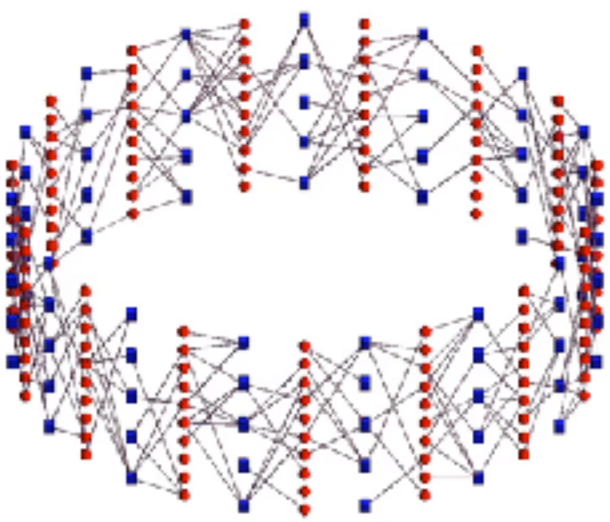
Spatial coupling: Algorithm and Proof Technique



Workshop on Local Algorithms - WOLA 2018

Boston, June 15th, 2018

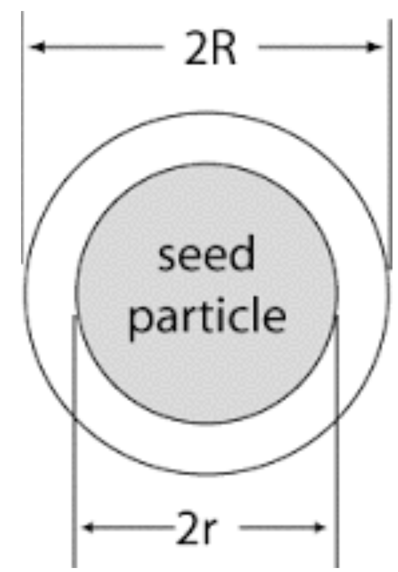
Physics inspiration:
nucleation, crystallization, meta-stability

Supercooled water

Heat packs

Sodium acetate, $C_2H_3NaO_2$,

Nucleation



Spatial-Coupling as an Algorithm

Introduction - Graphical Codes

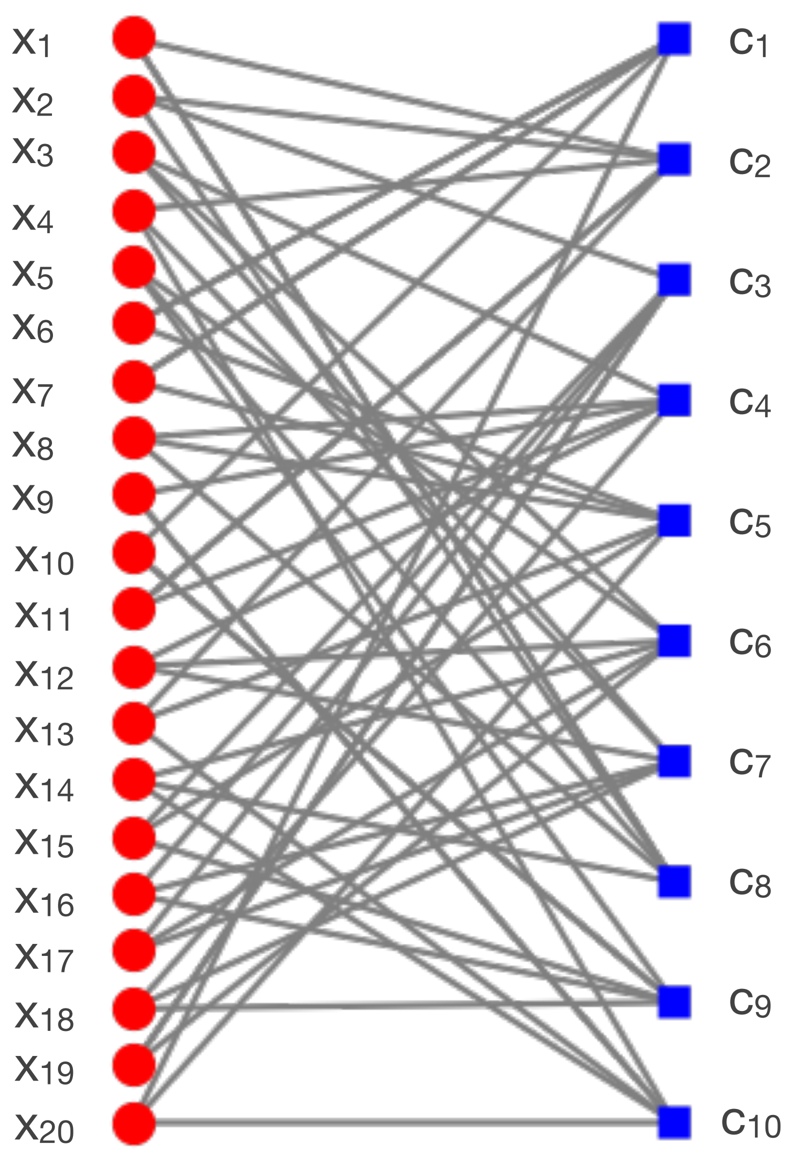
Low-Density Parity-Check Codes

Robert G. Gallager

1963

Low-density Parity-Check (LDPC) Codes

$d_l (= 3)$ $\quad$ $d_r (= 6)$



$c_1 \longleftrightarrow x_6 + x_7 + x_{10} + x_{20} = 0$

$$\text{rate} \geq r_{\text{design}} = \frac{\#\text{variables} - \#\text{checks}}{\#\text{variables}} = \frac{20 - 10}{20} = \frac{1}{2}$$

$$\text{rate} \sim r_{\text{design}}$$

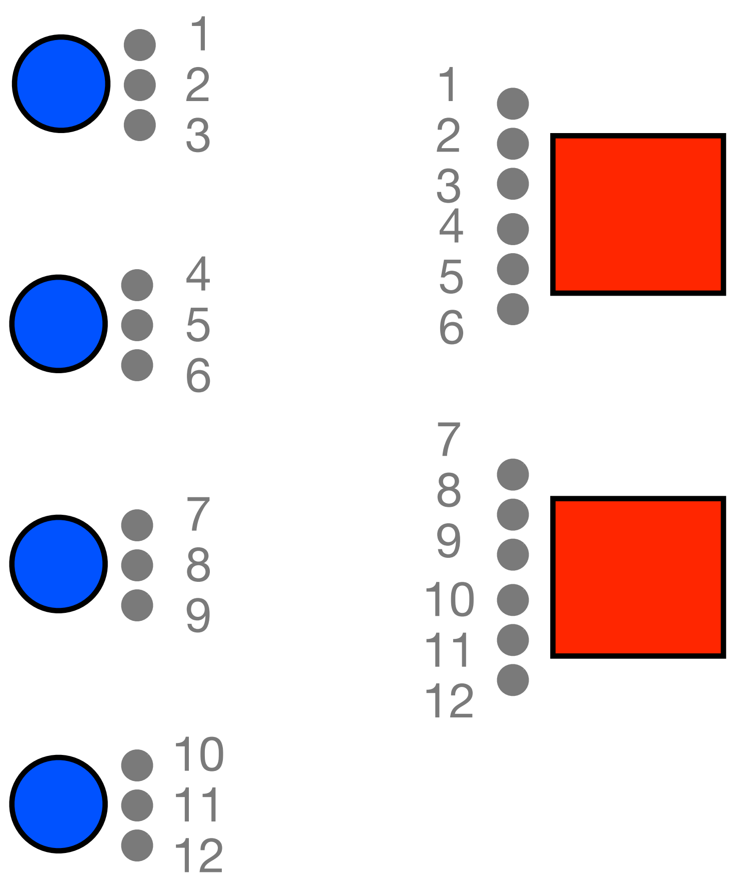
variable nodes

check nodes

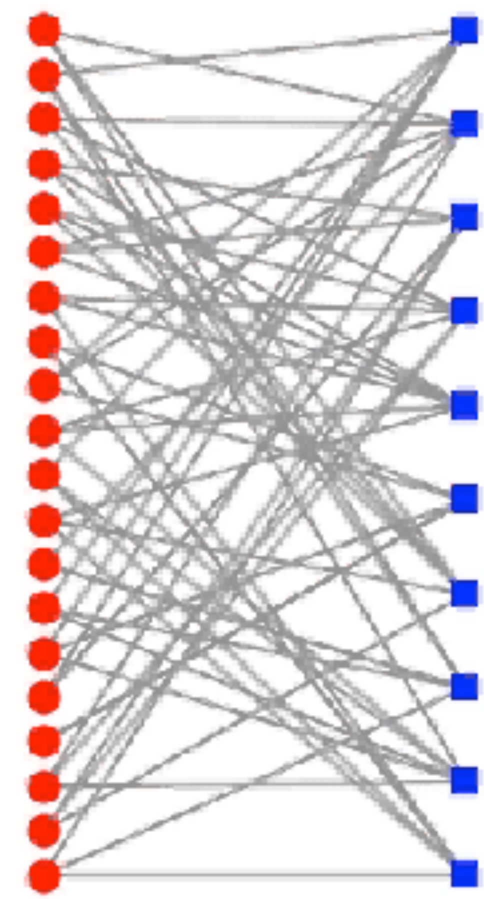
$c_{10} \longleftrightarrow x_4 + x_9 + x_{13} + x_{14} + x_{20} = 0$

