Set Cover in Sub-linear Time

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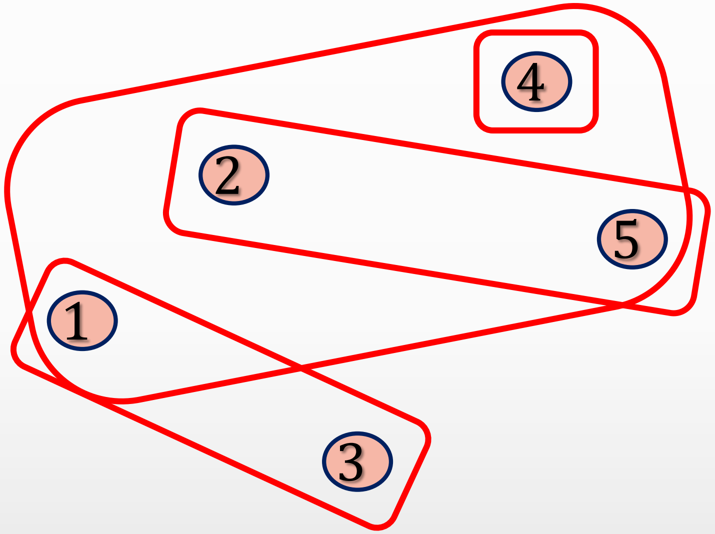
Set Cover Problem

Input: Collection $\mathcal{F}$ of sets $S_1, \dots, S_m$, each a subset of $\mathcal{U} = \{1, \dots, n\}$



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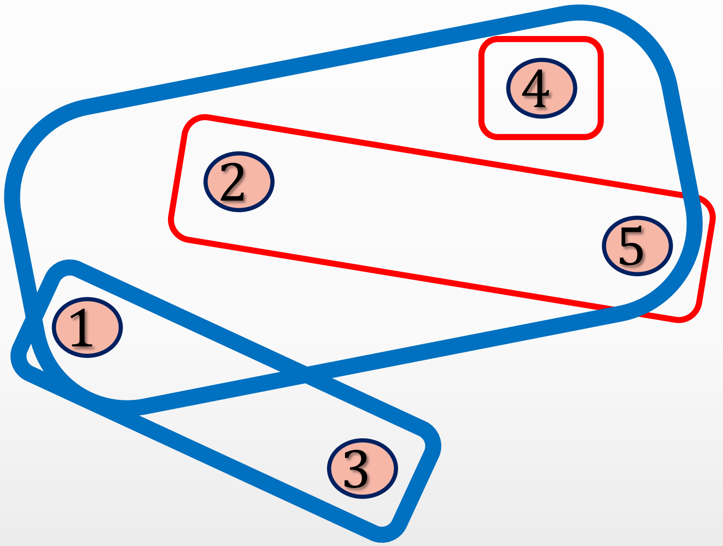

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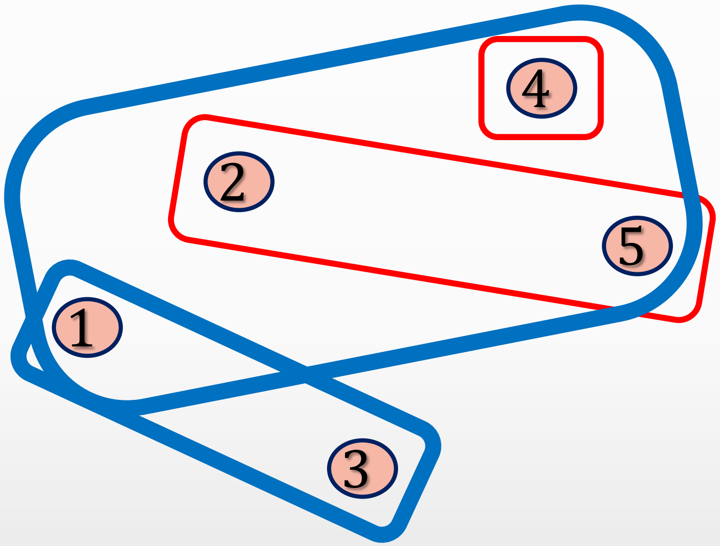

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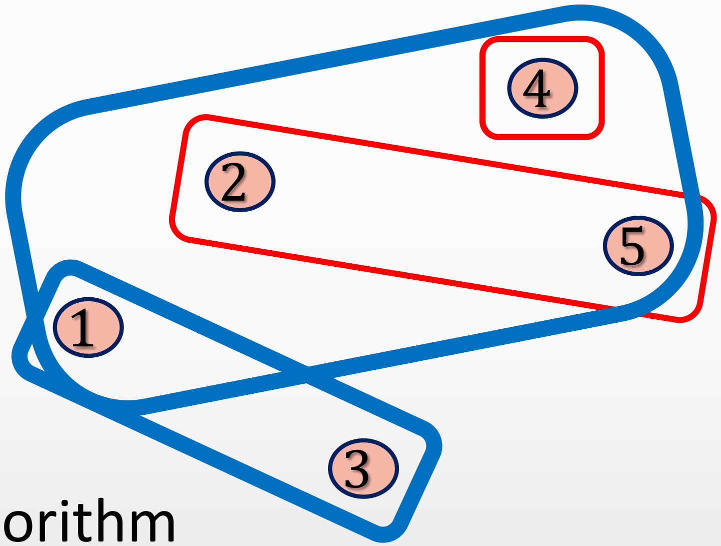

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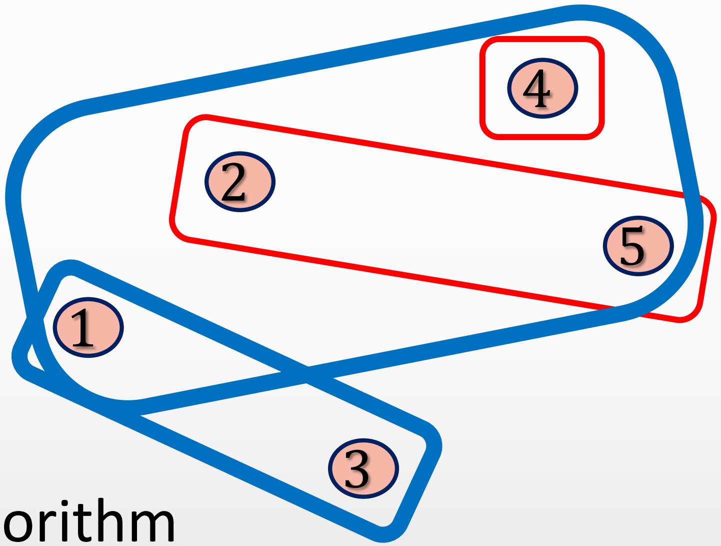

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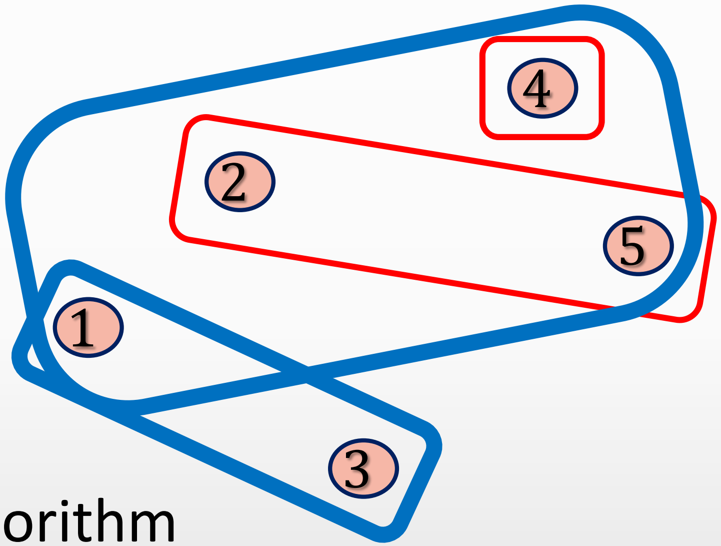

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*"Is it possible to solve minimum set cover in **sub-linear time**?"*

Sub-linear Time Set Cover

Data Access Model ?

Sub-linear Time Set Cover

Data Access Model [NO’08,YYI’12]

EltOf(S, i): ith element in $\boldsymbol{S}$
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  - $O(mk^2 + nk^2)$ (can be improved to $O(m + nk)$)

n = number of *elements* m = number of *sets* k = size of the optimal solution

Results

Problem	Approximation	Constraints	Query Complexity
Set Cover	$\alpha\rho + 1$	$\alpha \geq 2$	$\tilde{O}\left(m\left(\frac{n}{k}\right)^{\frac{1}{\alpha-1}} + nk\right)$
	$\rho + 1$	$-$	$\tilde{O}\left(\frac{mn}{k}\right)$
	α	$k \leq \left(\frac{n}{\log m}\right)^{\frac{1}{4\alpha+1}}$	$\tilde{\Omega}\left(m\left(\frac{n}{k}\right)^{\frac{1}{2\alpha}}\right)$
	α	$\alpha \leq 1.01$ $k = O(n/\log m)$	$\tilde{\Omega}\left(\frac{mn}{k}\right)$
Cover Verification	$-$	$k \leq n/2$	$\tilde{\Omega}(nk)$

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Theorem: There exists an algorithm that with high probability finds an $O(\rho\alpha)$-approximate cover which uses $\tilde{O}(\boldsymbol{mn^{1/\alpha}} + \boldsymbol{nk})$ number of queries.

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1. Two simple components used for coverage problems in massive data models.
   - Set Sampling
   - Element Sampling
2. The algorithm overview

Component I: set sampling

Set Sampling: After picking ℓ sets uniformly at random, all elements with degree at least $\frac{\mathrm{m} \log n}{\ell}$ are covered w.h.p.

