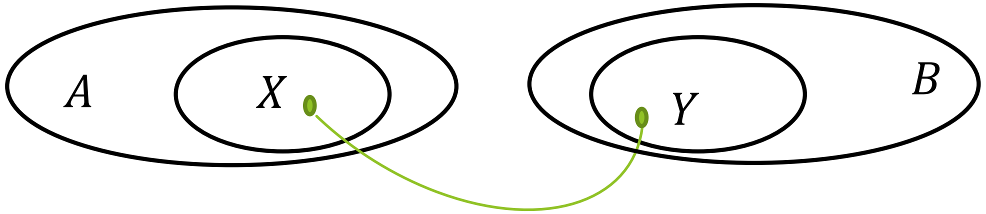
Our results - Observations

Results: Weak and Strong CRLCC for binary alphabet, constant error and information rate, $poly \log(n)$ queries, assuming the existence of collision-resistant hash function.

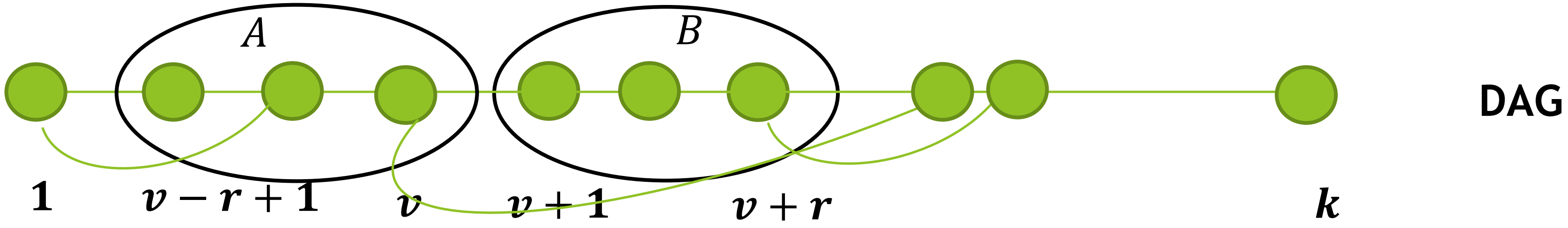
- Classical RLCCs [GRR18]: $q = (\log n)^{O(\log \log n)}$, constant information rate, subconstant error rate
- Previous constructions of RLCC in CBC need public/private-key crypto setup; our constructions don't.
- Our setup assumption: public seed chosen once
- **Key Idea:** local expander graphs

Local Expander Graphs and Their Properties

[ErdosGrahamSzemeredi75] (A, B) contains a δ-expander if
for all subsets $X \subseteq A$, $Y \subseteq B$ of fractional size δ, there is an edge between X and Y.



δ - local expander: G is a DAG such that for all vertices v, and radii r,
$(A = [v - r + 1, v], B = [v + 1, v + r])$ contains a δ-expander.



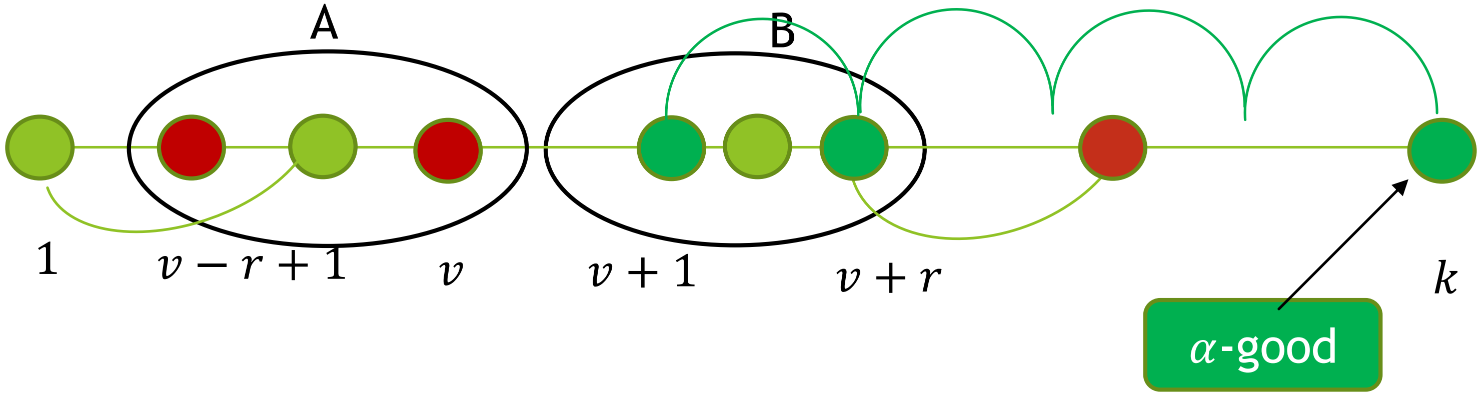
Local Expanders: Properties and Applications