Ensemble of Codes - Configuration Construction

(3, 6) ensemble

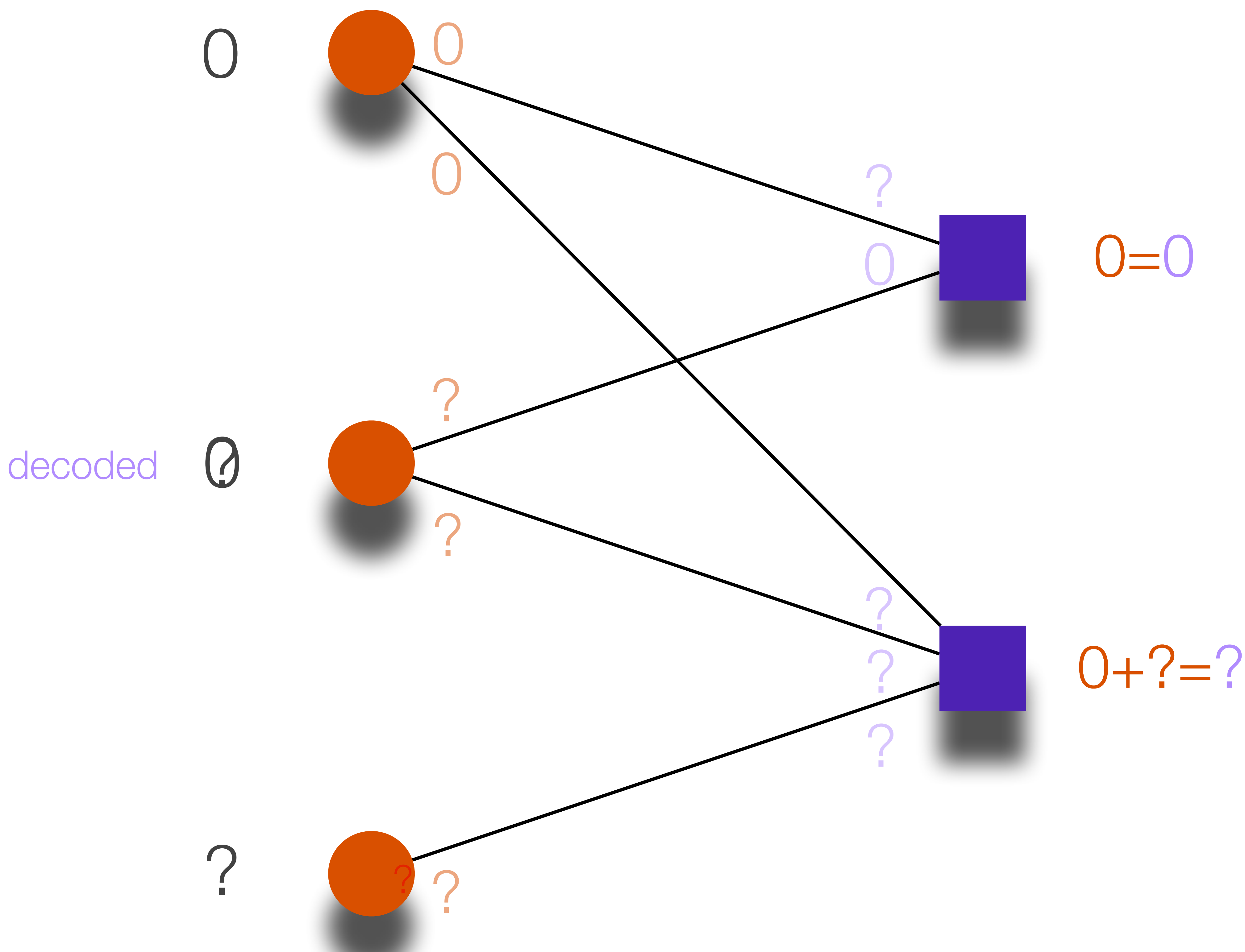


each configuration
has uniform
probability

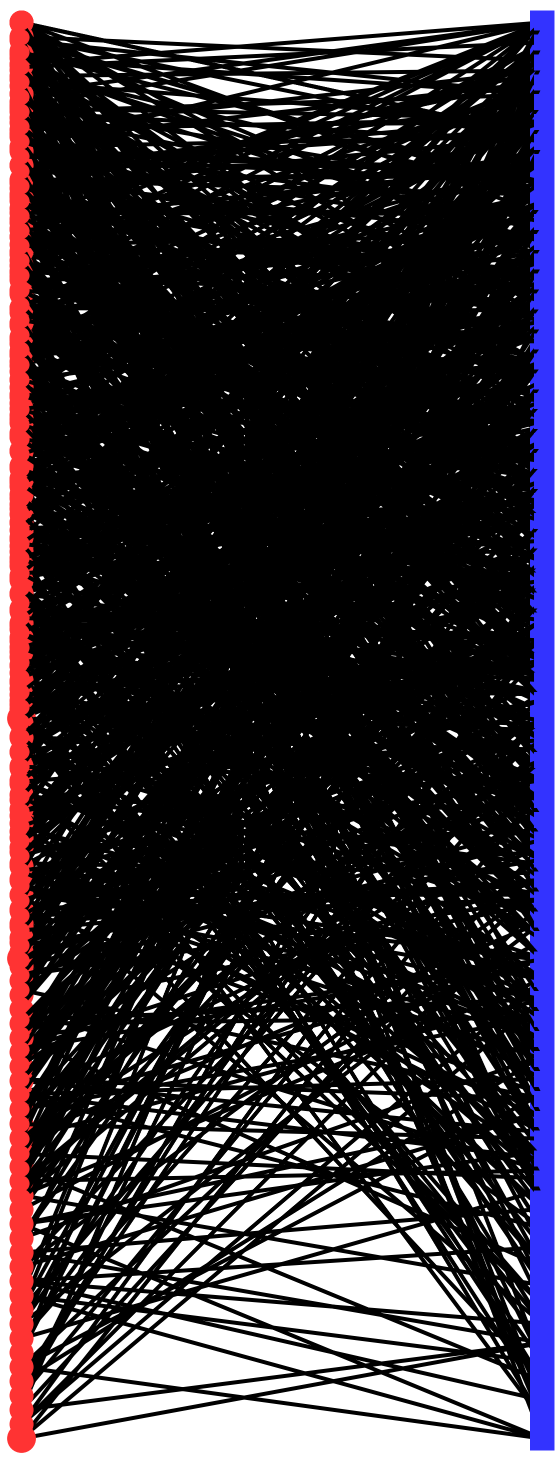


code is sampled u.a.r.
from the ensemble
and used for transmission

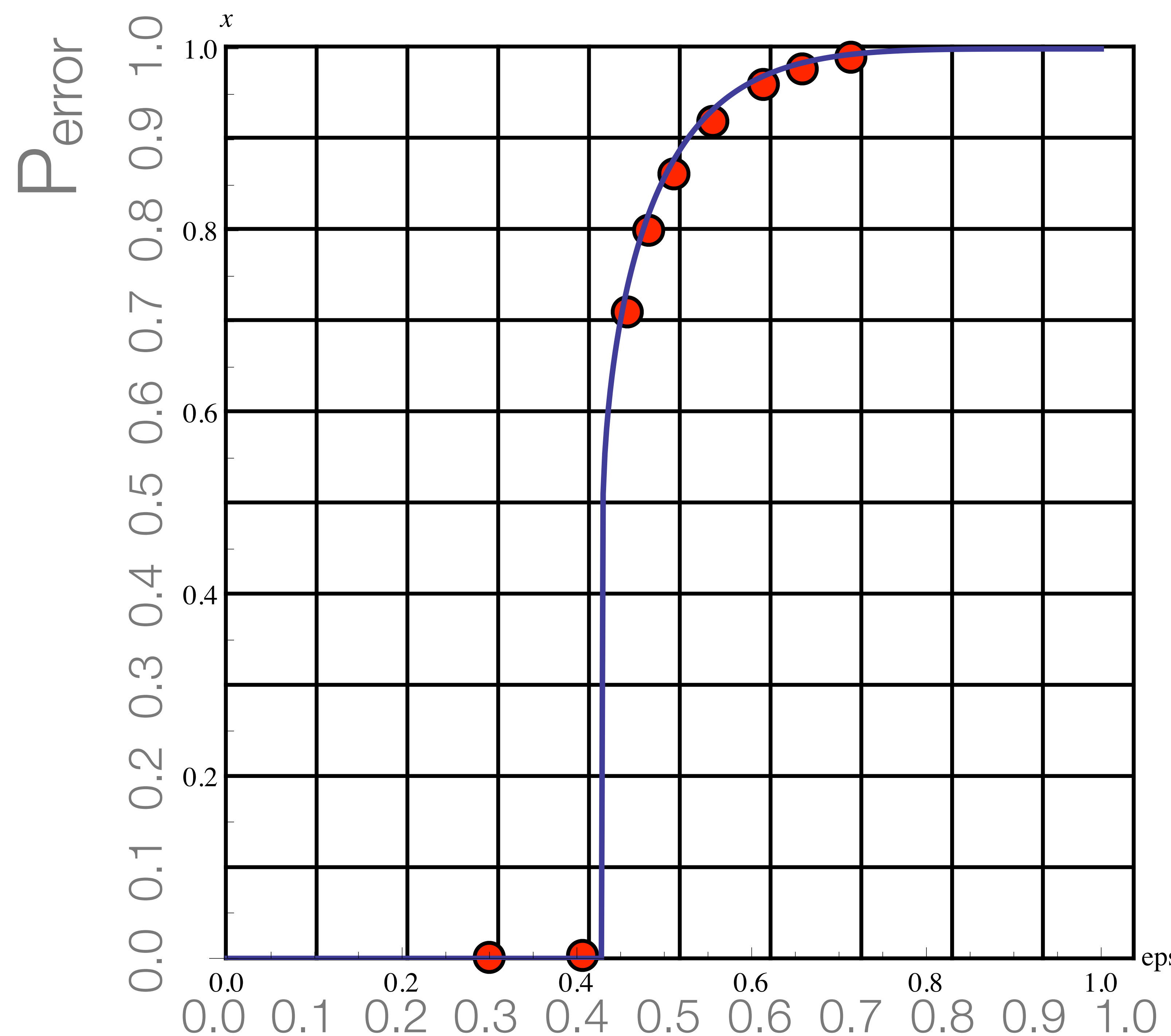
BP Decoder - BEC



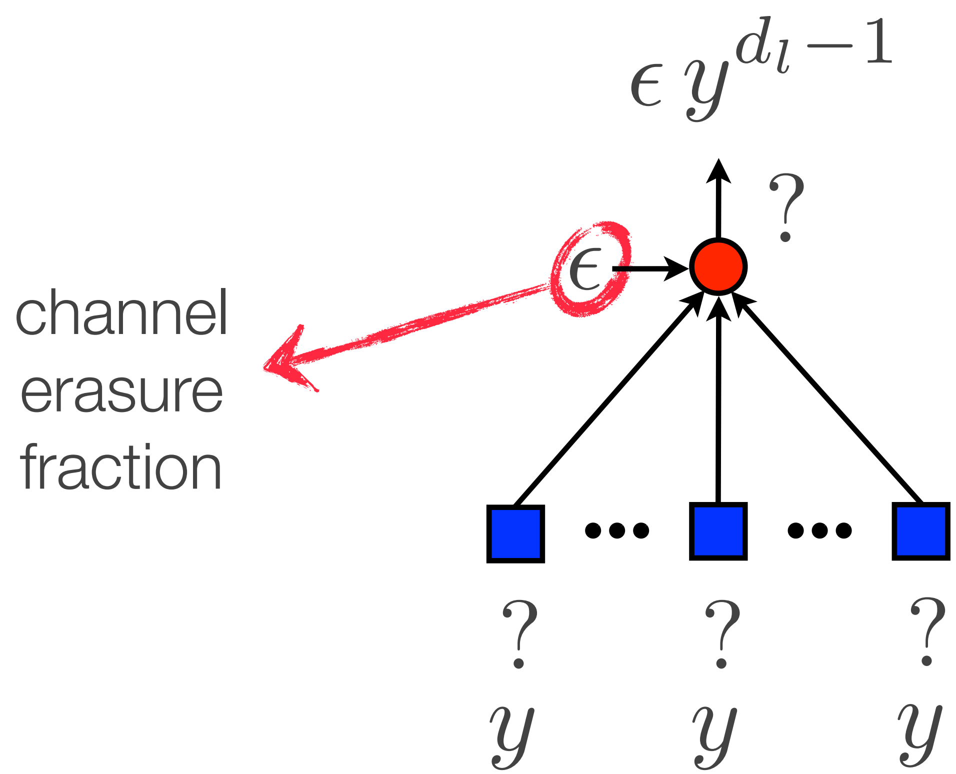
How does BP perform on the BEC?



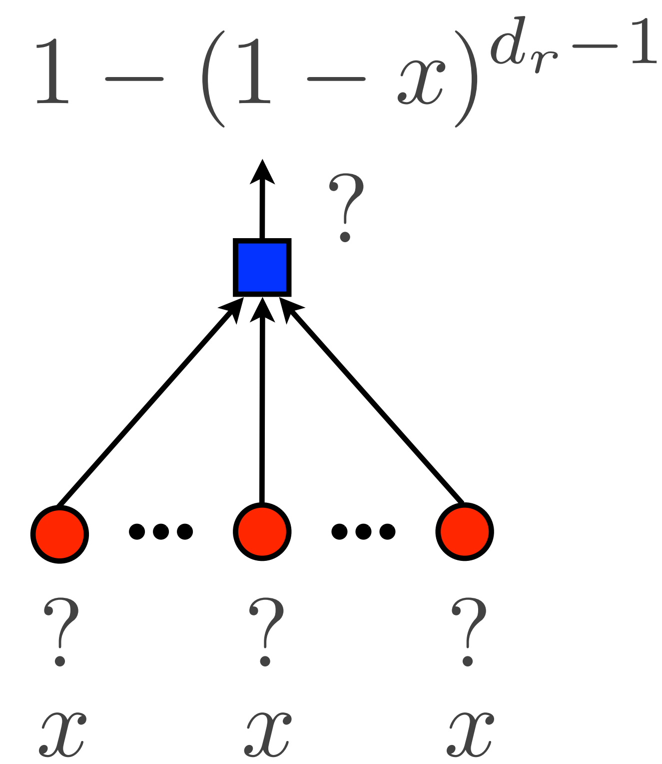
(3, 6) ensemble



Asymptotic Analysis - Density Evolution (DE)



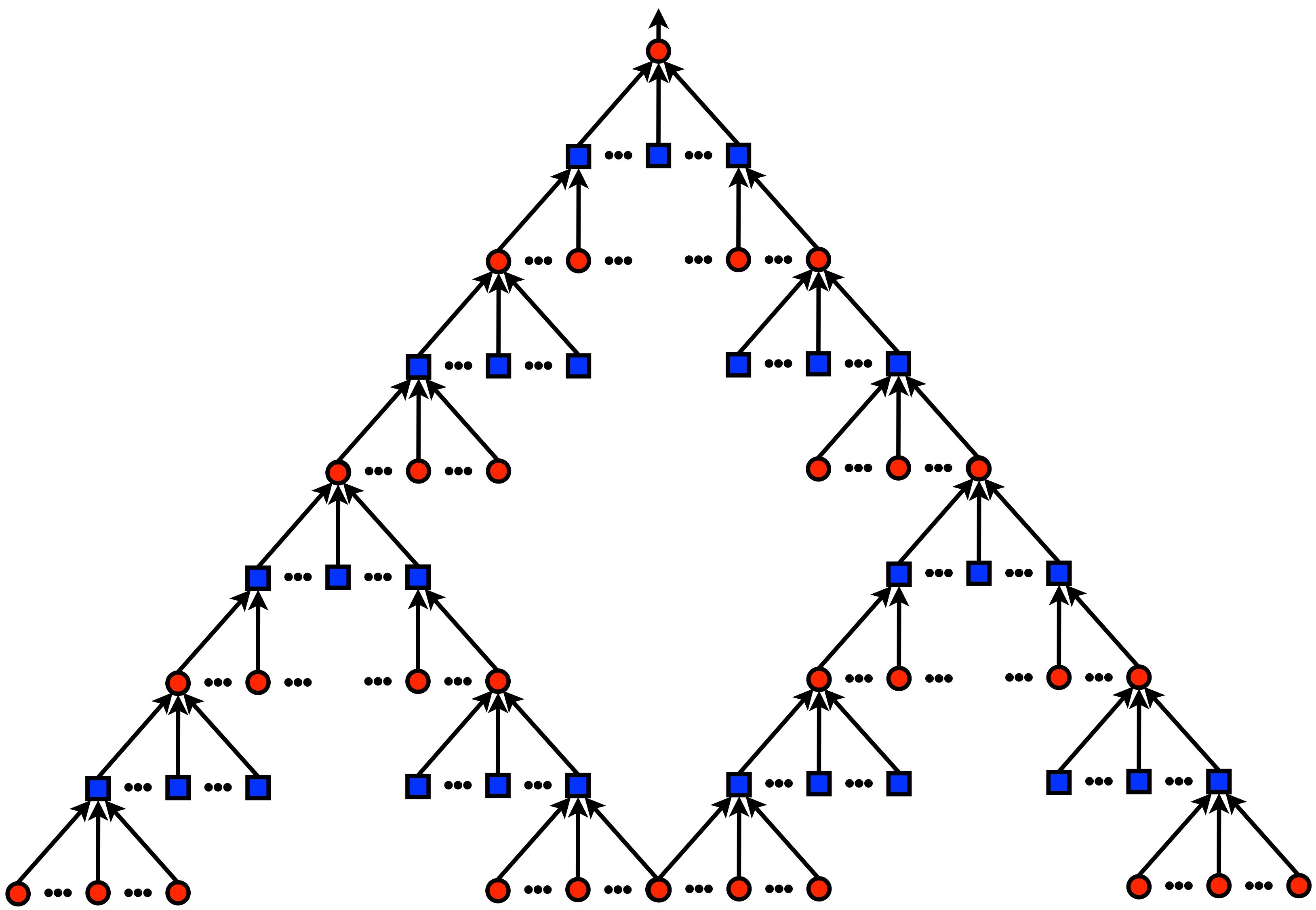
one iteration
of BP at variable
node



one iteration
of BP at check
node

Asymptotic Analysis - Density Evolution (DE)

erasure fraction
at the root after
ℓ iterations



IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 47, NO. 2, FEBRUARY 2001

Efficient Erasure Correcting Codes

Michael G. Luby, Michael Mitzenmacher, M. Amin Shokrollahi, and Daniel A. Spielman

Abstract—We introduce a simple erasure recovery algorithm for codes derived from cascades of sparse bipartite graphs and analyze the algorithm by analyzing a corresponding discrete-time random process. As a result, we obtain a simple criterion involving the fractions of nodes of different degrees on both sides of the graph which is necessary and sufficient for the decoding process to finish successfully with high probability. By carefully designing these graphs we can construct for any given rate R and any given real number ϵ a family of linear codes of rate R which can be encoded in time proportional to $\ln(1/\epsilon)$ times their block length n. Furthermore a codeword can be recovered with high probability from a portion of its entries of length $(1+\epsilon)Rn$ or more. The recovery algorithm also runs in time proportional to $n\ln(1/\epsilon)$. Our algorithms have been implemented and work well in practice; various implementation issues are discussed.

Index Terms—Erasure channel, large deviation analysis, low-density parity-check codes.