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How we use the lemma: set $\ell = O(k)$

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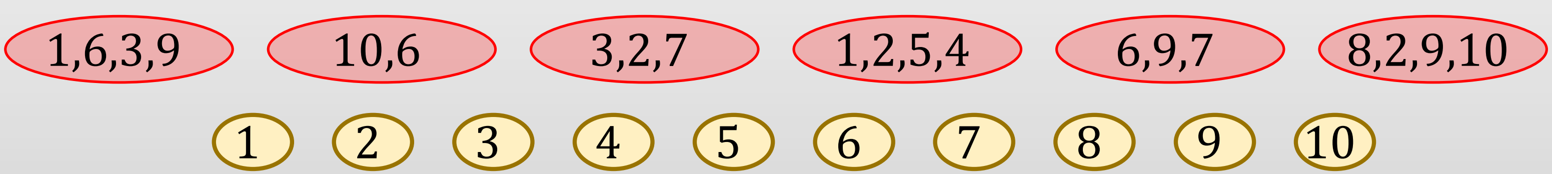

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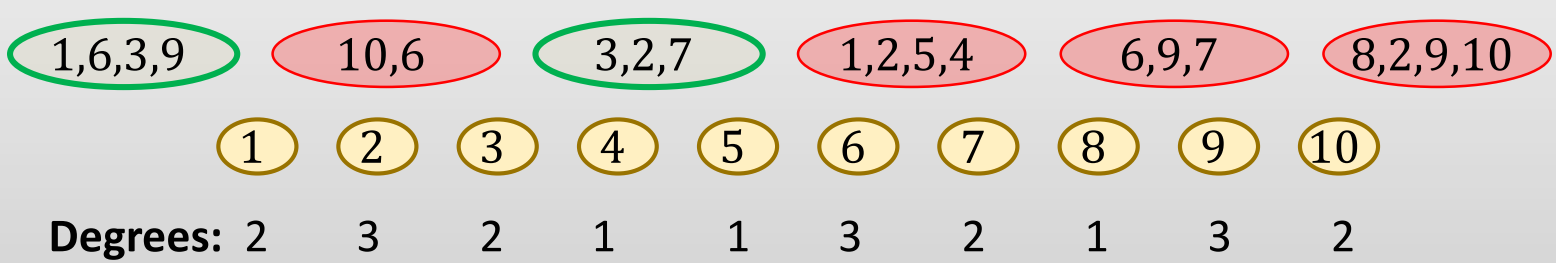

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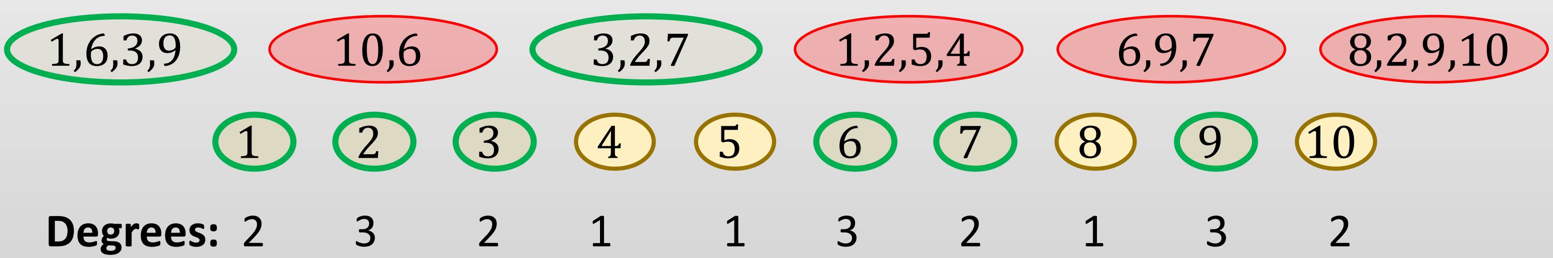

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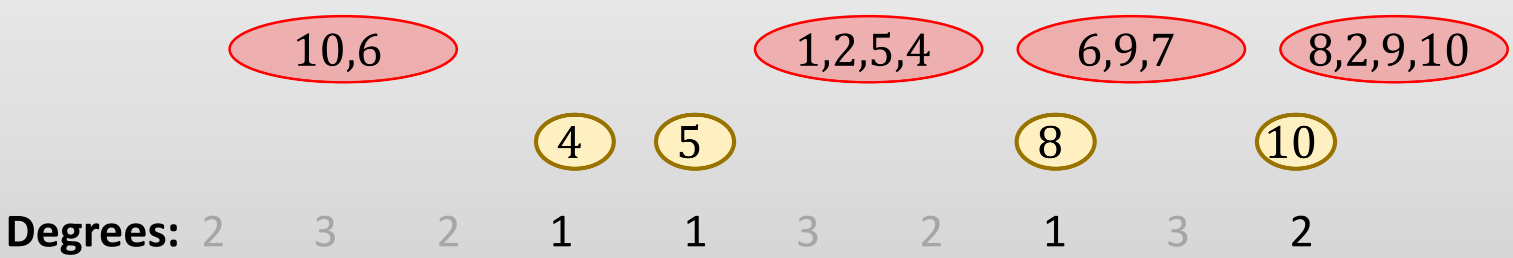

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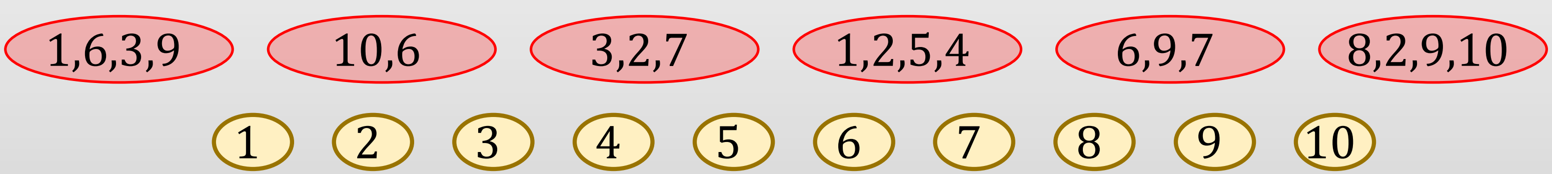

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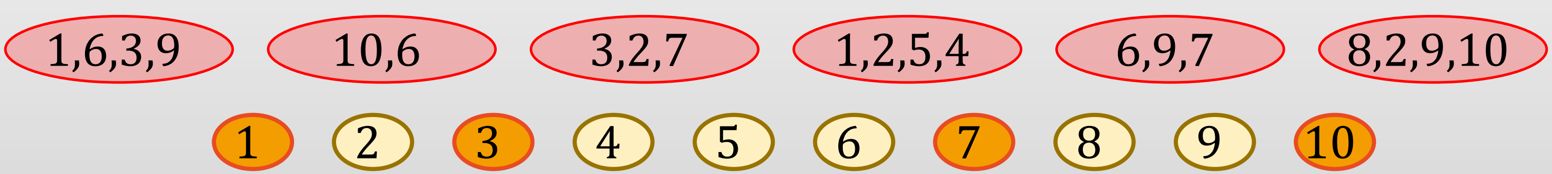

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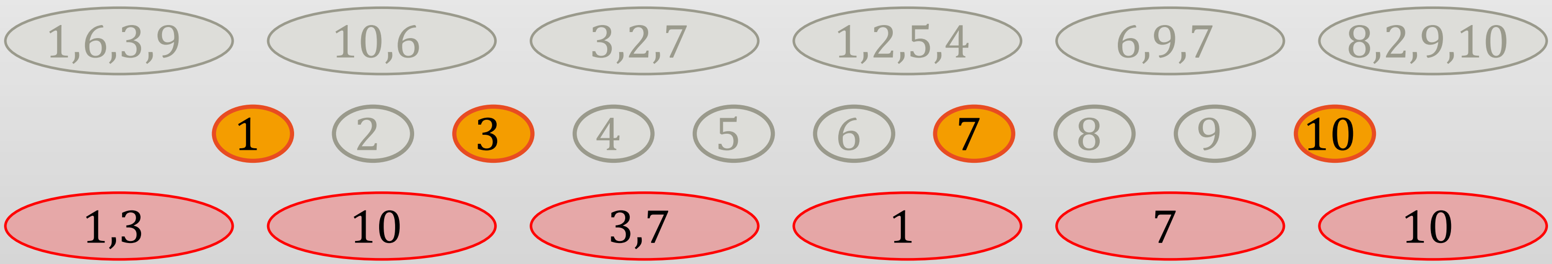

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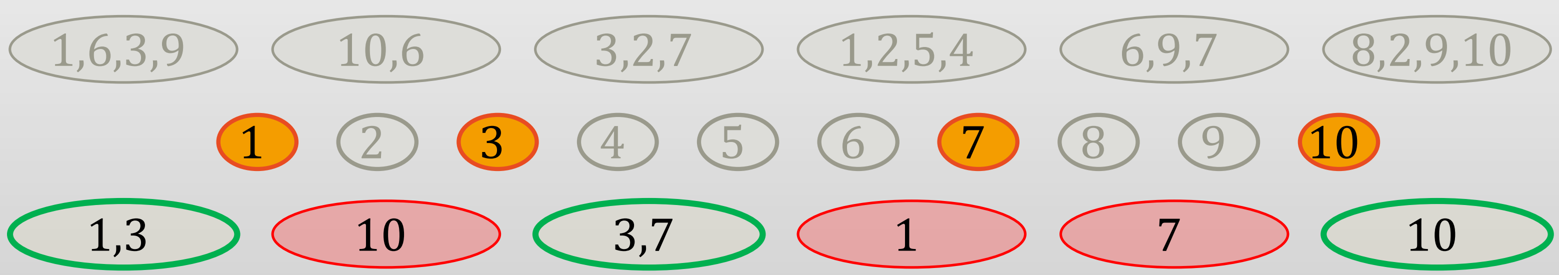

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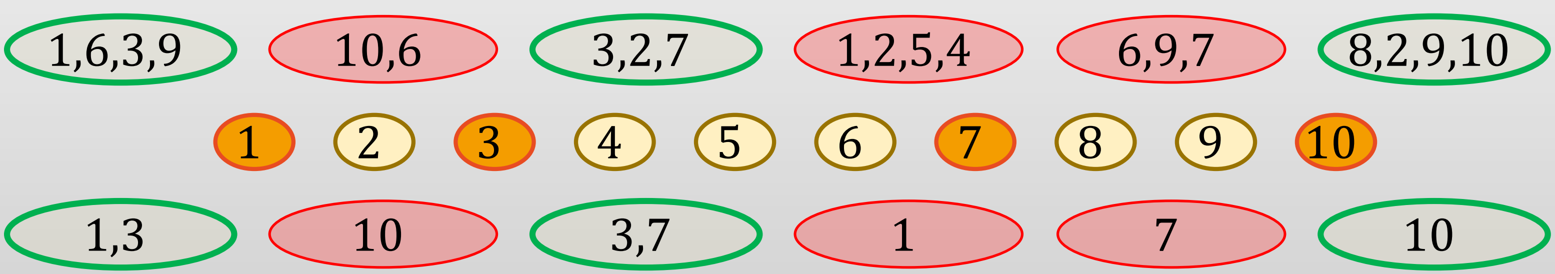

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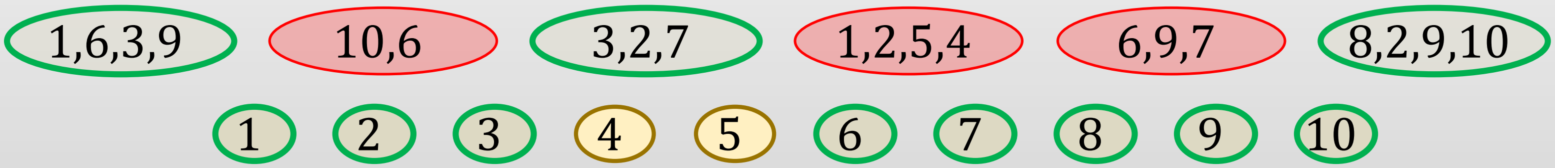

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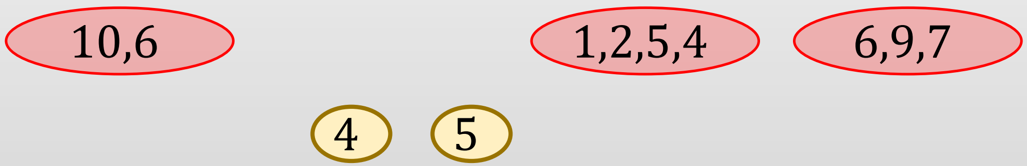

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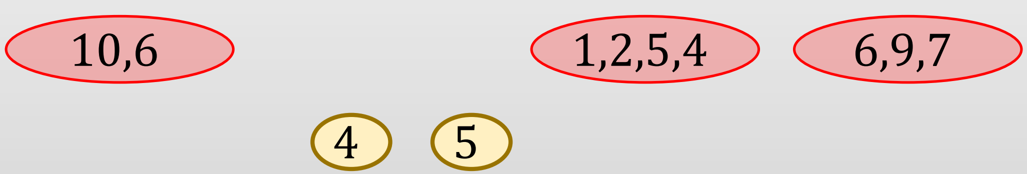

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Cover Verification	$-$	$k \leq n/2$	$\tilde{\Omega}(nk)$

Cover Verification: given a set system, verify whether a given sub-collection of sets covers the universe.