Thm [EGS75, ABP18]: For any $\delta > 0$, there exist explicit δ-local expanders G on n vertices with

indegree(G), outdegree(G) = $\boldsymbol{O(\log n)}$

Def: For set $\boldsymbol{S}$, vertex $\boldsymbol{v}$ is α-good if for any radius r, $|S \cap [v-r+1, v]| \le \alpha r$ and $|S \cap [v+r-1, v]| \le \alpha r$

Thm [EGS75, ABP18]: If we delete large set $\mathbf{S} \subseteq V$, all α-good vertices are on a path



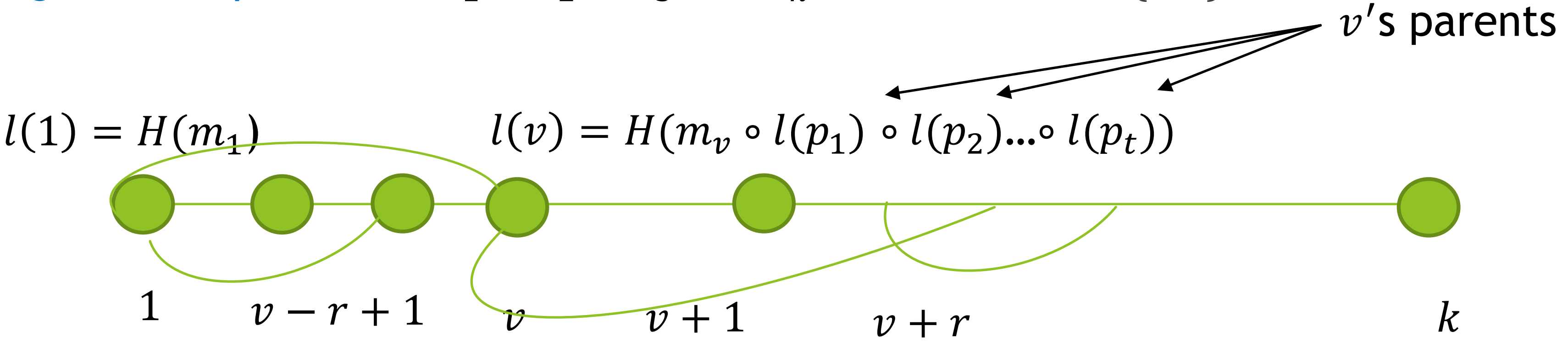
Applications:

- proof of sequential work [MMV13, CP18]
- time-lock puzzles and fair coin flipping protocols [BN00, JM10]
- design of memory hard functions [ABH17, ABP17, BZ17, ABP18]

(Weak) CRLCCs using local expander graphs

CRHF: $H_s\colon \{0,1\}^* \dashrightarrow \{0,1\}^{L(\lambda)}$ is collision-resistant if for all PPT adversaries A, $\Pr[A$ finds $H(x) = H(x')]$ is negl.