Low-Density Parity-Check Codes

Robert G. Gallager

1963

$$x^{(\ell)} = \epsilon(y^{(\ell)})^{d_l-1}$$

$$y^{(\ell)} = 1-(1-x^{(\ell-1)})^{d_r-1}$$

$$x^{(\ell=2)} = \epsilon(y^{(\ell=2)})^{d_l-1}$$

$$y^{(\ell=2)} = 1-(1-x^{(\ell=1)})^{d_r-1}$$

$$x^{(\ell=1)} = \epsilon(y^{(\ell=1)})^{d_l-1}$$

$$y^{(\ell=1)} = 1-(1-x^{(\ell=0)})^{d_r-1}$$

$$x^{(\ell=0)} = \epsilon$$

Asymptotic Analysis - Density Evolution (DE)

$$f(\epsilon, x) = \epsilon(1 - (1 - x)^{d_r - 1})^{d_l - 1}$$

$f(\epsilon, x)$ is increasing in both its arguments

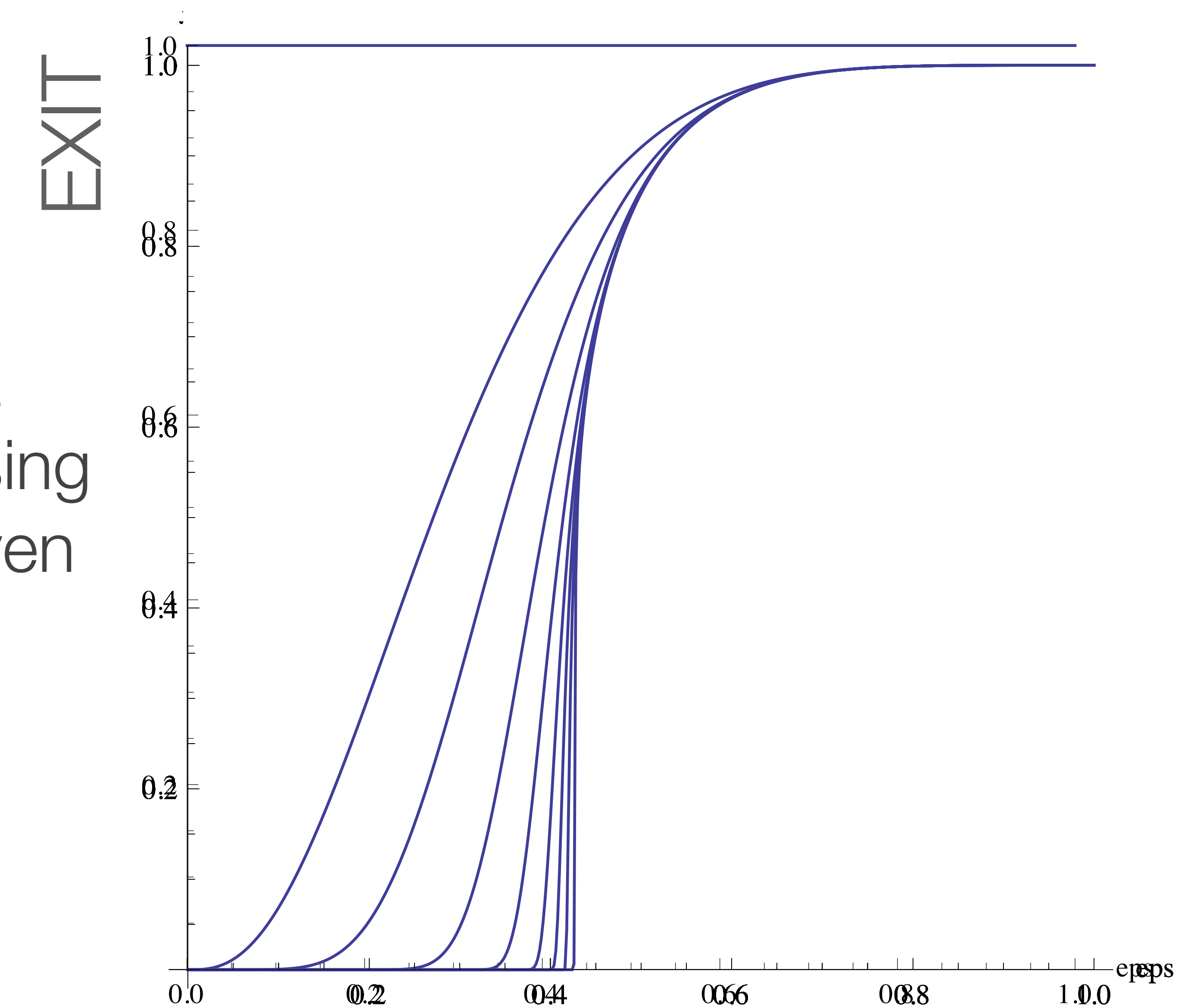
$$x^{(\ell+1)} = f(\epsilon, x^{(\ell)}) \overset{x^{(\ell)} \leq x^{(\ell-1)}}{\leq} f(\epsilon, x^{(\ell-1)}) = x^{(\ell)}$$

$$x^{(1)} = f(\epsilon, x^{(0)} = 1) = \epsilon \leq x^{(0)} = 1$$

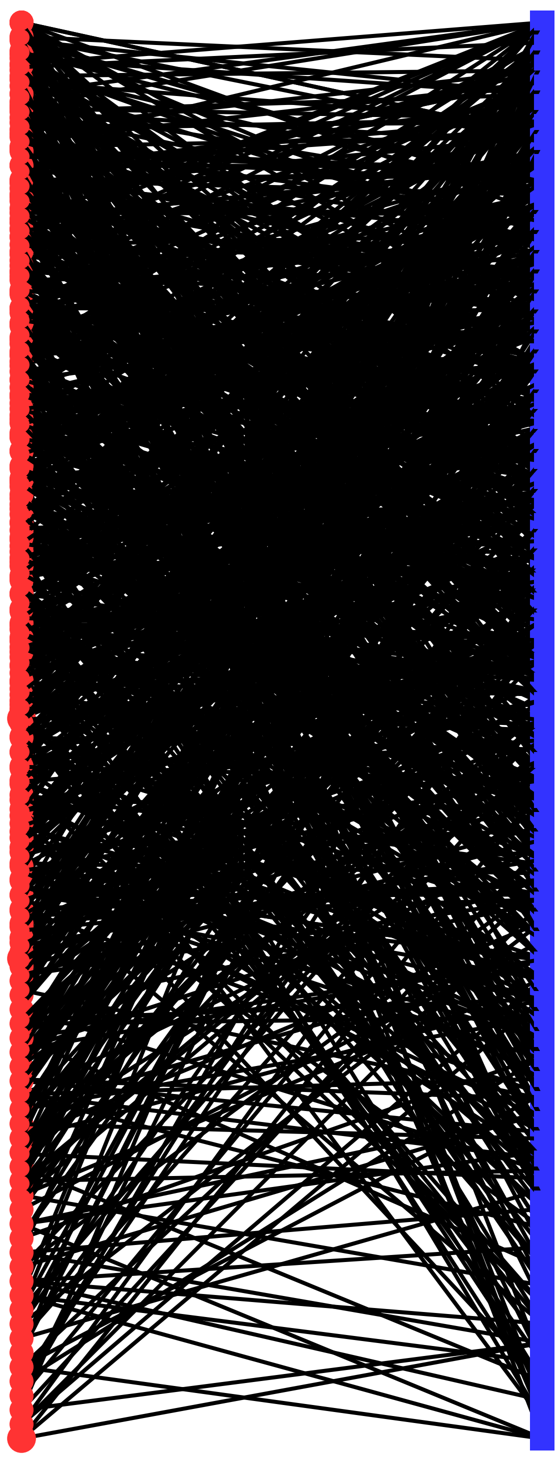
Note: DE sequence is decreasing and bounded from below ⇒ converges

EXIT Curve for (3, 6) Ensemble

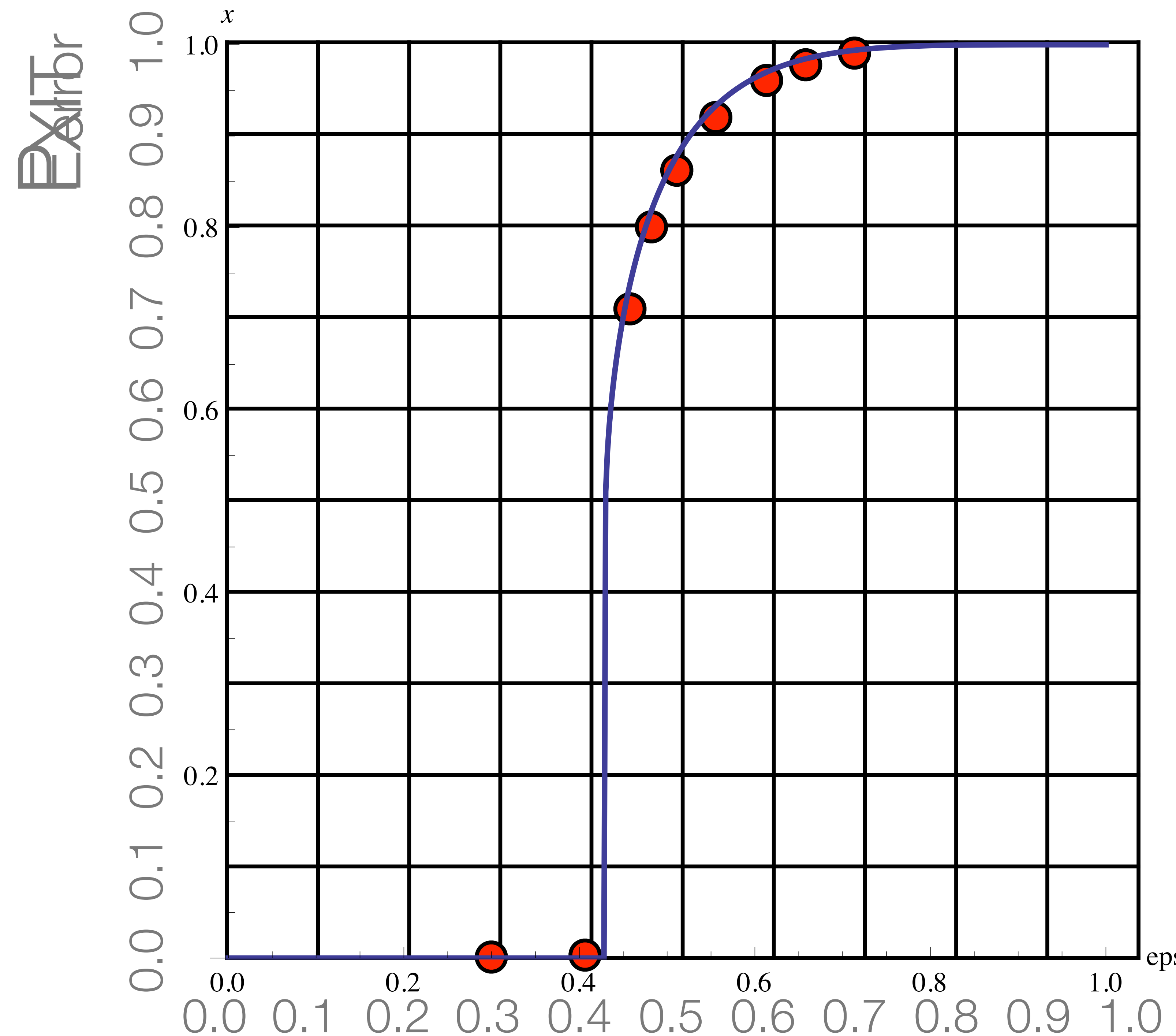
EXIT value as a function of increasing iterations for a given channel value



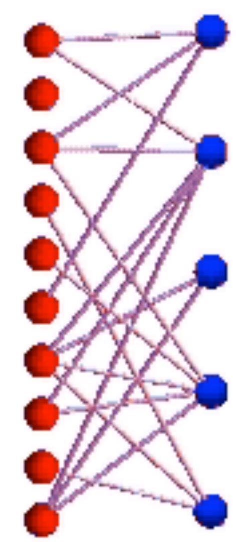
A look back ...