$\boldsymbol{\rho}$ = approximation factor for offline **Set Cover**

n = number of *elements* **m = number of *sets*** **k = Size of the optimal Solution**

Part two: lower bound

Theorem: Any randomized algorithm that with probability at least 2/3 distinguishes whether the minimum Set Cover size is 2 or at least 3 requires $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{mn})$ number of queries.

High Level Approach

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3. Any algorithm that can detect these two cases requires to query at least $\widetilde{\Omega}(mn)$ queries.

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Construction: is randomized. For every S, e the set S contains e with probability $1 - p_0$ where $p_0 = \sqrt{\frac{9 \log m}{n}}$

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Take one such instance I^* with the above properties

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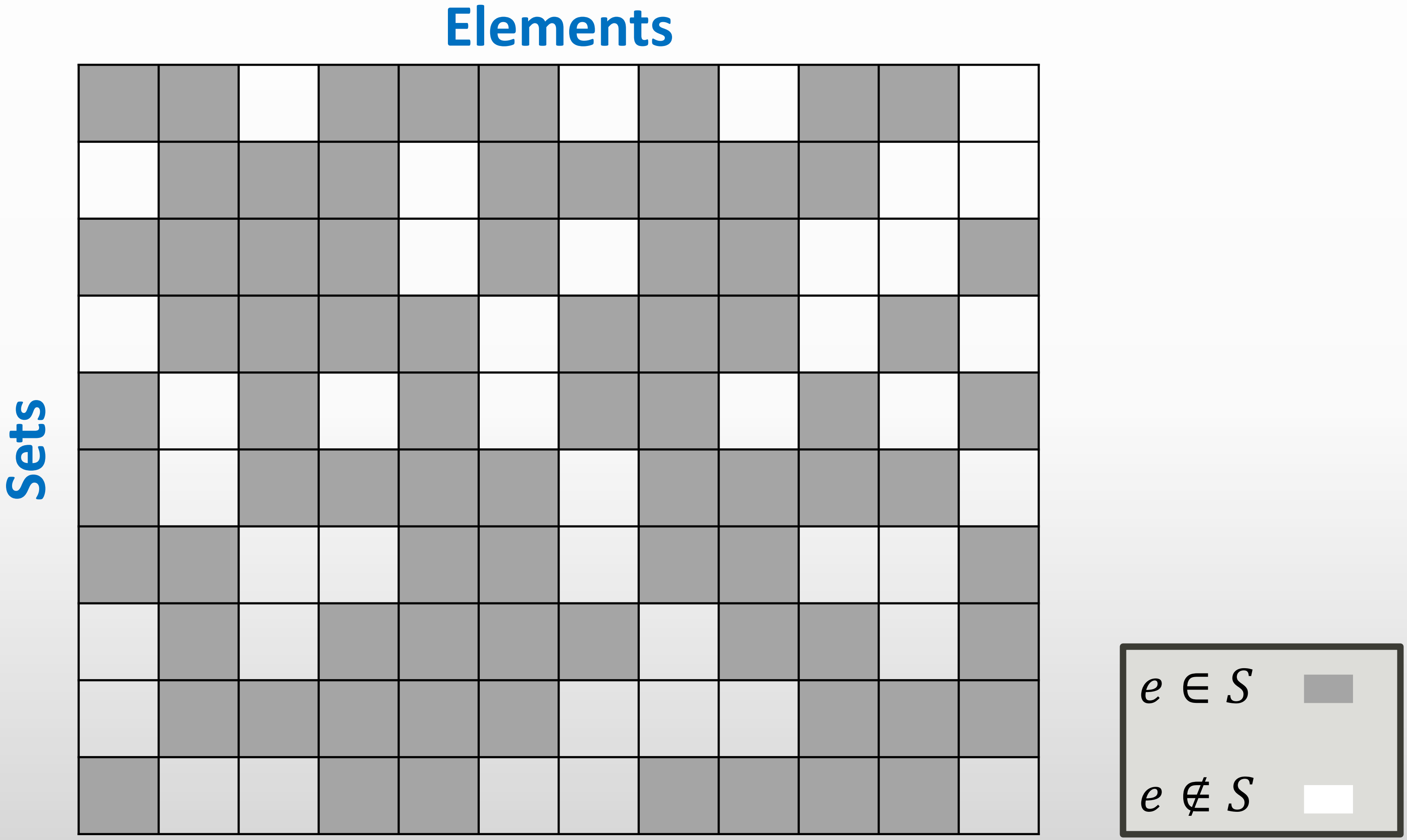

Generating a Modified Instance

Pick two random sets S_1 and S_2 and turn them into a set cover.
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+
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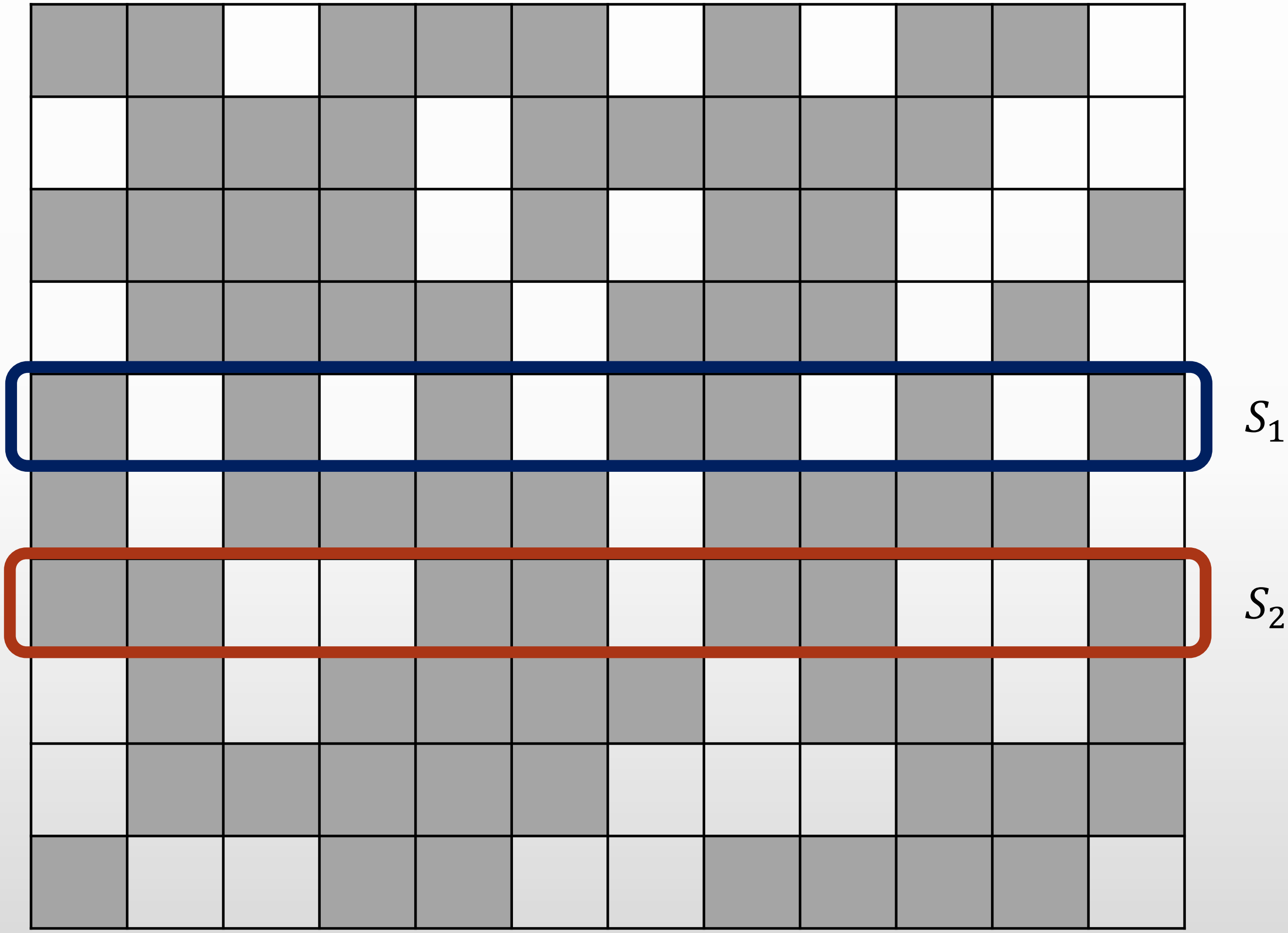