Labeling graph G using H and input $m = (m_1 \circ m_2 \circ m_3 \ldots \circ m_k) \in \Sigma^k$, where Σ=$\{0,1\}^{L(\lambda)}$

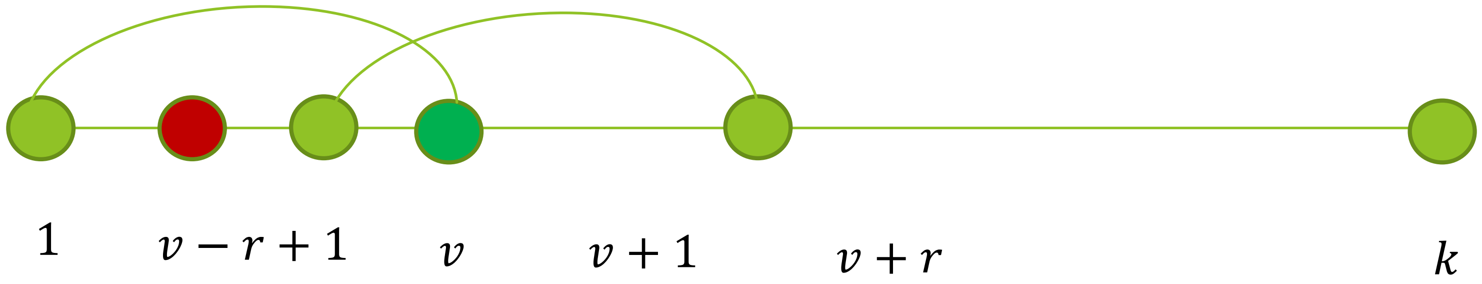


Encoding of $m = (m_1 \circ m_2 \circ m_3 \ldots \circ m_k)$ is the concatenation of 3 parts

1. $(ECC(m_1) \circ ECC(m_2) \circ ECC(m_3) \ldots \circ ECC(m_k))$ ECC is good and efficiently decodable (eg., Justesen)
2. $(ECC(l(1)) \circ ECC(l(2)) \circ ECC(l(3)) \ldots ECC(l(k))$ underlying G is δ-local expander
3. $(ECC(l(k)) \circ ECC(l(k)) \circ ECC(l(k)) \ldots\ldots ECC(l(k)))$ k copies of last label

Ingredients of the Local Decoder

- Testing consistent labeling:



After decoding the ECCs, check if v's label is consistent with parents' labels $\boldsymbol{l'(v) = H(m'_v \circ l'(p_1) \circ \ldots \circ l'(p_t))}$.

Else $\boldsymbol{v}$ is inconsistent.

$O(\log n)$ vertex queries

- Testing α-goodness:

Recall: Vertex $\boldsymbol{v}$ is α-good w.r.t set **S** if for any radius r,
$|S \cap [v-r+1, v]| \leq \alpha r$ and $|S \cap [v+r-1, v]| \leq \alpha r$

Test if vertex $\boldsymbol{v}$ of G is $\alpha/4$-good with respect to set **S** of inconsistent nodes.

Test guarantees: *accepts* if $\boldsymbol{v}$ is $\alpha/4$-good (hence also α-good) (whp)
rejects if $\boldsymbol{v}$ is not α-good (whp)

$\text{poly}(\log n)$ vertex queries

Local Decoding

Encoding of labeling of δ-expander G

$$w = (\underbrace{ECC(m'_1) \circ ECC(m'_2) \ldots \circ ECC(m'_k)}_{} \circ \boxed{ECC(l'(1)) \circ ECC(l'(2)) \circ \cdots \mathbf{ECC(l'(k))}} \circ \underbrace{ECC(l'(k)) \circ \ldots ECC(l'(k))}_{})$$

$\boldsymbol{D(i)}$: Output same answer as for the corresponding vertex

Ensures that the last block is decoded correctly

$\boldsymbol{D(i)}$: Decode by majority vote

$\boldsymbol{D(i)}$: Decode by majority vote
Test consistency of vertex $\boldsymbol{k}$ in G
Test if vertex $\boldsymbol{k}$ is α-good w.r.t set of inconsistent nodes
Output $\perp$ if tests fail; o/w output decoded bit