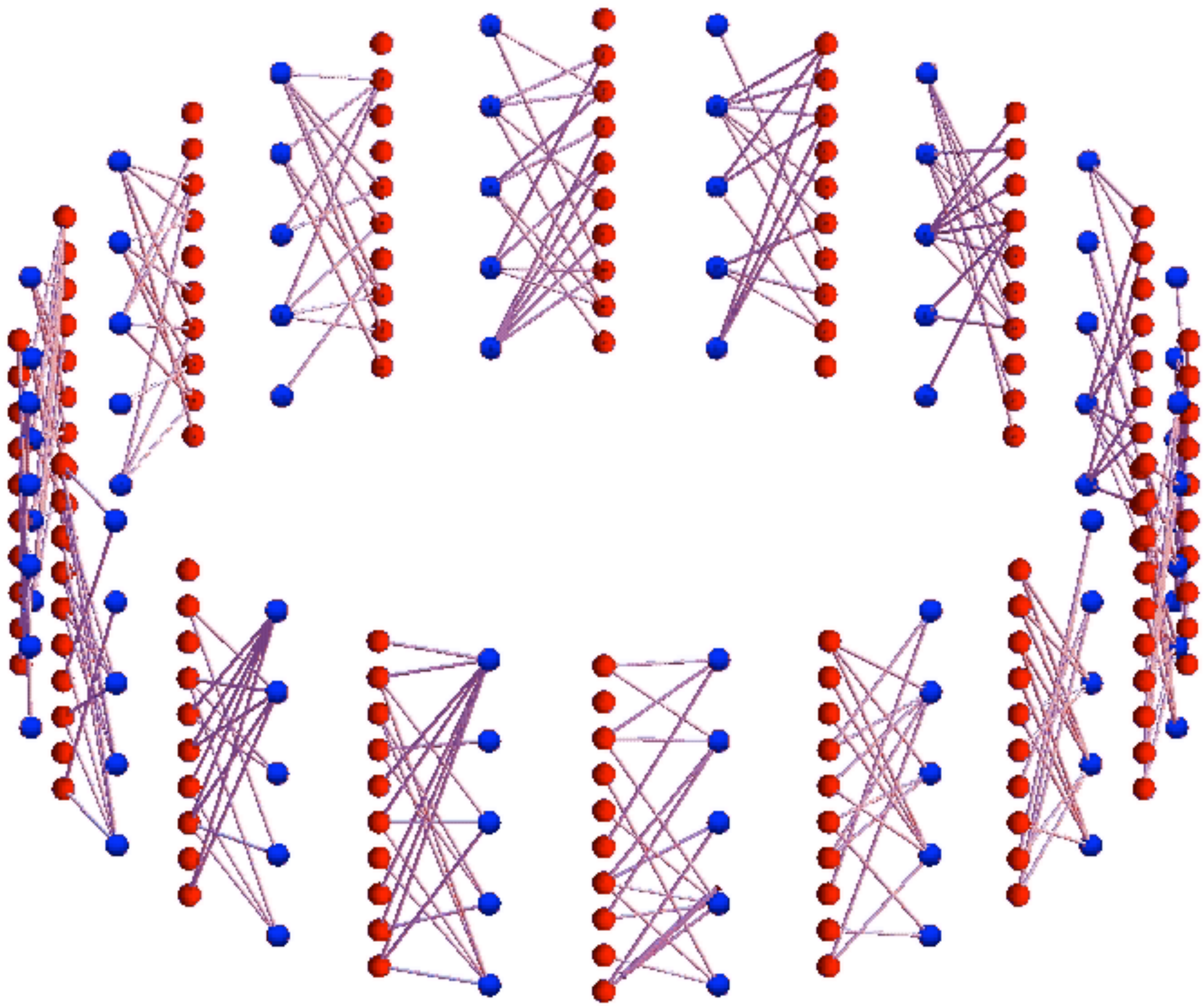


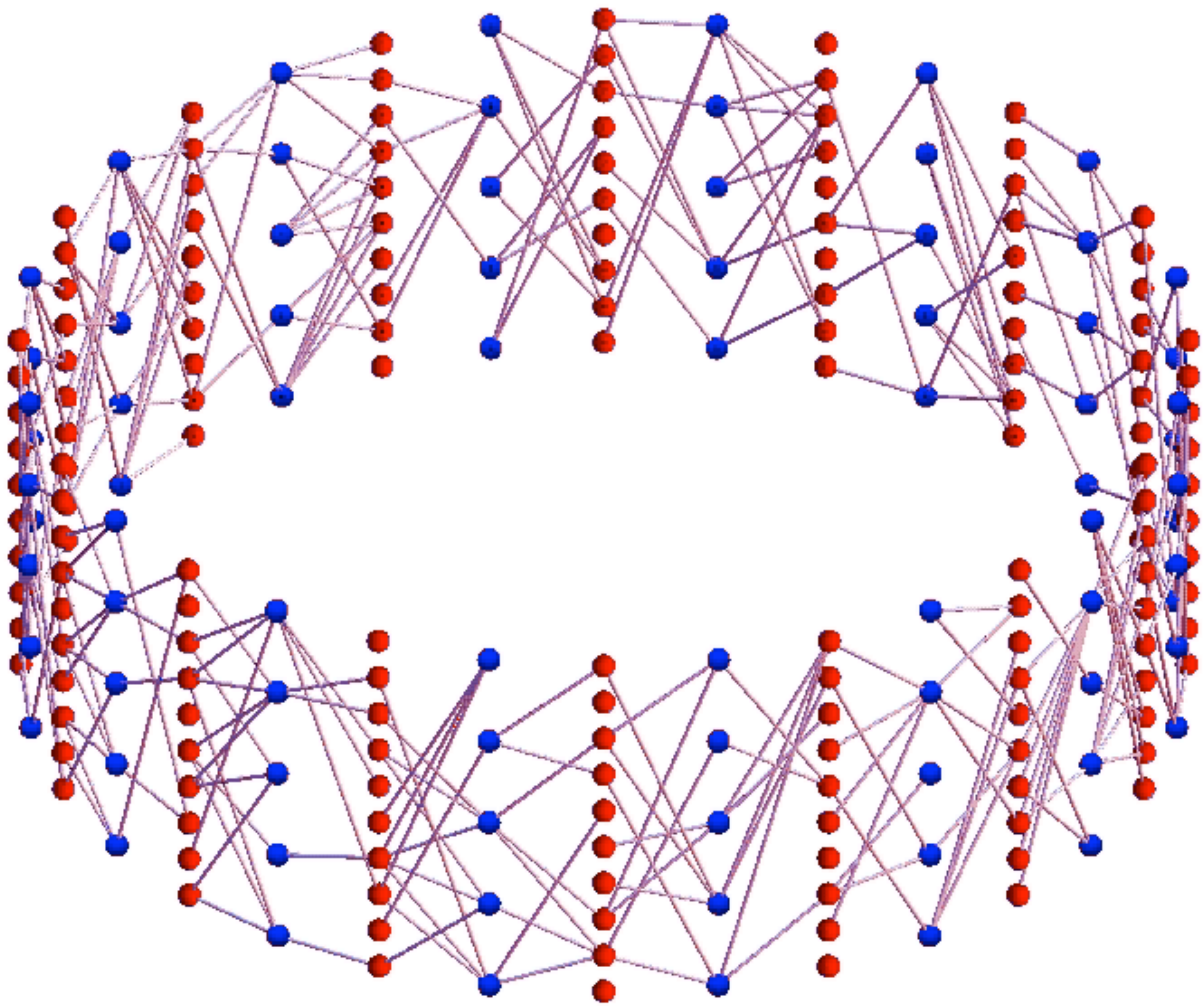
(3, 6) ensemble

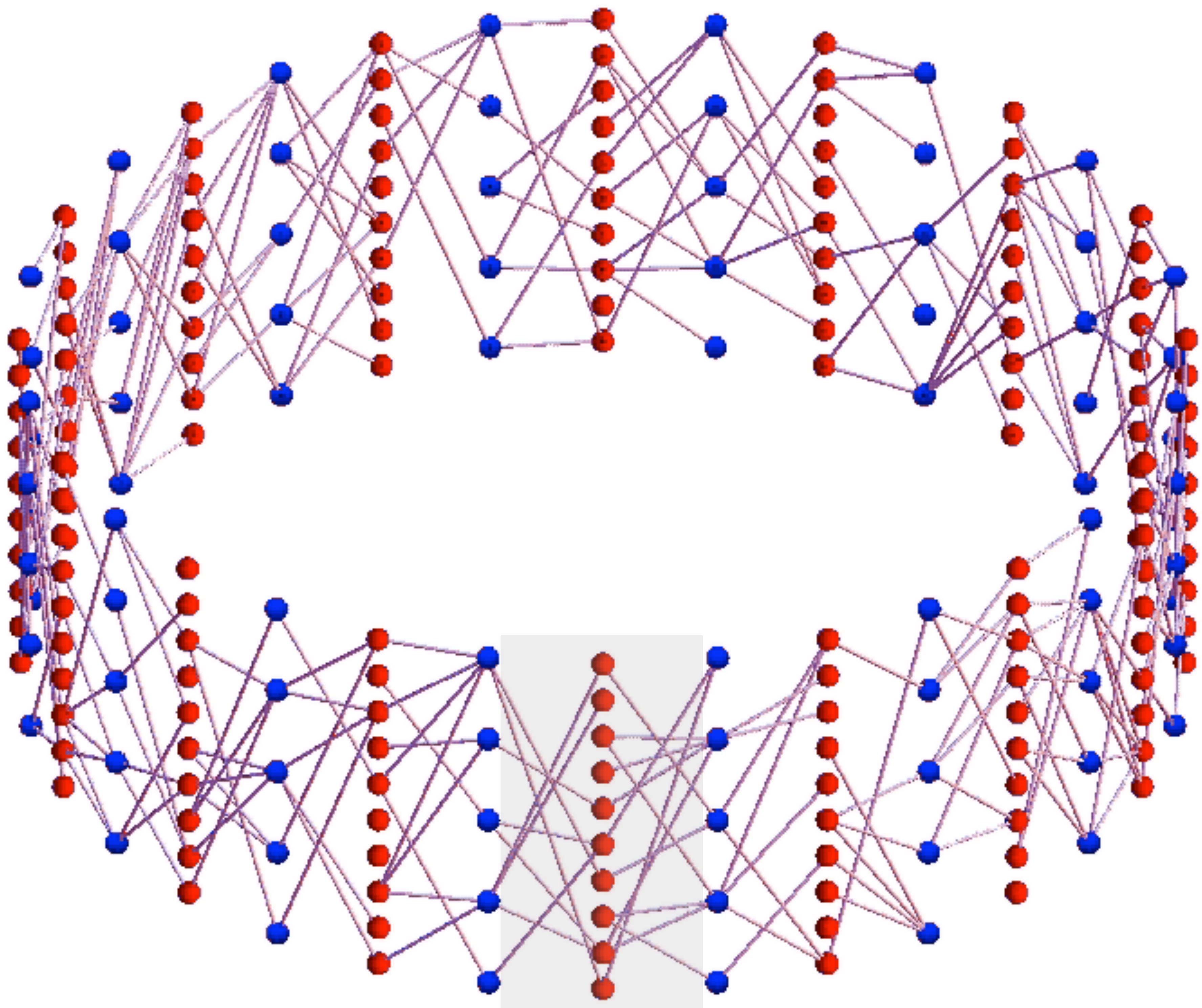


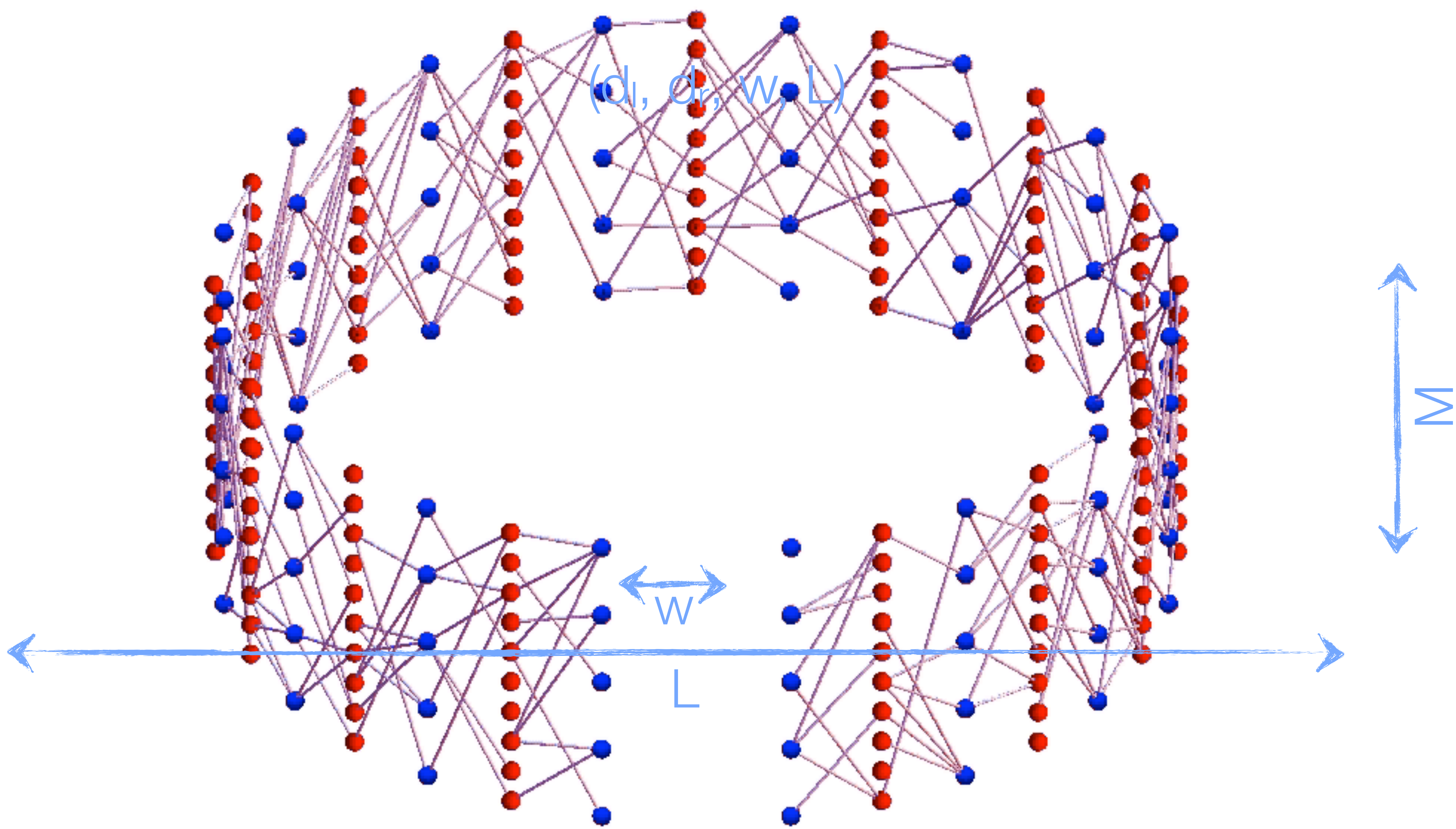
BP decoder ends up in meta-stable state.
Optimal (MAP) decoder would reach stable state.
Can we use nucleation?