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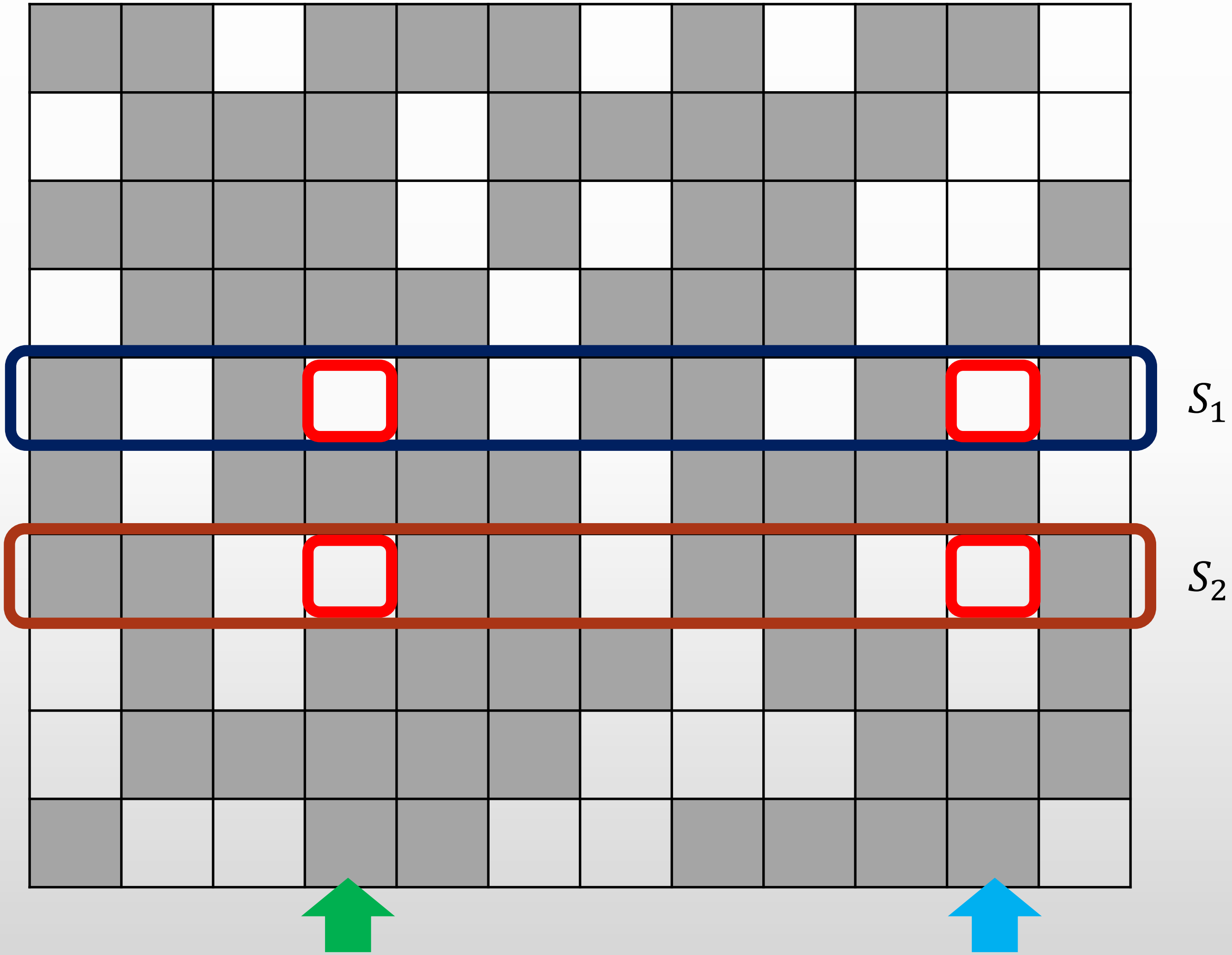

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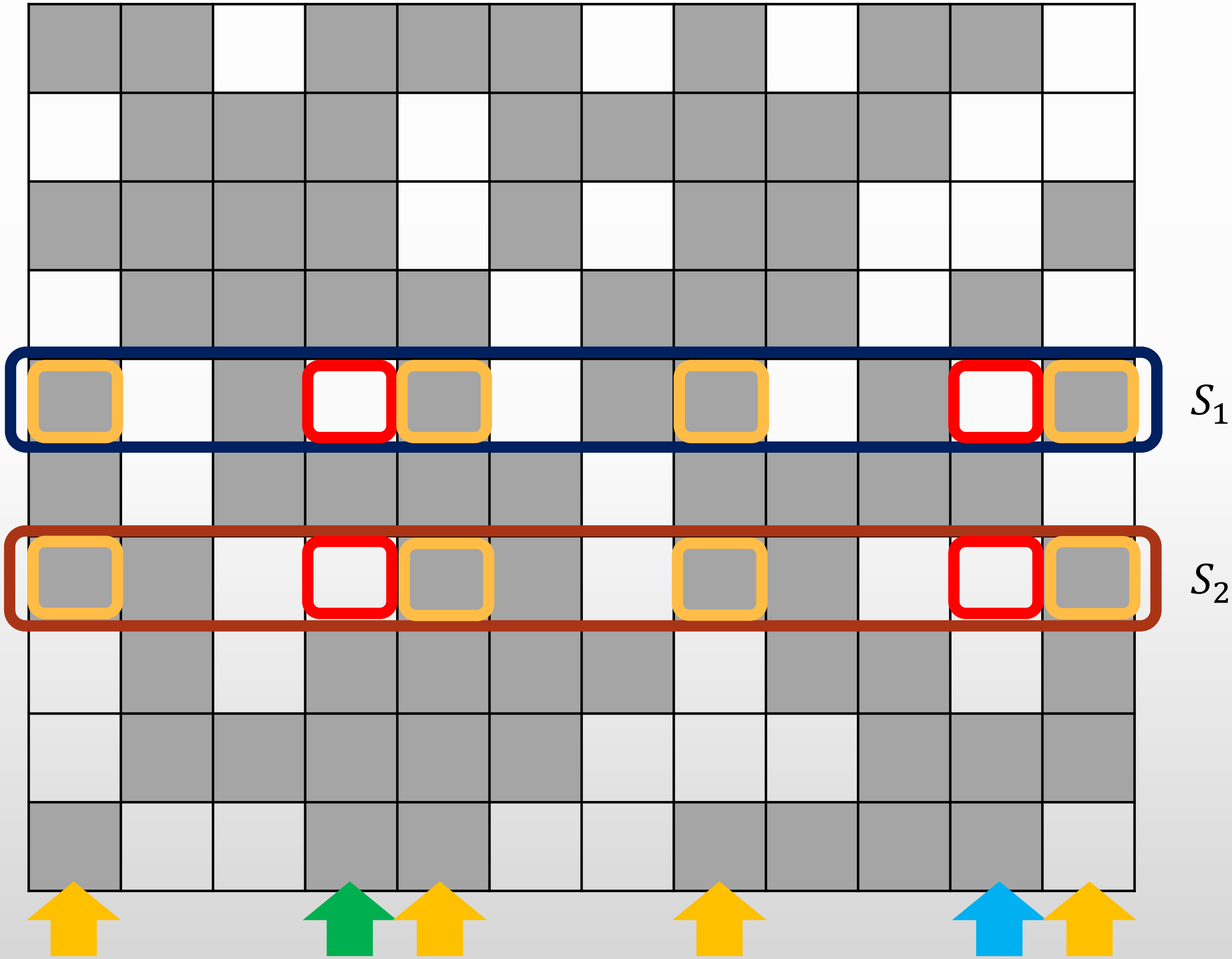

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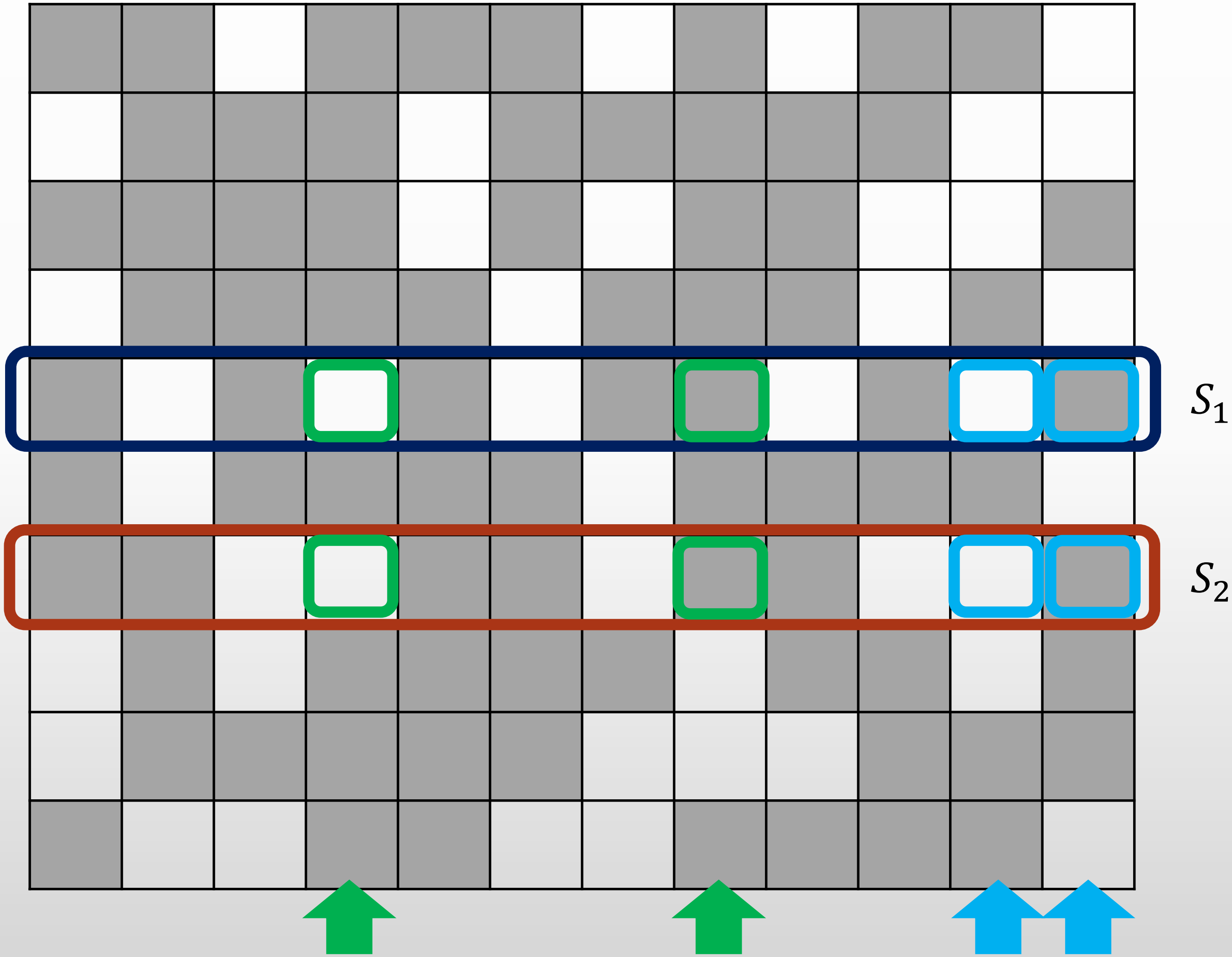