$\boldsymbol{D(i)}$: If i is in $ECC(l'(j))$, test if vertex j of G is α-good with respect to set of inconsistent nodes
If the answer is **yes**, output the decoding of $ECC(l'(j))$; o/w output $\perp$

Analysis: Key Ideas

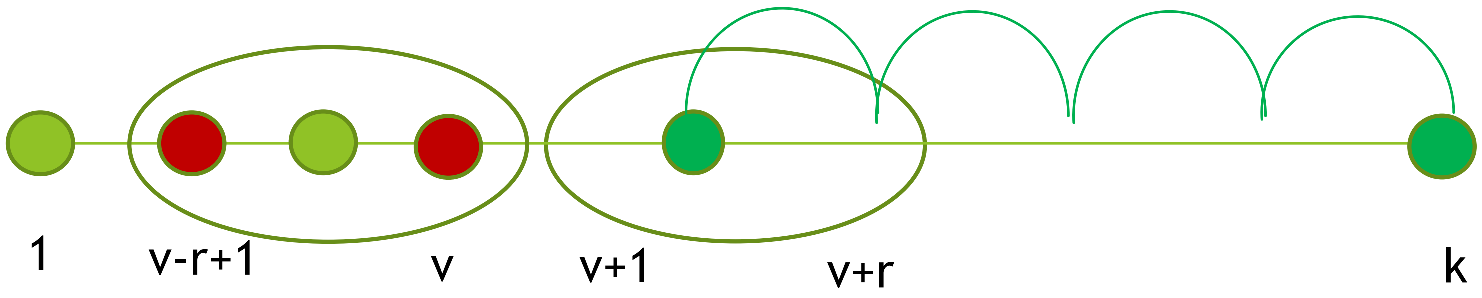
If vertex is consistent and **correctly decoded** then ← Want there properties from the last vertex

$$l(v) = l'(v) = H(m_v \circ l(p_1) \circ l(p_2) \ldots \circ l(p_t)) = H(m'_v \circ l'(p_1) \circ l'(p_2) \ldots \circ l'(p_t))$$

Implies $m_v = m'_v$ and $l(p_1) = l'(p_1), l(p_2) = l'(p_2), \ldots, l(p_t) = l'(p_t)$ **[correct decoding of parent label!]**

or colliding pair was found!

Hence, if a parent is consistent, then can iteratively backtrack along a path of consistent nodes and deduce correct decoding of a label!



Recall: Thm [EGS75, ABP18] If we delete large set $S \subseteq V$, **all α-good** vertices remain on a path.

Conclusion: The test only returns the decoded bit when it thinks that block is correctly decoded (and α-good.)

Extensions: Strong CRLCCs

Def: (Gen, E, D) is a CRLCC with parameters q queries, τ error rate, ρ, against a class of PPT adversaries if D makes q queries to input w and

1) For all s, if $w = E(s,m)$ then $D(s,m,i) = \big(E(s,m)\big)_i$
2) For all A in the class, Pr[[$\Pr[b \notin \{w_i, \perp\}]$> negl.]<negl.
3) For all A in the class, Pr[$\text{Good}_{\text{A,s}}$ < ρ]<negl.

- Need to ensure that the adversary cannot corrupt the entire codeword and obtain a new encoding in which all tests check
- **Idea:** Reduce the degree of the graphs by a composition of δ-expanders and path-like graphs and encode `metanodes' as blocks
- Use the extra fact that there are many α-good nodes (long paths)

Conclusions and Further Directions

Our results: Weak and Strong CRLCC/CRLDC for binary alphabet,
constant error and information rate,
$poly \log(n)$ queries,
assuming the existence of collision-resistant hash function.

Open directions: Better tradeoffs: $q = \Theta(1)$?

Other local models in computationally bounded channels (non-relaxed LCCs, testing)?

THANK YOU!