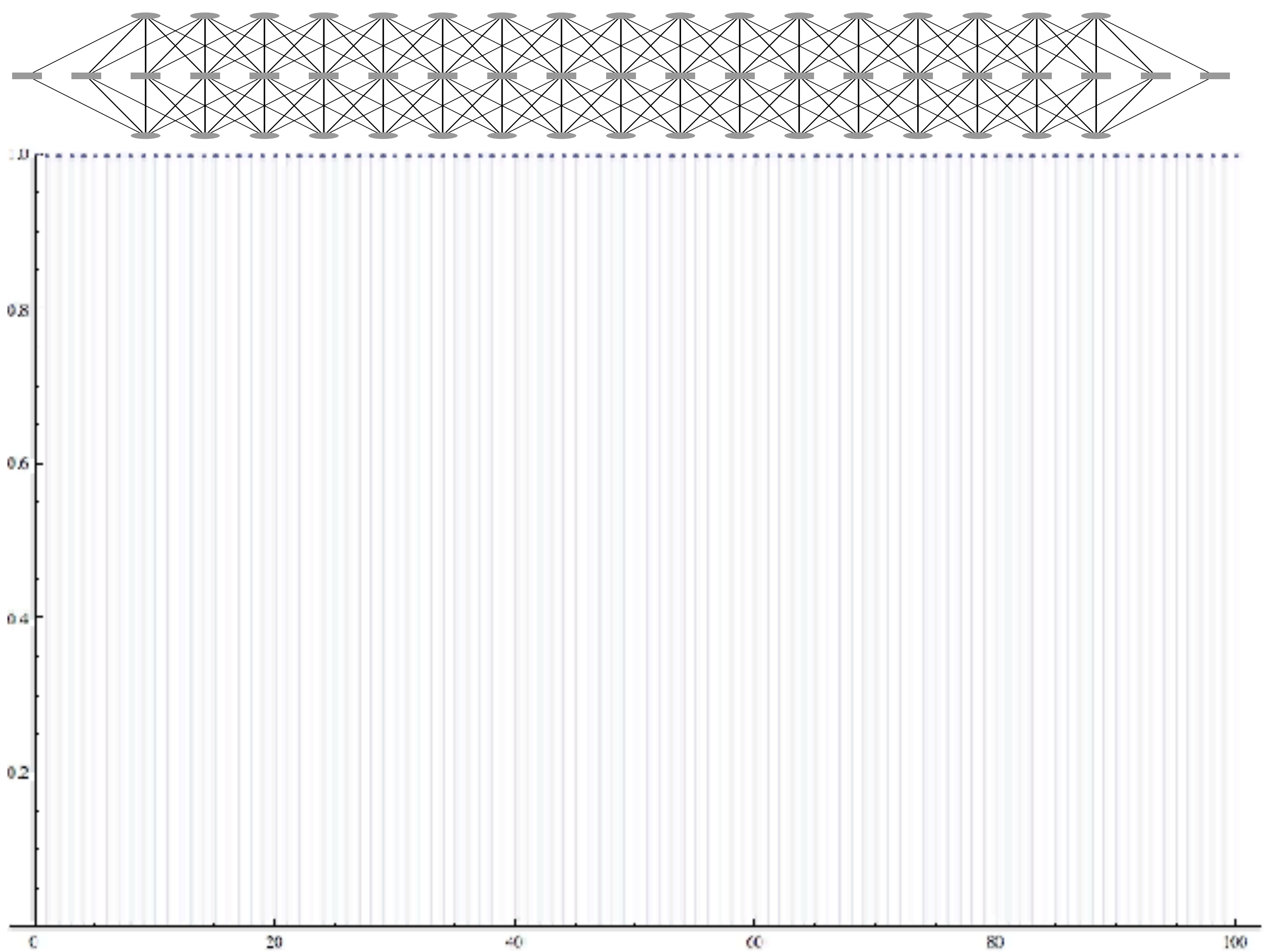


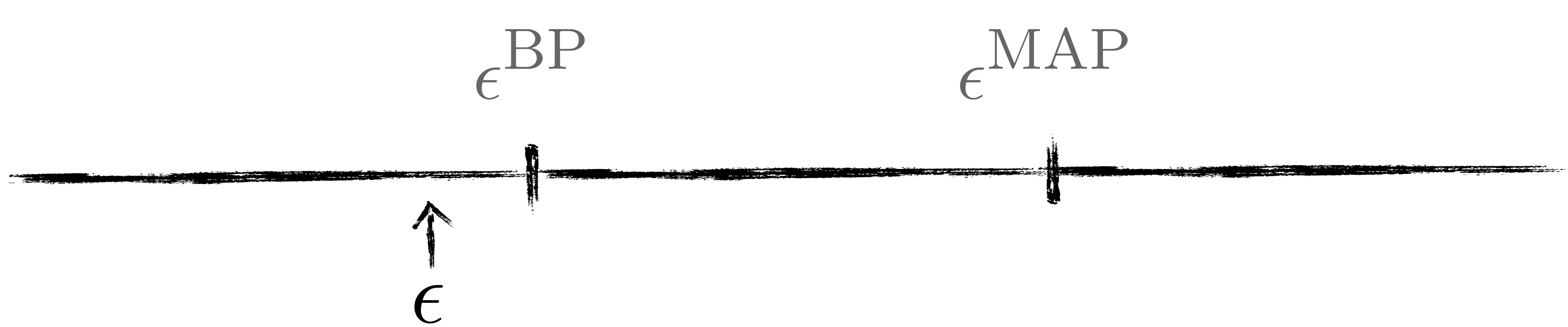




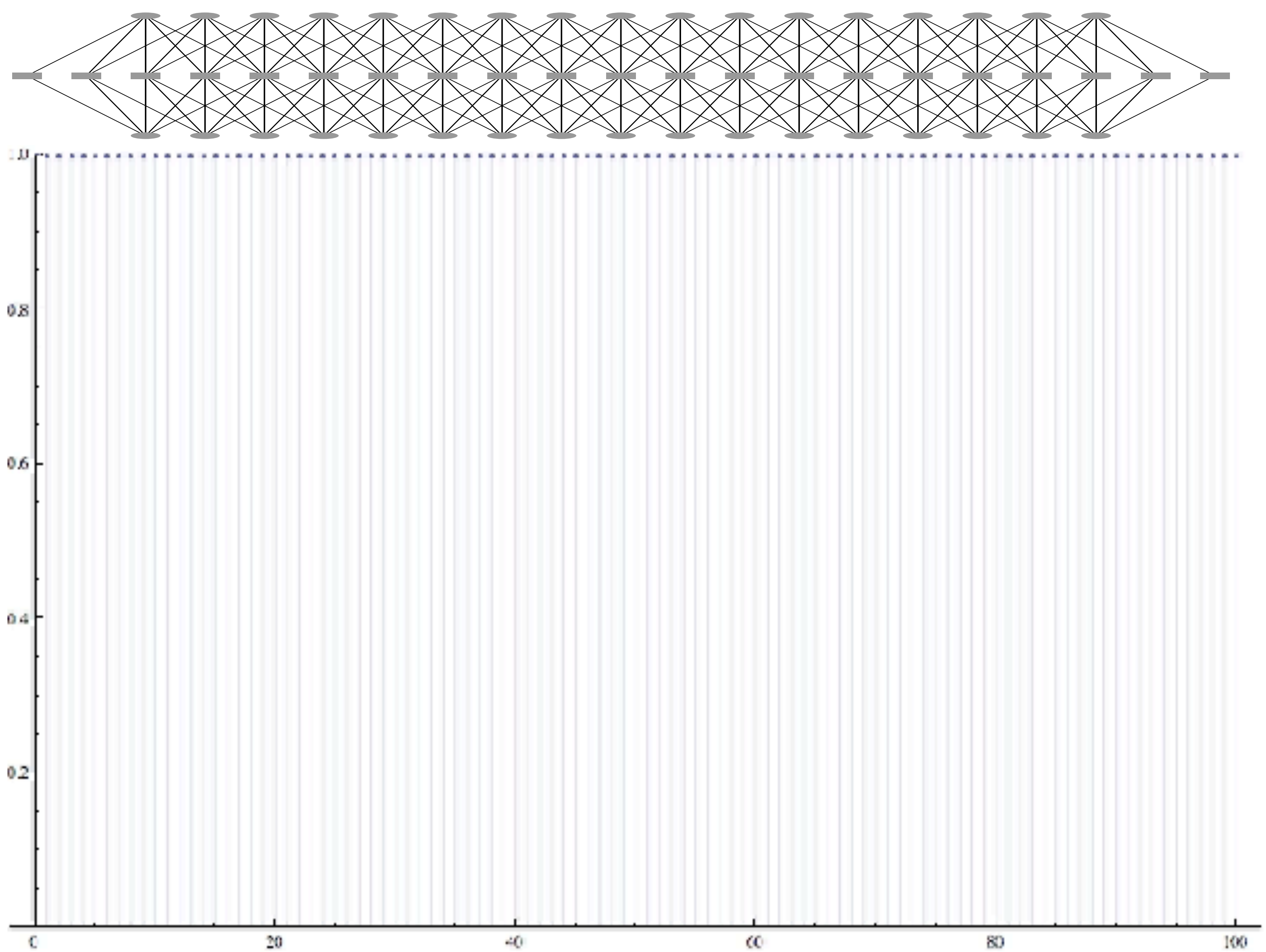

(dl, dr, w, L)
w
L
M

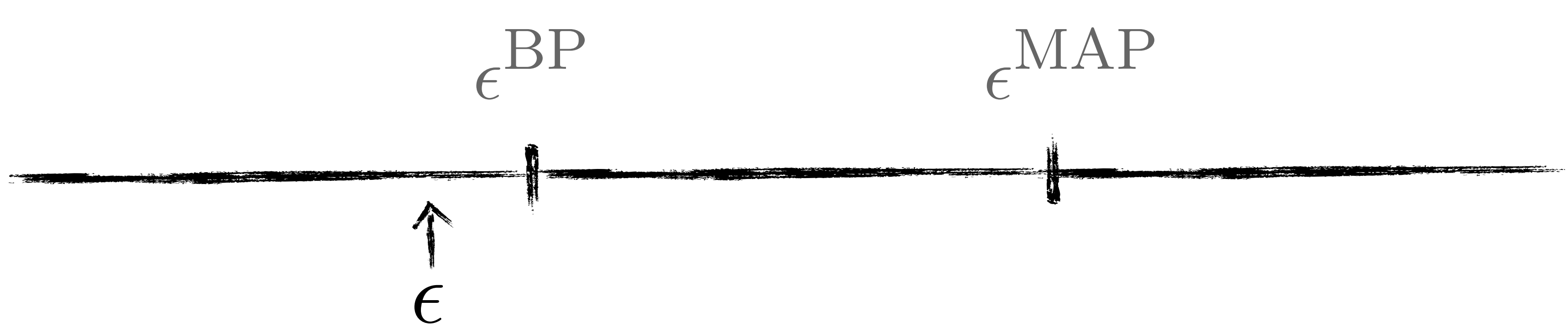
DE for Coupled Ensemble



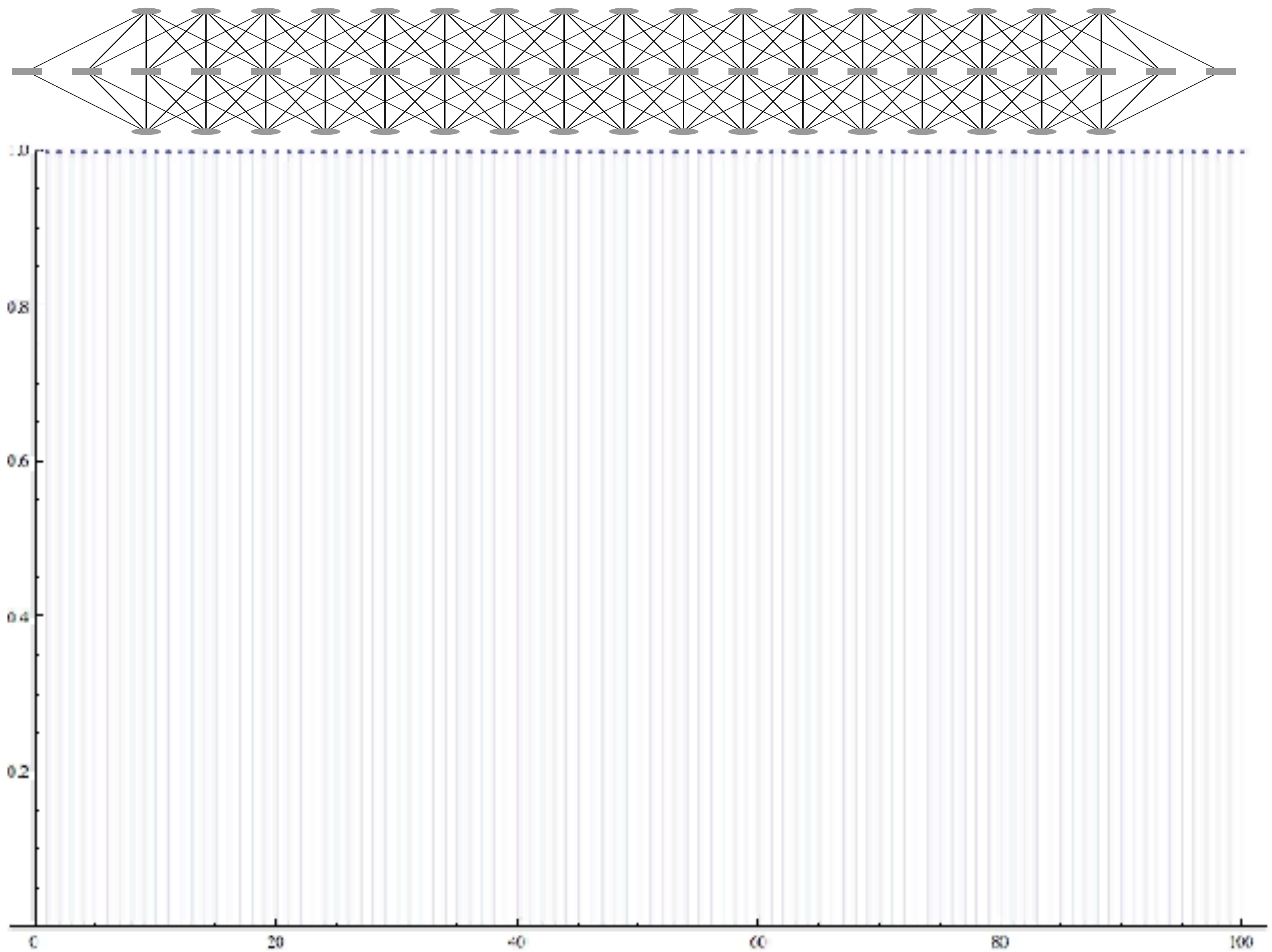


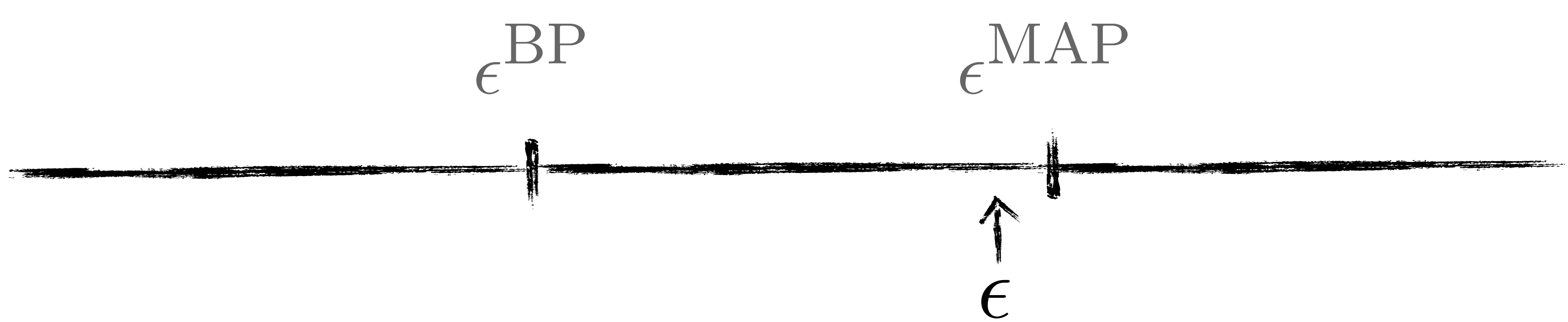
DE for Coupled Ensemble





DE for Coupled Ensemble

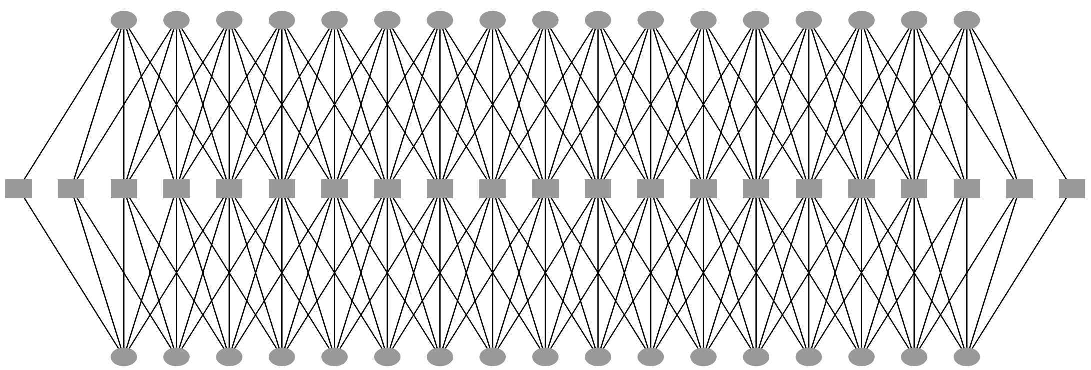


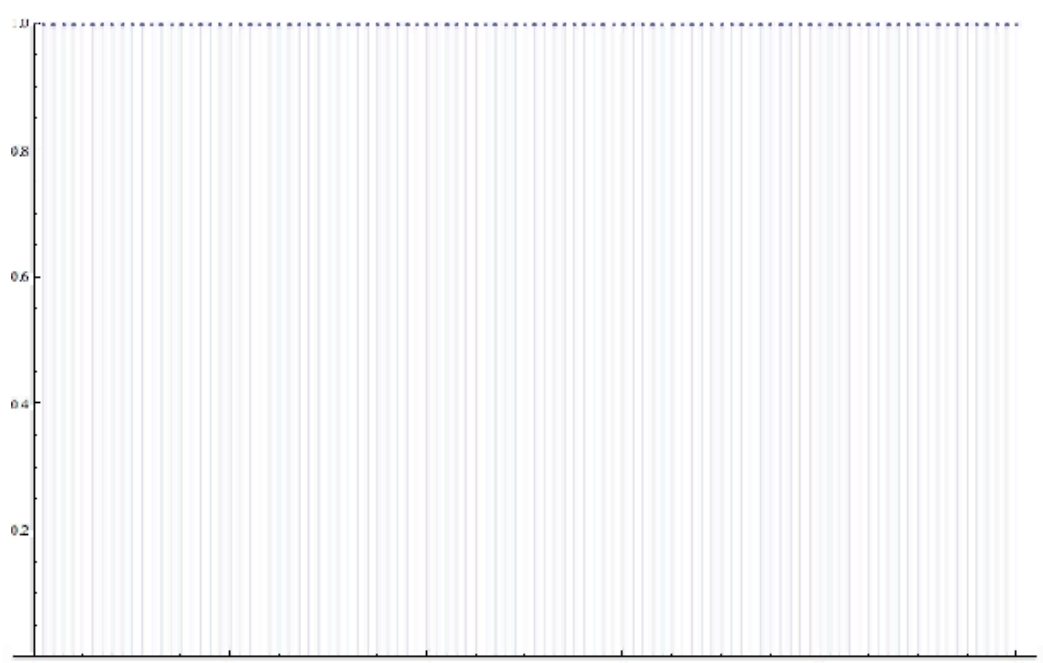


Thresholds

capacity 1/2	BEC	BAWGNC	BSC
(3, 6)	0.488	0.48	0.468
(4, 8)	0.498	0.496	0.491
(5, 10)	0.499	0.499	0.497
(6, 12)	0.4999	0.4996	0.499

Back to the Physics Interpretation

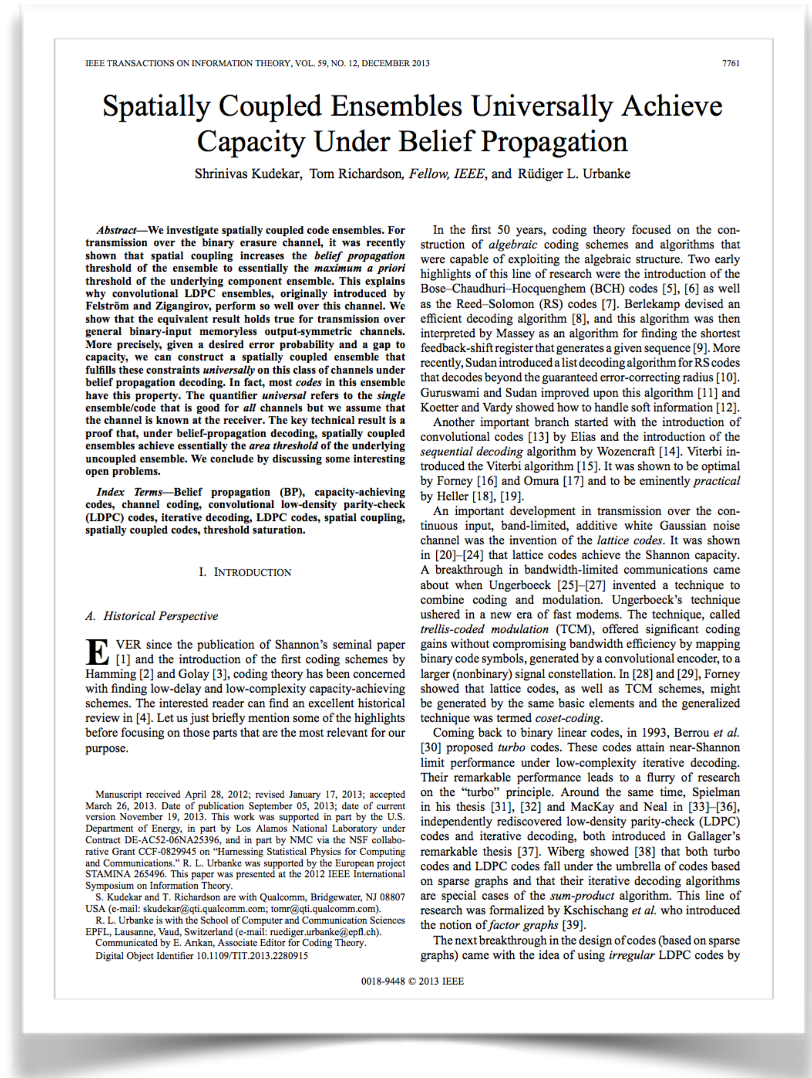




Krzakala, Mezard, Sausset, Sun, and Zdeborova

metastability and nucleation

Spatially Coupled Ensembles — Summary



IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 59, NO. 12, DECEMBER 2013 7761

Spatially Coupled Ensembles Universally Achieve Capacity Under Belief Propagation

Shrinivas Kudekar, Tom Richardson, *Fellow, IEEE*, and Rüdiger L. Urbanke

***Abstract*—We investigate spatially coupled code ensembles. For transmission over the binary erasure channel, it was recently shown that spatial coupling increases the *belief propagation* threshold of the ensemble to essentially the *maximum a priori* threshold of the underlying component ensemble. This explains why convolutional LDPC ensembles, originally introduced by Felström and Zigangirov, perform so well over this channel. We show that the equivalent result holds true for transmission over general binary-input memoryless output-symmetric channels. More precisely, given a desired error probability and a gap to capacity, we can construct a spatially coupled ensemble that fulfills these constraints *universally* on this class of channels under belief propagation decoding. In fact, most *codes* in this ensemble have this property. The quantifier *universal* refers to the *single* ensemble/code that is good for *all* channels but we assume that the channel is known at the receiver. The key technical result is a proof that, under belief-propagation decoding, spatially coupled ensembles achieve essentially the *area threshold* of the underlying uncoupled ensemble. We conclude by discussing some interesting open problems.**

***Index Terms*—Belief propagation (BP), capacity-achieving codes, channel coding, convolutional low-density parity-check (LDPC) codes, iterative decoding, LDPC codes, spatial coupling, spatially coupled codes, threshold saturation.**

I. INTRODUCTION

A. Historical Perspective

EVER since the publication of Shannon's seminal paper [1] and the introduction of the first coding schemes by Hamming [2] and Golay [3], coding theory has been concerned with finding low-delay and low-complexity capacity-achieving schemes. The interested reader can find an excellent historical review in [4]. Let us just briefly mention some of the highlights before focusing on those parts that are the most relevant for our purpose.

In the first 50 years, coding theory focused on the construction of *algebraic* coding schemes and algorithms that were capable of exploiting the algebraic structure. Two early highlights of this line of research were the introduction of the Bose–Chaudhuri–Hocquenghem (BCH) codes [5], [6] as well as the Reed–Solomon (RS) codes [7]. Berlekamp devised an efficient decoding algorithm [8], and this algorithm was then interpreted by Massey as an algorithm for finding the shortest feedback-shift register that generates a given sequence [9]. More recently, Sudan introduced a list decoding algorithm for RS codes that decodes beyond the guaranteed error-correcting radius [10]. Guruswami and Sudan improved upon this algorithm [11] and Koetter and Vardy showed how to handle soft information [12].

Another important branch started with the introduction of convolutional codes [13] by Elias and the introduction of the *sequential decoding* algorithm by Wozencraft [14]. Viterbi introduced the Viterbi algorithm [15]. It was shown to be optimal by Forney [16] and Omura [17] and to be eminently *practical* by Heller [18], [19].