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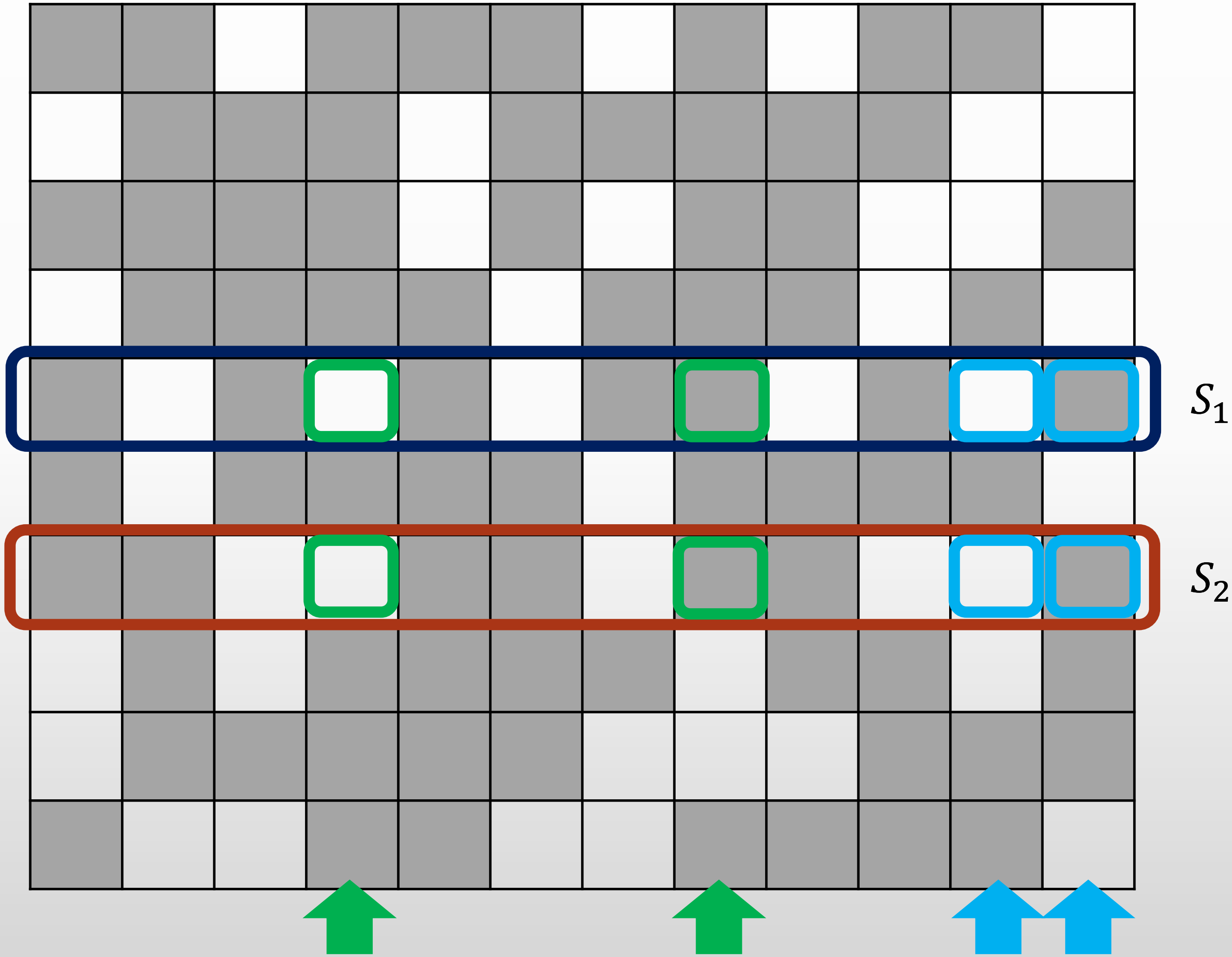

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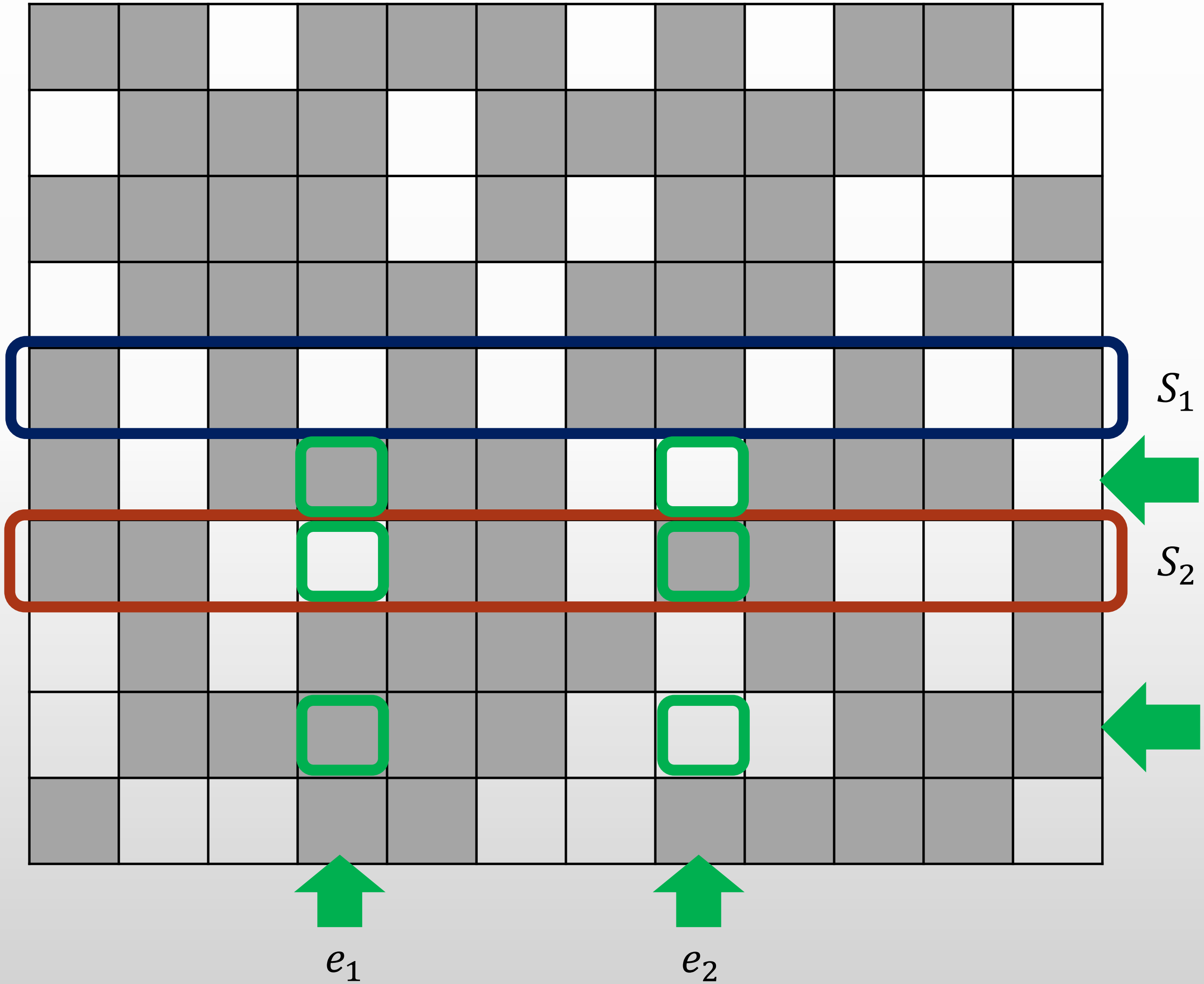

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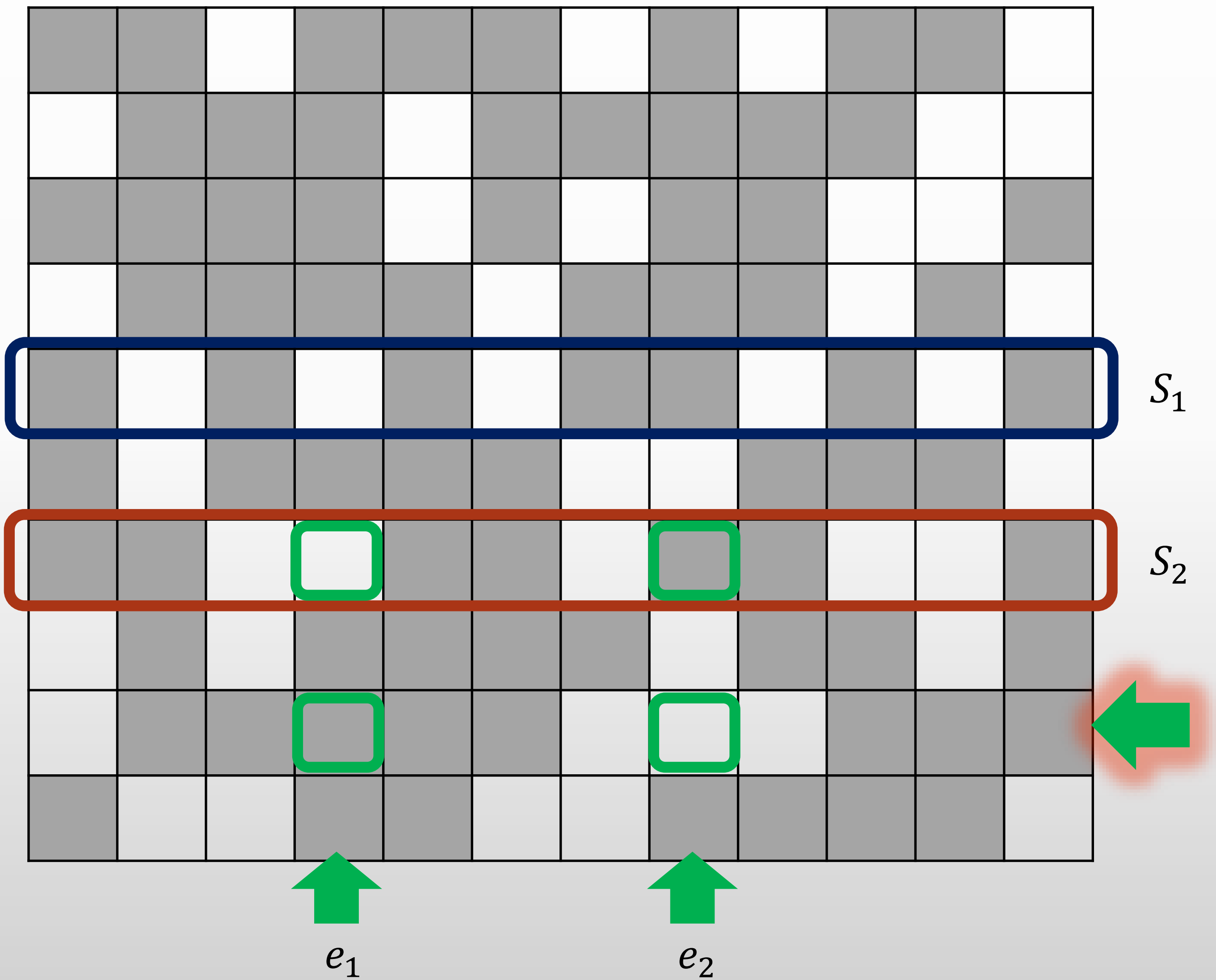

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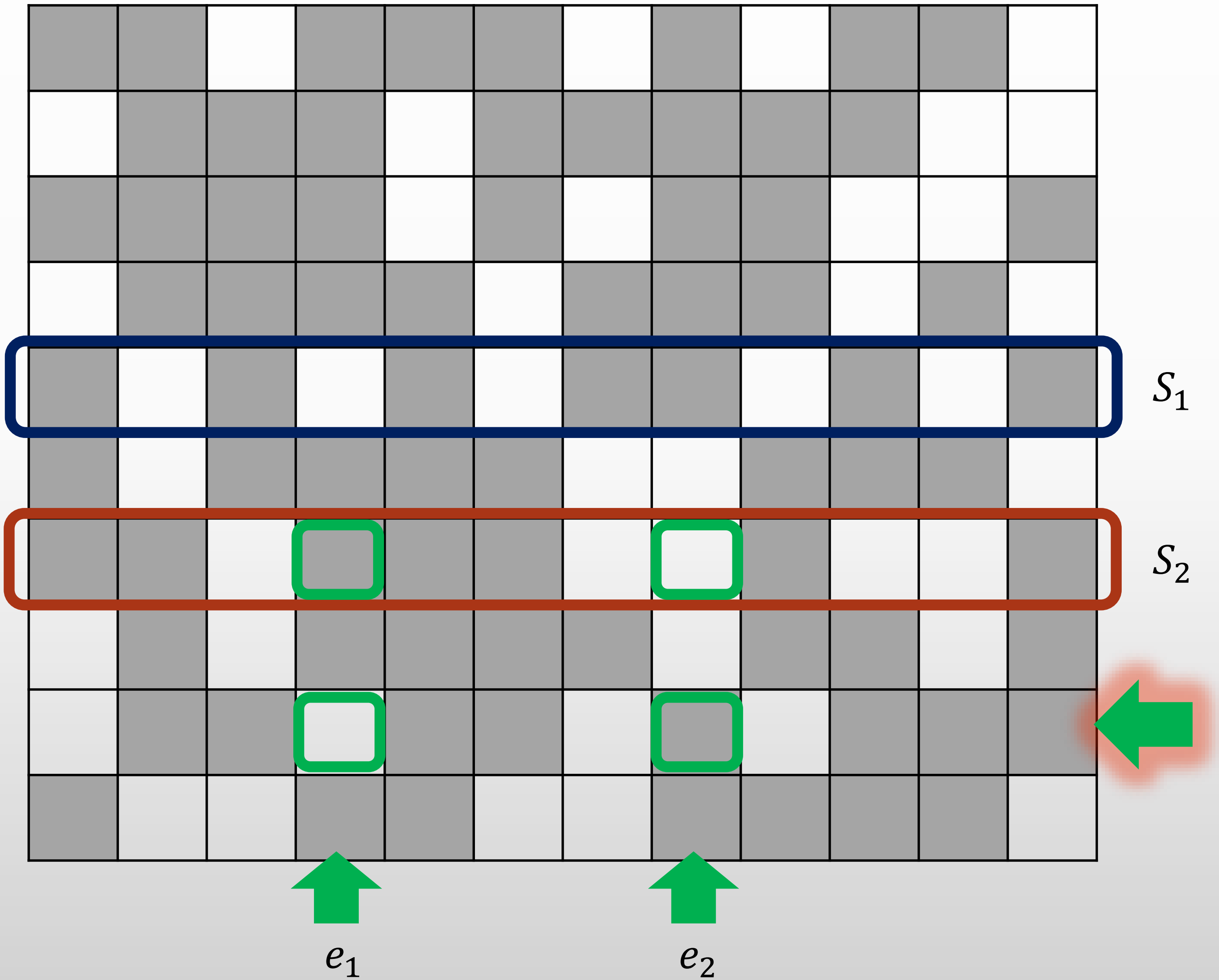

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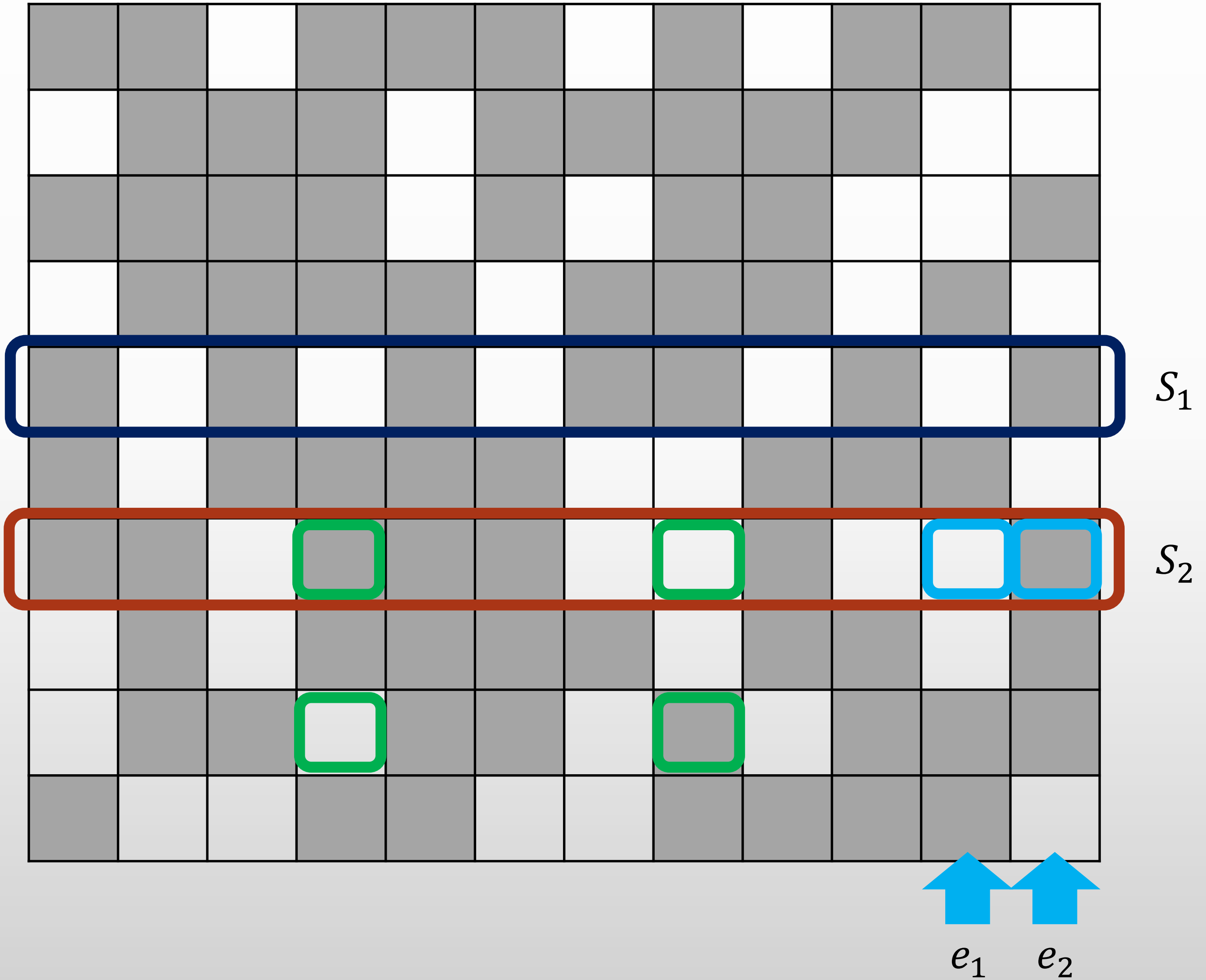

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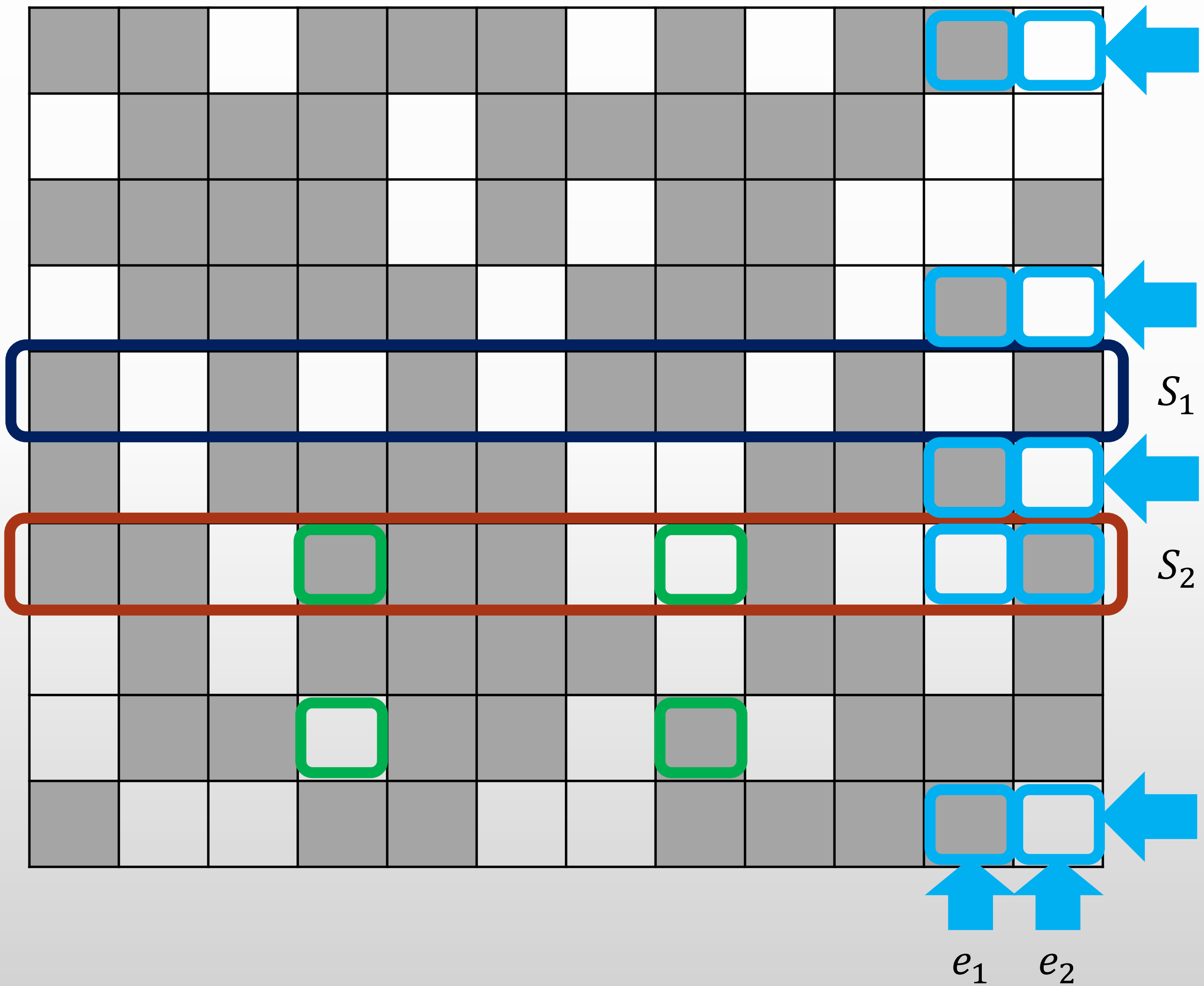