An important development in transmission over the continuous input, band-limited, additive white Gaussian noise channel was the invention of the *lattice codes*. It was shown in [20]–[24] that lattice codes achieve the Shannon capacity. A breakthrough in bandwidth-limited communications came about when Ungerboeck [25]–[27] invented a technique to combine coding and modulation. Ungerboeck's technique ushered in a new era of fast modems. The technique, called *trellis-coded modulation* (TCM), offered significant coding gains without compromising bandwidth efficiency by mapping binary code symbols, generated by a convolutional encoder, to a larger (nonbinary) signal constellation. In [28] and [29], Forney showed that lattice codes, as well as TCM schemes, might be generated by the same basic elements and the generalized technique was termed *coset-coding*.

Coming back to binary linear codes, in 1993, Berrou *et al.* [30] proposed *turbo* codes. These codes attain near-Shannon limit performance under low-complexity iterative decoding. Their remarkable performance leads to a flurry of research on the "turbo" principle. Around the same time, Spielman in his thesis [31], [32] and MacKay and Neal in [33]–[36], independently rediscovered low-density parity-check (LDPC) codes and iterative decoding, both introduced in Gallager's remarkable thesis [37]. Wiberg showed [38] that both turbo codes and LDPC codes fall under the umbrella of codes based on sparse graphs and that their iterative decoding algorithms are special cases of the *sum-product* algorithm. This line of research was formalized by Kschischang *et al.* who introduced the notion of *factor graphs* [39].

The next breakthrough in the design of codes (based on sparse graphs) came with the idea of using *irregular* LDPC codes by

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- achieve capacity for any BMS channel
- block length: $O(1/\delta^3)$
- encoding complexity per bit: $O(\log(1/\delta))$
- number of iterations: $O(1/\delta)$ (educated guess :-))
- number of bits required for processing of messages: $O(\log(1/\delta))$
- decoding complexity per bit: $O(1/\delta \log^2(1/\delta))$ bit operations

Main Message

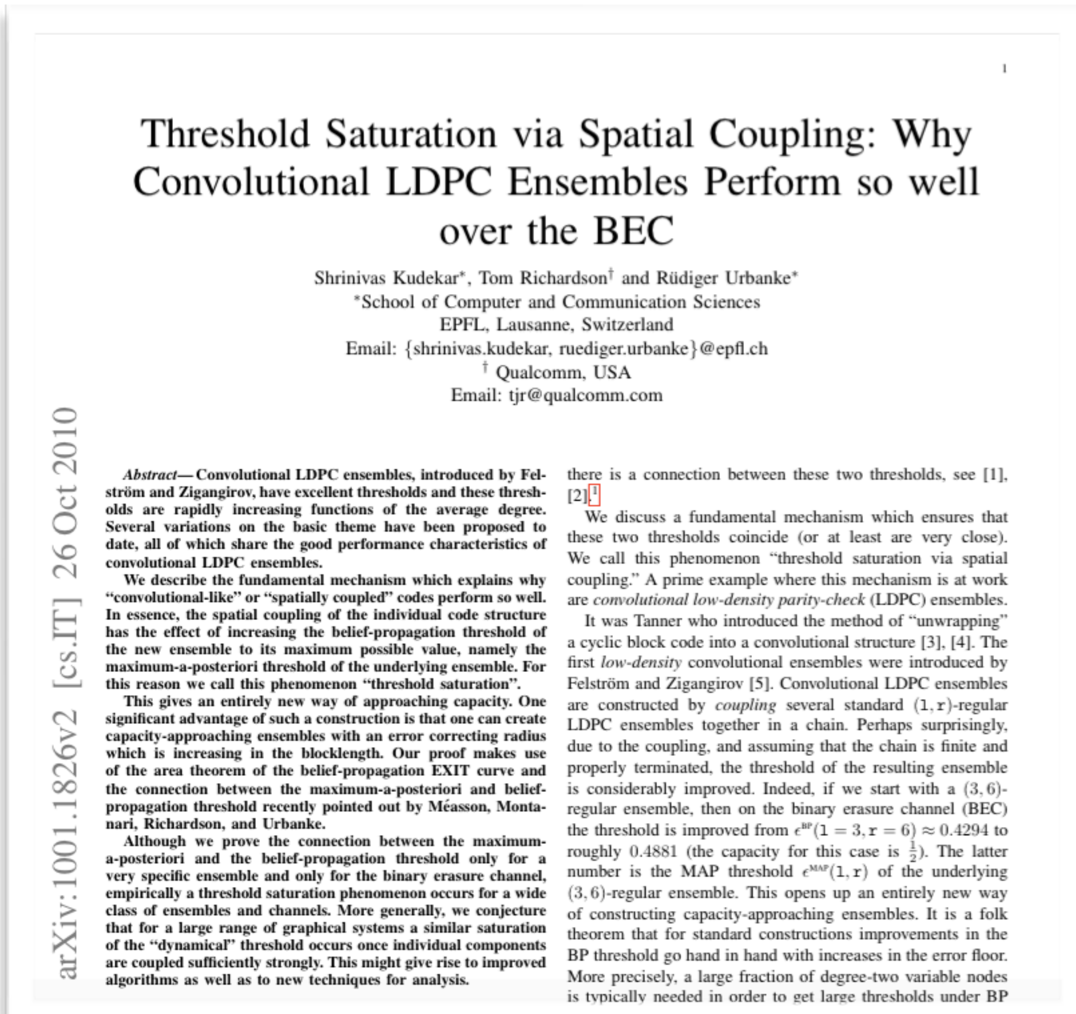
Coupled ensembles under BP decoding behave
like uncoupled ensembles under MAP decoding.

Since coupled ensemble achieve the highest threshold they can achieve (namely the MAP threshold) under BP we speak of the threshold saturation phenomenon.

Via spatial coupling we can construct codes which are capacity-achieving *universally* across the whole set of BMS channels.

On the downside, due to the termination which is required, we loose in rate. We hence have to take the chain length large enough in order to amortize this rate loss. Therefore, the blocklength has to be reasonably large.

Spatial Coupling as a Proof Technique (coding)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Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform so well over the BEC

Shrinivas Kudekar*, Tom Richardson† and Rüdiger Urbanke*
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***Abstract*— Convolutional LDPC ensembles, introduced by Felström and Zigangirov, have excellent thresholds and these thresholds are rapidly increasing functions of the average degree. Several variations on the basic theme have been proposed to date, all of which share the good performance characteristics of convolutional LDPC ensembles.**

We describe the fundamental mechanism which explains why "convolutional-like" or "spatially coupled" codes perform so well. In essence, the spatial coupling of the individual code structure has the effect of increasing the belief-propagation threshold of the new ensemble to its maximum possible value, namely the maximum-a-posteriori threshold of the underlying ensemble. For this reason we call this phenomenon "threshold saturation".