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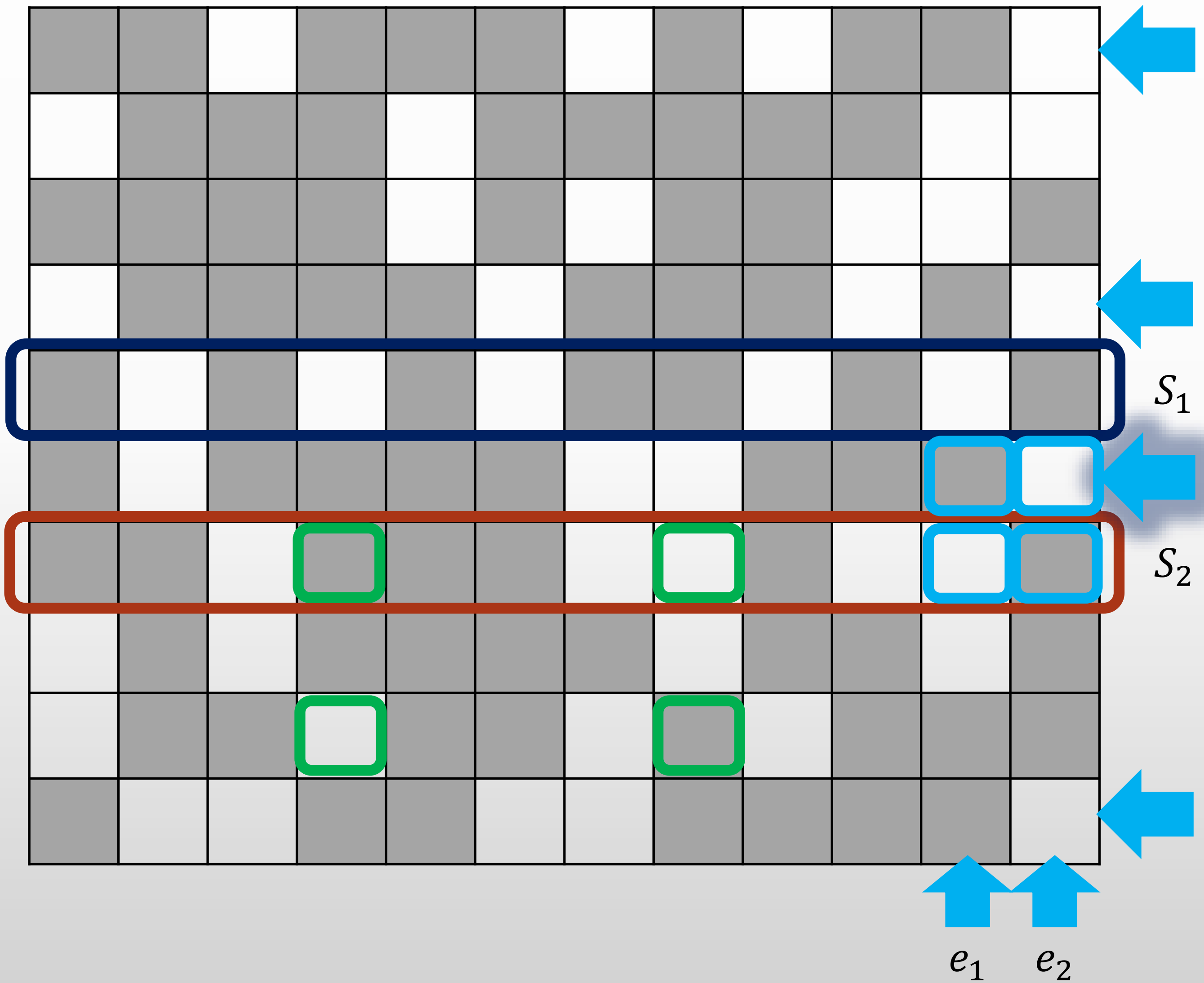

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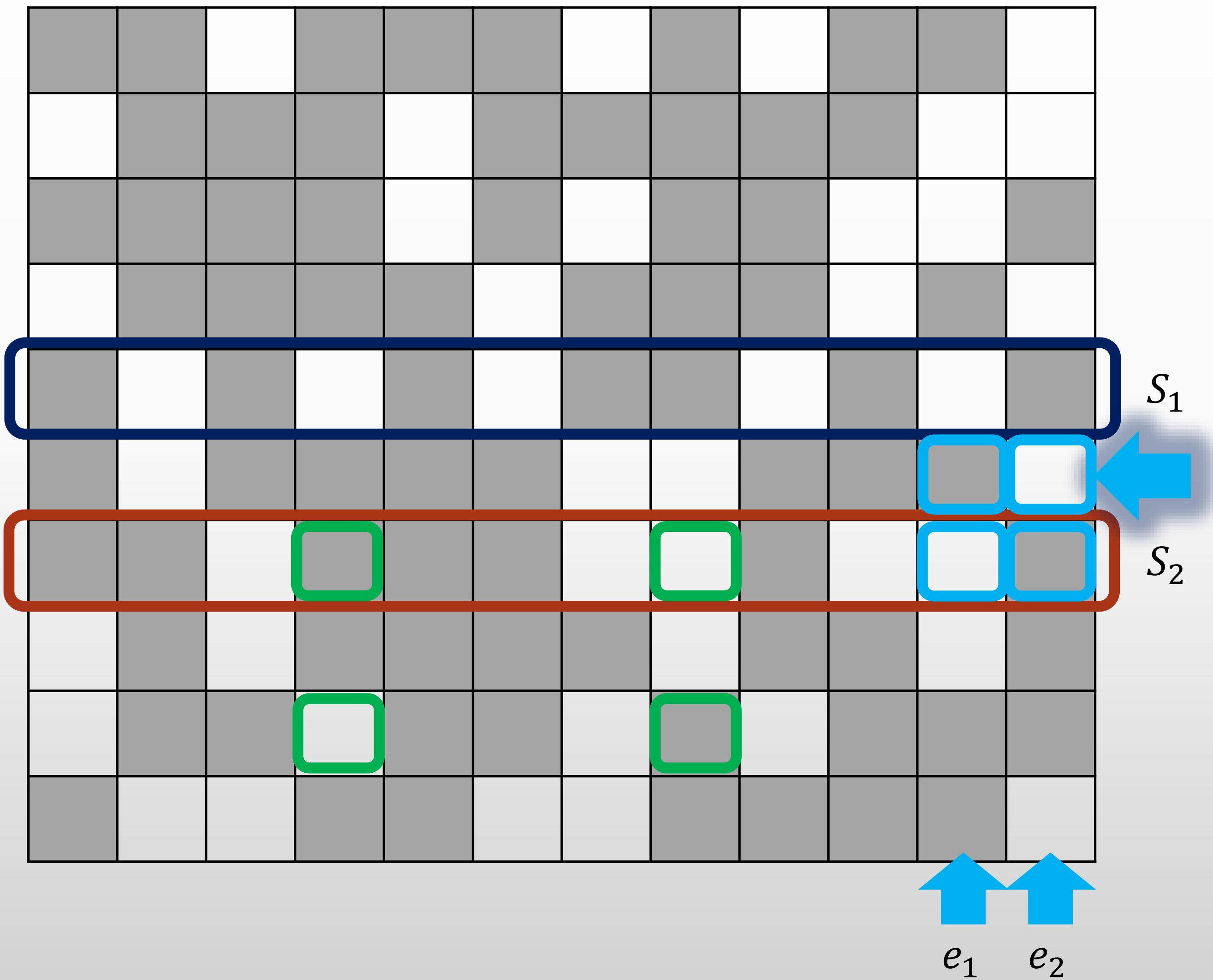

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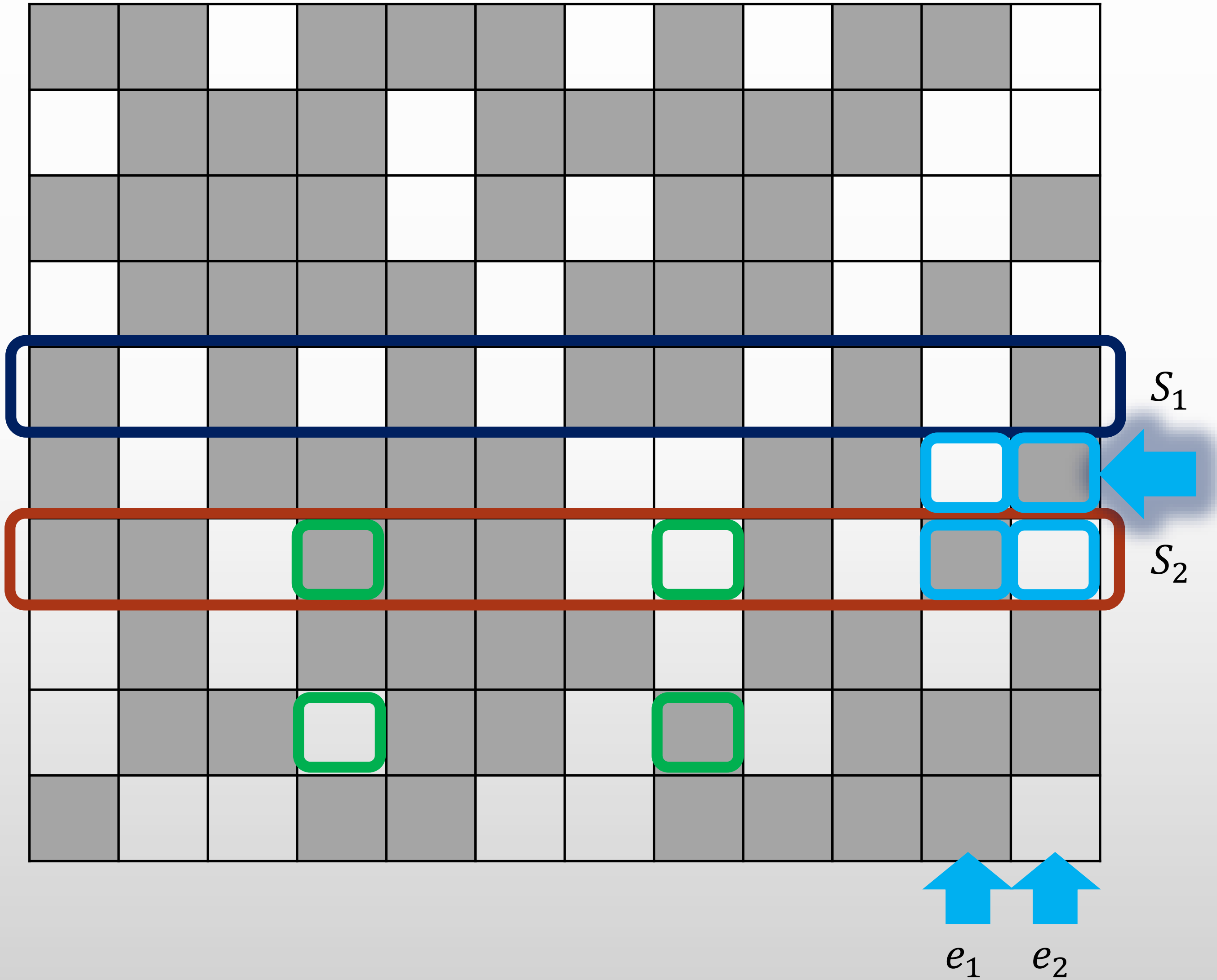

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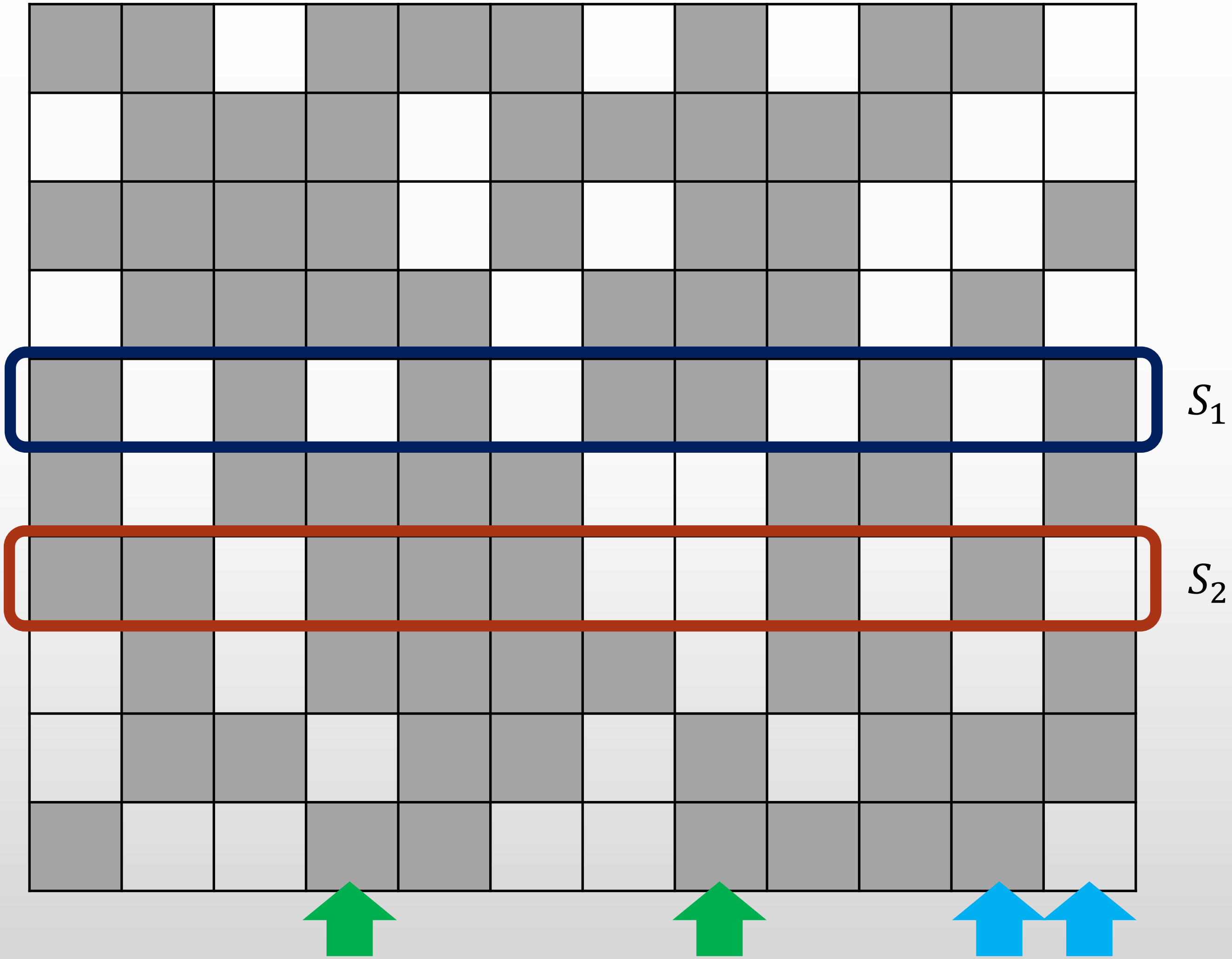

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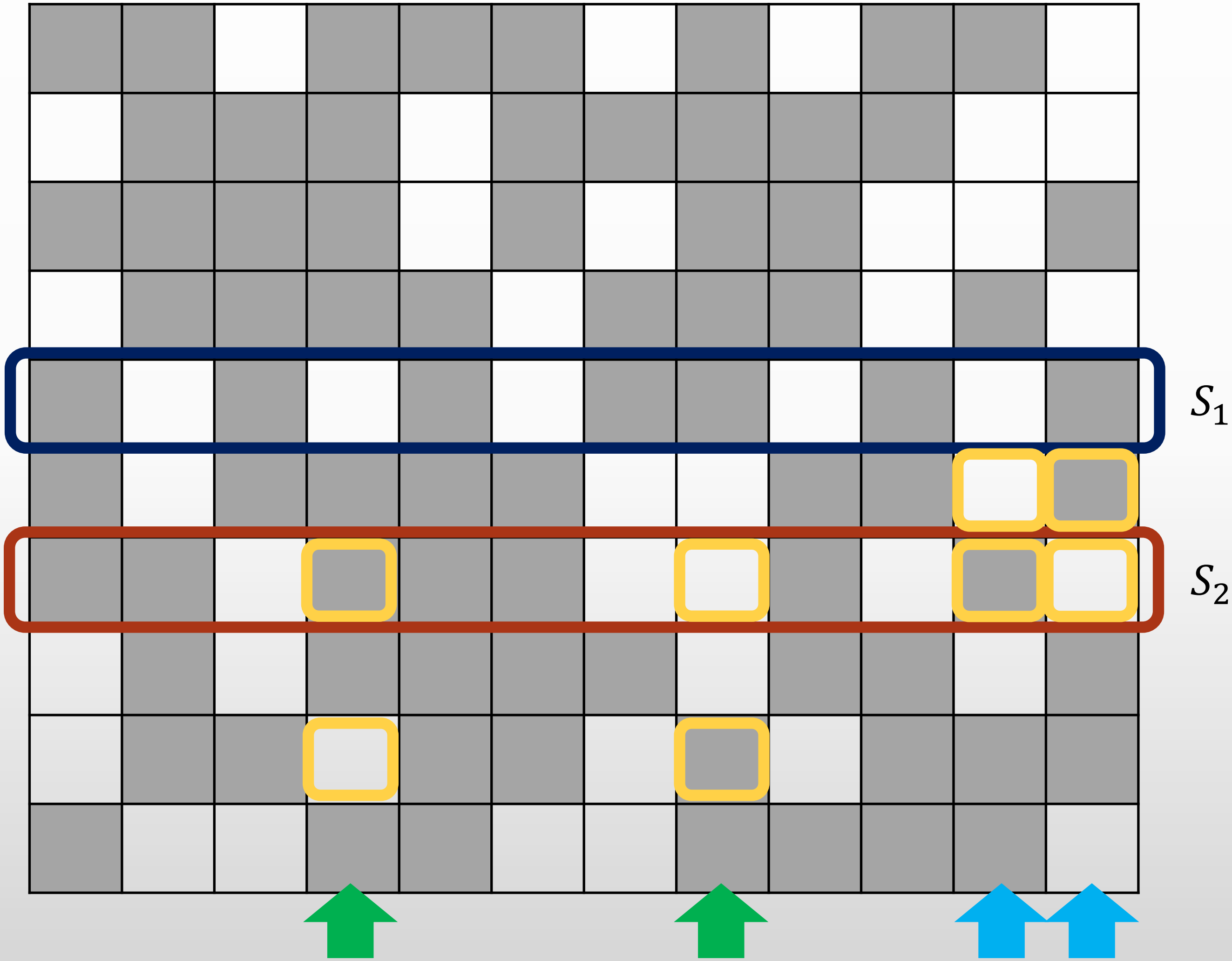

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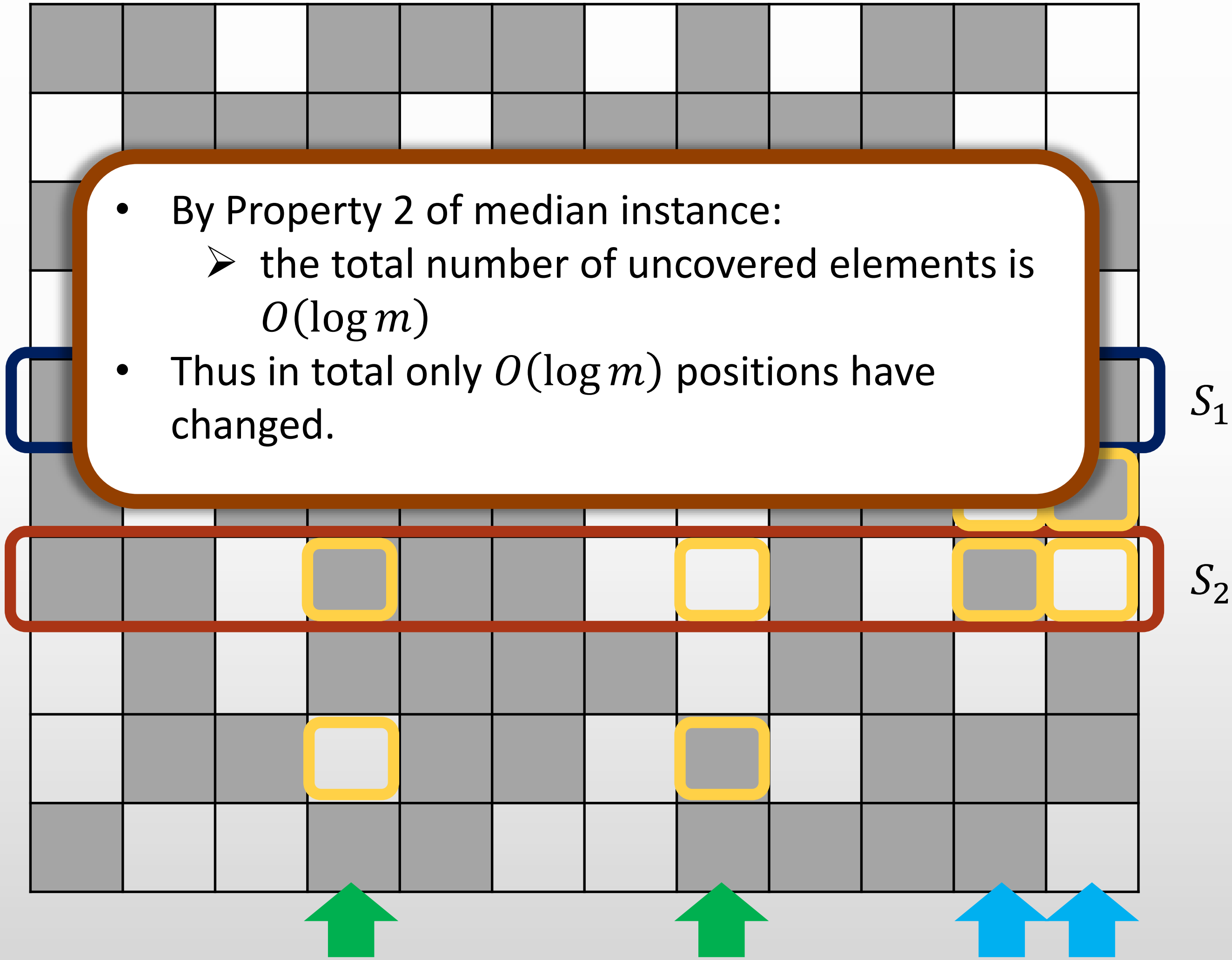

- By Property 2 of median instance:
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- Thus in total only $O(\log m)$ positions have changed.

Overall Argument

Lemma: For any element e and any set S, the probability that pair participate in a swap is almost uniform, i.e., $O(\frac{\log m}{mn})$.

- Using other properties of the median instances

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