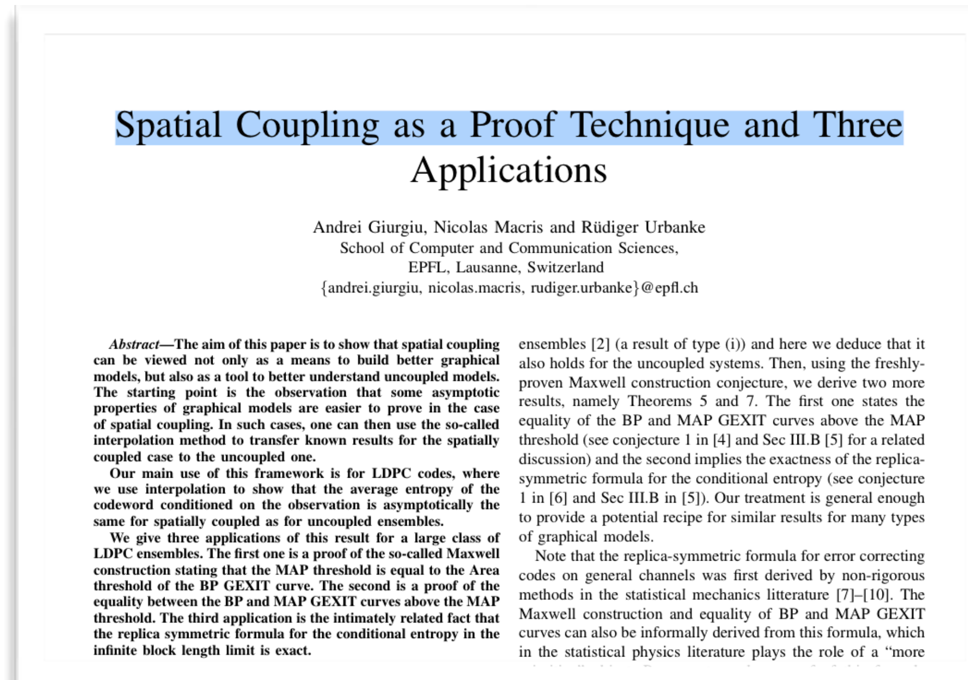
This gives an entirely new way of approaching capacity. One significant advantage of such a construction is that one can create capacity-approaching ensembles with an error correcting radius which is increasing in the blocklength. Our proof makes use of the area theorem of the belief-propagation EXIT curve and the connection between the maximum-a-posteriori and belief-propagation threshold recently pointed out by Méasson, Montanari, Richardson, and Urbanke.

Although we prove the connection between the maximum-a-posteriori and the belief-propagation threshold only for a very specific ensemble and only for the binary erasure channel, empirically a threshold saturation phenomenon occurs for a wide class of ensembles and channels. More generally, we conjecture that for a large range of graphical systems a similar saturation of the "dynamical" threshold occurs once individual components are coupled sufficiently strongly. This might give rise to improved algorithms as well as to new techniques for analysis.

there is a connection between these two thresholds, see [1], [2].[1]

We discuss a fundamental mechanism which ensures that these two thresholds coincide (or at least are very close). We call this phenomenon "threshold saturation via spatial coupling." A prime example where this mechanism is at work are *convolutional low-density parity-check* (LDPC) ensembles.

It was Tanner who introduced the method of "unwrapping" a cyclic block code into a convolutional structure [3], [4]. The first *low-density* convolutional ensembles were introduced by Felström and Zigangirov [5]. Convolutional LDPC ensembles are constructed by *coupling* several standard $(\mathtt{l}, \mathtt{r})$-regular LDPC ensembles together in a chain. Perhaps surprisingly, due to the coupling, and assuming that the chain is finite and properly terminated, the threshold of the resulting ensemble is considerably improved. Indeed, if we start with a $(3, 6)$-regular ensemble, then on the binary erasure channel (BEC) the threshold is improved from $\epsilon^{\text{BP}}(\mathtt{l}=3, \mathtt{r}=6) \approx 0.4294$ to roughly 0.4881 (the capacity for this case is $\frac{1}{2}$). The latter number is the MAP threshold $\epsilon^{\text{MAP}}(\mathtt{l}, \mathtt{r})$ of the underlying $(3, 6)$-regular ensemble. This opens up an entirely new way of constructing capacity-approaching ensembles. It is a folk theorem that for standard constructions improvements in the BP threshold go hand in hand with increases in the error floor. More precisely, a large fraction of degree-two variable nodes is typically needed in order to get large thresholds under BP


Spatial Coupling as a Proof Technique and Three Applications

Andrei Giurgiu, Nicolas Macris and Rüdiger Urbanke
School of Computer and Communication Sciences,
EPFL, Lausanne, Switzerland
{andrei.giurgiu, nicolas.macris, rudiger.urbanke}@epfl.ch

***Abstract*—The aim of this paper is to show that spatial coupling can be viewed not only as a means to build better graphical models, but also as a tool to better understand uncoupled models. The starting point is the observation that some asymptotic properties of graphical models are easier to prove in the case of spatial coupling. In such cases, one can then use the so-called interpolation method to transfer known results for the spatially coupled case to the uncoupled one.**

Our main use of this framework is for LDPC codes, where we use interpolation to show that the average entropy of the codeword conditioned on the observation is asymptotically the same for spatially coupled as for uncoupled ensembles.

We give three applications of this result for a large class of LDPC ensembles. The first one is a proof of the so-called Maxwell construction stating that the MAP threshold is equal to the Area threshold of the BP GEXIT curve. The second is a proof of the equality between the BP and MAP GEXIT curves above the MAP threshold. The third application is the intimately related fact that the replica symmetric formula for the conditional entropy in the infinite block length limit is exact.

ensembles [2] (a result of type (i)) and here we deduce that it also holds for the uncoupled systems. Then, using the freshly-proven Maxwell construction conjecture, we derive two more results, namely Theorems 5 and 7. The first one states the equality of the BP and MAP GEXIT curves above the MAP threshold (see conjecture 1 in [4] and Sec III.B [5] for a related discussion) and the second implies the exactness of the replica-symmetric formula for the conditional entropy (see conjecture 1 in [6] and Sec III.B in [5]). Our treatment is general enough to provide a potential recipe for similar results for many types of graphical models.

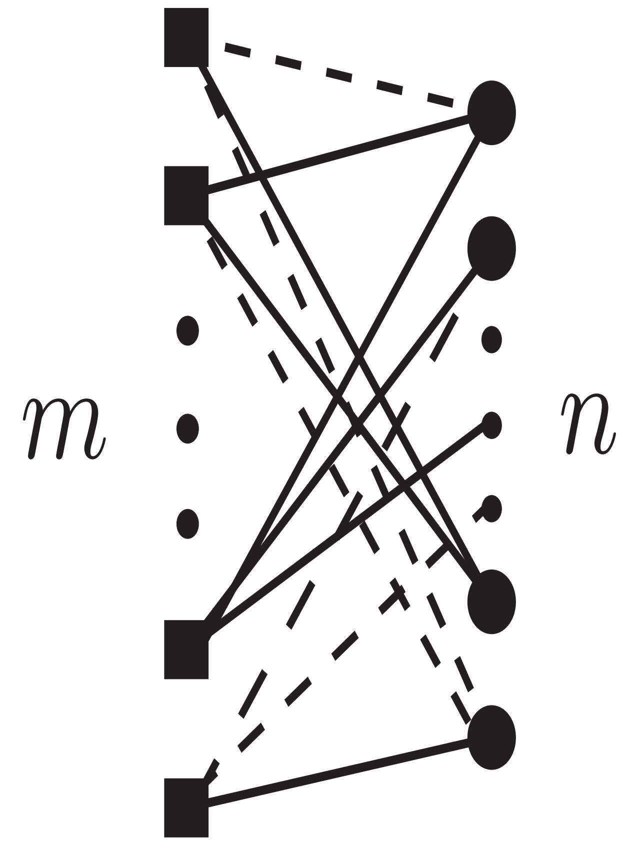
Note that the replica-symmetric formula for error correcting codes on general channels was first derived by non-rigorous methods in the statistical mechanics litterature [7]–[10]. The Maxwell construction and equality of BP and MAP GEXIT curves can also be informally derived from this formula, which in the statistical physics literature plays the role of a "more

shows that MAP threshold is given by Maxwell conjecture

Spatial Coupling as a Proof Technique

Paradigmatic CSP: random K-SAT

- Random graph with n variable nodes and m clauses.
- Each variable node is connected to K clauses u.a.r by an edge.
- Edge is dashed or full with probability $1/2$. Degree of variable nodes is Poisson(αK).



- Boolean variables: $x_i \in \{T, F\}$ or $\in \{0, 1\}$, $i = 1, \cdots, n$
- Clauses: $(\vee_{i=1}^{K} x_{a_i}^{n(a_i)})$, $a = 1, \cdots, m$
- $F_{n,\alpha,K} = \wedge_{a=1}^{M} (\vee_{i=1}^{K} x_{a_i}^{s(a_i)})$

Control parameter $\alpha = \frac{\#(\text{clauses})}{\#(\text{variables})} = \frac{m}{n}$**: Phase Transitions.**

- Friedgut 1999: $\exists \alpha_s(n,K)$ s.t $\forall \epsilon > 0$

$$\lim_{n\to\infty} \Pr\{F_{n,\alpha,K} \text{is SAT}\} = \begin{cases} 1 & \text{if } \alpha < (1-\epsilon)\alpha_s(n,K), \\ 0 & \text{if } \alpha > (1-\epsilon)\alpha_s(n,K). \end{cases}$$

Existence of $\lim_{n\to+\infty} \alpha_s(n,K)$ is still an open problem.

- This talk: MAX-SAT or Hamiltonian version of the problem:

$$H_F(\underline{x}) = \sum_{a=1}^{m} \left(1 - \mathbf{1}\left(\vee_{i=1}^{K} x_{a_i}^{s(a_i)}\right)\right),$$

the MAX-SAT/UNSAT threshold is defined as:

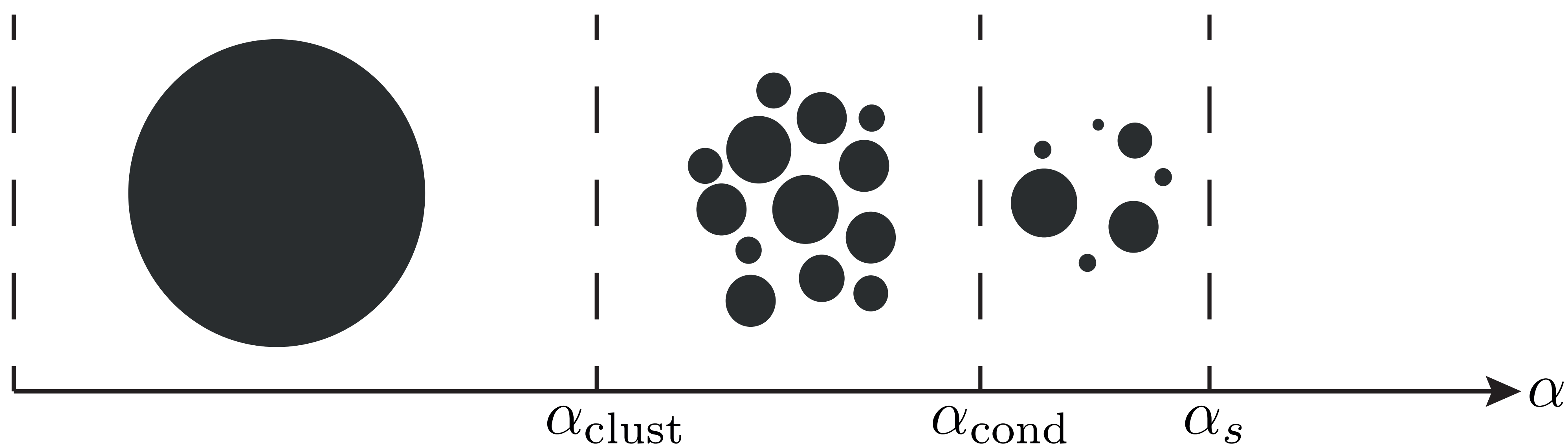
$$\alpha_s(K) \equiv \inf\{\ \alpha \ \mid \ \underbrace{\lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[\min_{\underline{x}} H_F(\underline{x})]}_{\text{exists and continuous function of } \alpha} \ > 0\ \}$$

In particular α_s exists. [Interpolation methods: Franz-Leone, Panchenko, Gamarnik-Bayati-Tetali].

The Physics Picture

Parisi-Mezard-Zechina 2001

Semerjian-RicciTersenghi-Montanari, Krazkala-Zdeborova 2008



Known Lower bounds on the SAT-UNSAT threshold

- Algorithmic lower bounds: find analyzable algorithm and find solutions for $\alpha_{\rm alg}(K) < \alpha_s(K)$. [long history ...]
- Second Moment lower bounds, weighted s.m with cavity inspired weights [long history, ... Achlioptas - Coja Oghlan].

K	3	4	$\cdots$	large K
best lower bound	$3.52^{\rm alg}$	$7.91^{\rm s.m}$	$\cdots$	$2^K \ln 2 - \frac{3}{2}\ln 2 + o(1)^{\rm s.m}$
best algor bound	3.52	5.54	$\cdots$	$\frac{2^K \ln K}{K}(1+o(1))$
$\alpha_{\rm dyn}$	3.86	9.38	$\cdots$	$\frac{2^K \ln K}{K}(1+o(1))$
$\alpha_{\rm cond}$	3.86	9.55	$\cdots$	$2^K \ln 2 - \frac{3}{2}\ln 2 + o(1)$
$\alpha_{\rm s}$	4.26	9.93	$\cdots$	$2^K \ln 2 - \frac{1}{2}(1+\ln 2) + o(1)$

New Lower bounds by the Spatial Coupling Method

Recall:

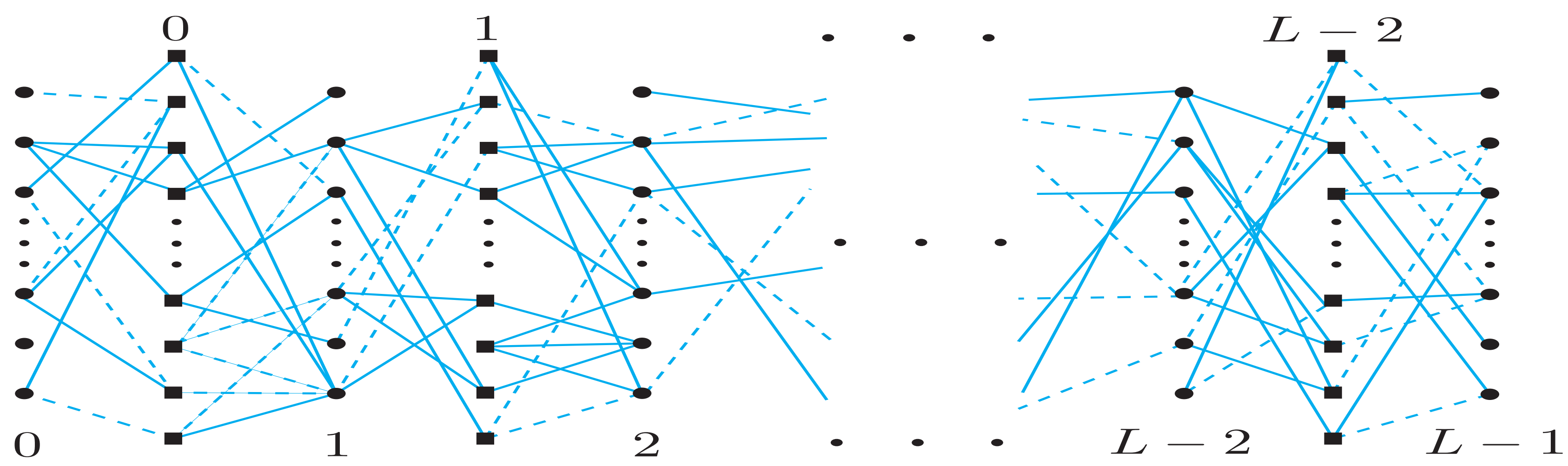
$H_F(\underline{x})$ = number of UNSAT clauses of F for $\underline{x} \in \{0,1\}^n$

and $\alpha_s = \inf\{\alpha \mid \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[\min_{\underline{x}} H_F(\underline{x})] > 0\}$

K	3	4	$\cdots$	large K
α_{new}	3.67	7.81	$\cdots$	$2^K \times \frac{1}{2}$
best algor bound	3.52	5.54	$\cdots$	$\frac{2^K \ln K}{K}(1+o(1))$
best lower bound	3.52^{alg}	$7.91^{\text{s.m}}$	$\cdots$	$2^K \ln 2 - \frac{3}{2}\ln 2 + o(1)^{\text{s.m}}$
α_{dyn}	3.86	9.38	$\cdots$	$\frac{2^K \ln K}{K}(1+o(1)$
α_{cond}	3.86	9.55	$\cdots$	$2^K \ln 2 - \frac{3}{2}\ln 2 + o(1)$
α_{s}	4.26	9.93	$\cdots$	$2^K \ln 2 - \frac{1}{2}(1+\ln 2) + o(1)$

Strategy

construct spatially coupled model



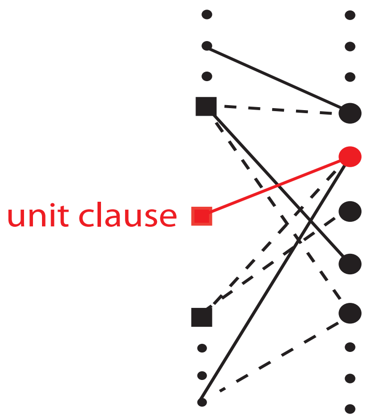
$$\alpha_{SAT}^{\text{coupled}} = \alpha_{SAT}^{\text{uncoupled}}$$

$$\alpha_{alg}^{\text{uncoupled}} \leq \alpha_{alg}^{\text{coupled}} \leq \alpha_{SAT}^{\text{(un)coupled}}$$

Unit Clause Propagation algorithm

1. Repeat until all variables are set:

2. Forced Step: If F contains unit clauses choose one at random and satisfy it by setting unique variable. Remove or shorten other clauses that contain this variable.



3. Free Step: If there are no unit clauses choose a variable at random and set it at random. Remove or shorten clauses that contain this variable.

Analysis by differential equations [Chao-Franco 1986]

A "Round" $=$ "free step immediately followed by forced steps and ends when all forced steps have ended".

(Rescaled) time t is number of rounds. For $K = 3$:

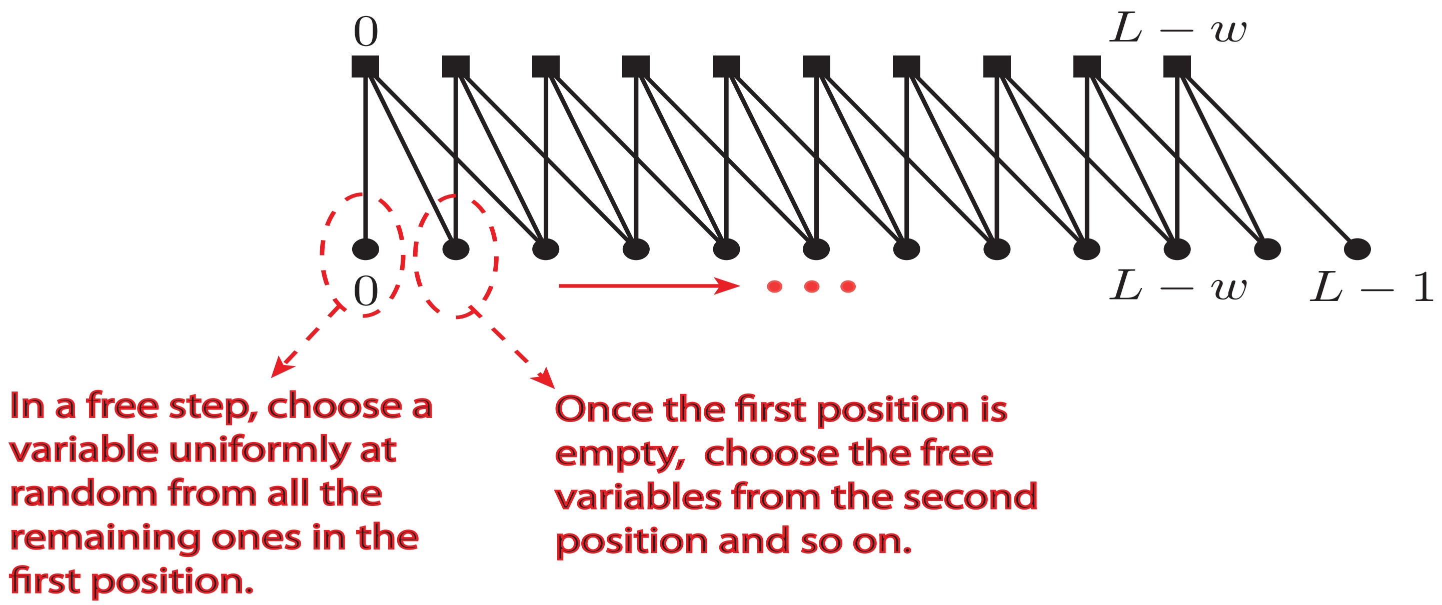
$$\begin{cases} \frac{d\ell(t)}{dt} = -2\beta(t), \qquad \beta(t) = \#(\text{variables set in a round}) \\ \frac{dc_3(t)}{dt} = -\beta(t)\left(\frac{3c_3(t)}{\ell(t)/2}\right) \\ \frac{dc_2(t)}{dt} = +\beta(t)\left(\frac{3c_3(t)}{\ell(t)/2}\right)\frac{1}{2} - \beta(t)\left(\frac{2c_2(t)}{\ell(t)/2}\right) \end{cases}$$

$$\rightarrow \quad \frac{d\ell(t)}{dt} = -\frac{2}{\ell(t)(1-\frac{3\alpha}{4}(1-\frac{\ell(t)}{2})} = -\frac{1}{1-r_1(t)}$$

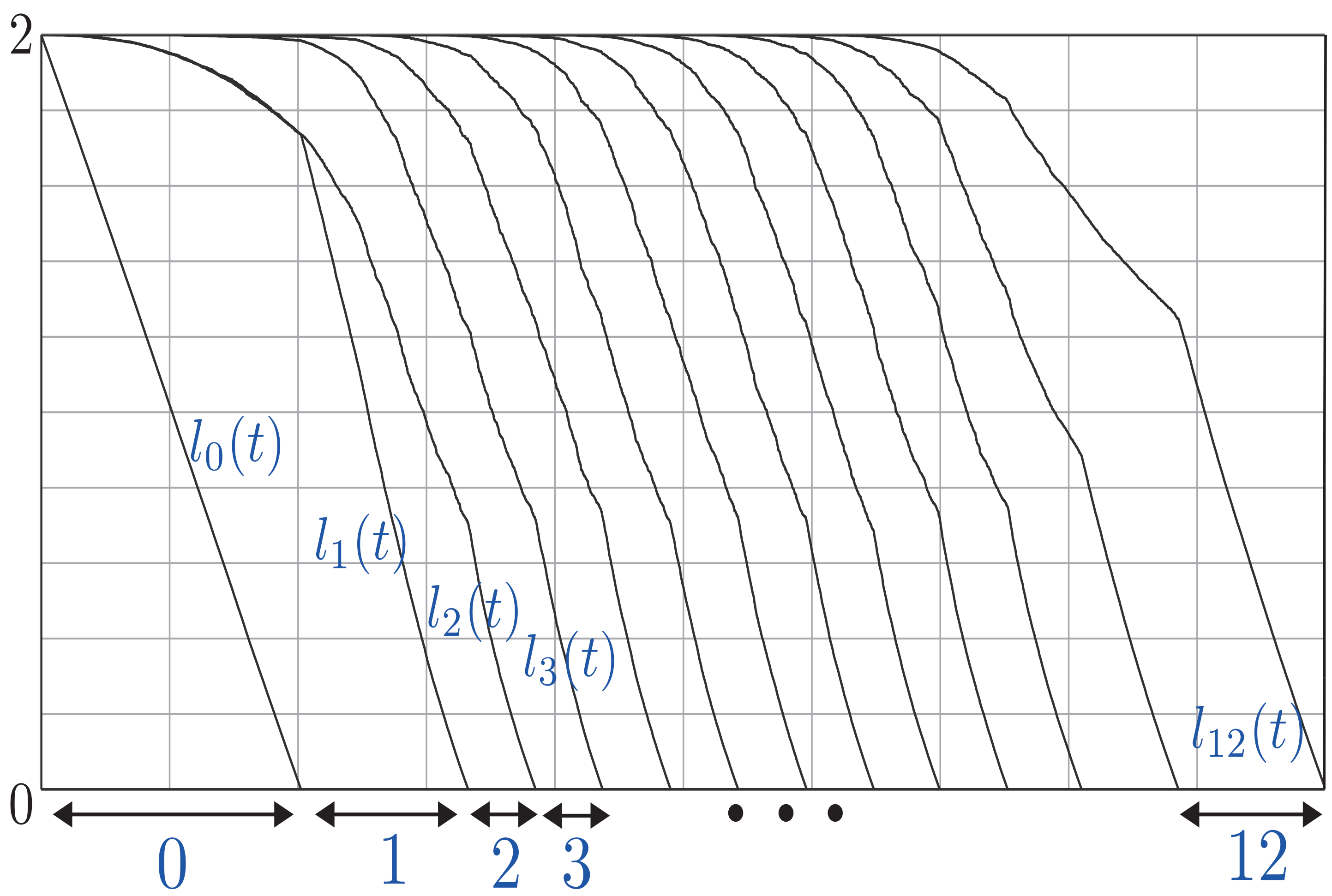
For $\alpha \to \frac{8}{3} \approx 2.66$, $\frac{d\ell(t)}{dt} \to +\infty$ and rate $r_1(t)$ of unit clauses production $\to 1$; $\implies \alpha_{\mathrm{UC}}(3) = \frac{8}{3} \approx 2.66$.

Unit Clause Propagation for coupled Formulas:

- **Forced step:** as long as ∃ unit clause, then satisfy it by setting the variable. Remove or shorten clauses containing this variable.
- **Free step:**



Evolution of number of variables per position



Algorithm runs in "phases" $p = 0, 1, 2, 3, \ldots$ which terminate each time all variables have been set in a position p.

At $\alpha \approx 3.67$ the curves develop vertical slopes: explosion of unit clauses.

Proposition: Let $\alpha_{\rm UC}^{\rm coupled}(K) \equiv \lim_{w\to\infty}\lim_{L\to\infty}\alpha_{\rm UC}^{\rm coupled}(K,L,w)$

K	3	4	...	large K
$\alpha_{{\rm UC}(K)}$	2.67	4.50	...	$\frac{e}{K}2^{K-1}$
$\alpha_{\rm UC}^{\rm coupled}(K)$	3.67	7.81	...	$2^{K-1}+\cdots$

Exact formula:

$$\alpha_{\rm UC}^{\rm coupled}(K) = \max\{\alpha \geq 0 | \min_{\ell\in[0,2]} \Phi_{\alpha,K}(\ell)\}$$

with

$$\Phi_{\alpha,K}(\ell) = 2 - \ell(1-\frac{\ln\ell}{2}) - \frac{\alpha}{2^{K-2}}(1-\frac{\ell}{2})^K$$

Differential Equations for Coupled-UC

Phase p ($i \geq p$). Round $\equiv$ free step followed by forced steps.

$$\frac{d\ell_i(t)}{dt} \equiv -2\beta_i(t) = -2 \text{ rate of removal of nodes at pos } i$$

$$\begin{cases} \frac{dc_i^{(3)}(t,\vec{\tau})}{dt} = -2\sum_{d=0}^{w-1} \beta_{i+d}(t)\frac{\tau_d c_i^{(3)}(t,\vec{\tau})}{\ell_{i+d}(t)} \\ \frac{dc_i^{(2)}(t,\vec{\tau})}{dt} = -2\sum_{d=0}^{w-1} \beta_{i+d}(t)\frac{\tau_d c_i^{(2)}(t,\vec{\tau})}{\ell_{i+d}(t)} + \sum_{d=0}^{w-1}(1+\tau_d)\beta_{i+d}(t)\frac{c_i^{(3)}(t,\vec{\tau}^d)}{\ell_{i+d}(t)} \end{cases}$$

Conclusion

- Lower bounds for CSP's by algorithmic lower bounds on coupled-CSP's.
- Applies to many problems: K-SAT, COL, XORSAT, Error Correcting LDPC codes, Rate-Distortion theory.
- For XORSAT and Error Correcting codes it gives optimal lower bounds $\alpha_{\rm alg} < \alpha_{\rm coupled-alg} = \alpha_s$.
- For SAT, COL, can we perform better with more sophisticated local rule instead of free step ?
- Above some K we find that $\alpha_{\rm UC}^{\rm coupled} > \alpha_{\rm dyn}^{\rm uncoupled}$.
- Sometimes we go above condensation threshold. E.g coloring with $Q \geq 4$.

Summary

Spatial coupling can be used in two different ways.

Algorithmic: spatially coupled graphs are particularly suited for message passing

Proof technique: extend problem to spatially coupled version
proof desired property for this version
show that original problem is equivalent to spatially coupled
with respect to this property;