LP Decoding Corrects a Constant Fraction of Errors

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Abstract — **We show that for low-density parity-check (LDPC) codes with sufficient expansion, the *Linear Programming (LP) Decoder* corrects a constant fraction of errors.**

I. Introduction

The *Linear Programming (LP) Decoder* [2, 3, 4, 6] provides an alternative to message-passing decoding that is more amenable to finite-length analysis. The pseudocodewords for an LP decoder are the vertices of a linear polytope whose constraints depend on the structure of the code, and they unify many known notions of pseudocodewords for various codes and decoders (see [2]). For the case of LDPC codes, Koetter and Vontobel [7, 8] described these LP pseudocodewords as "graph covers," and established a connection to the pseudocodewords of the message-passing "min-sum" algorithm.

In this paper, we show that LDPC codes using LP decoders can correct up to a constant fraction of error, which implies WER $\leq e^{-\Omega(n)}$ in the BSC. This is the first proof that LP decoding has an inverse-exponential WER on a constant rate code. As far as we are aware, no such bound is known for message-passing decoders such as min-sum and sum-product (belief propagation) on finite-length LDPC codes.

Our main theorem is given below, where a Tanner graph G is a (k, Δ)-*expander* if for all sets S of variable nodes where $|S| \leq k$, at least $\Delta|S|$ check nodes are incident to S:

Theorem 1 *Let $\mathcal{C}$ be a low-density parity-check code with length n and rate at least $1-m/n$ described by a Tanner graph G with n variable nodes, m check nodes, and regular left degree c. Suppose G is an $(\alpha n, \delta c)$-expander, where $\delta > 2/3 + 1/(3c)$ and δc is an integer. Then the LP decoder succeeds, as long at most $\frac{3\delta-2}{2\delta-1}(\alpha n - 1)$ bits are flipped by the channel.*

This result matches Sipser and Spielman [9] for the case $\delta = 3/4$. Random Tanner graphs will meet the conditions of this theorem with high probability, and recent work by Capalbo *et al.* [1] gives efficient deterministic constructions of such graphs.

II. Using a Dual Witness to Prove Success

We use the LP decoder from [6], which decodes any LDPC code described by a bipartite Tanner graph $G = (V \cup C, E)$, where codewords are settings of bits y_i to nodes $i \in V$ s.t. the neighborhood $N(j)$ of every check $j \in C$ has even parity. For each code bit i, let $\gamma_i = +1/-1$ if a 0/1 is received. We assume that the codeword 0^n is sent, an assumption that can be made without loss of generality (see [5]).

The decoder in [6] solves a particular LP relaxation of the ML decoding problem. If the solution y is integral (in $\{0,1\}^n$), it must be the ML codeword, and so it is output; otherwise the decoder declares an error. Decoding succeeds if 0^n is the unique optimal LP solution. To prove decoding success, it suffices to exhibit a **dual witness**: a solution to the dual LP with value zero. This notion leads to the following proposition, in which the dual variables play the role of edge weights:

Proposition 2 *A setting of weights τ_{ij} to every edge $(i, j) \in E$ is* feasible *if* **(i)** *for all checks $j \in C$ and distinct $i, i' \in N(j)$, we have $\tau_{ij} + \tau_{i'j} \geq 0$, and* **(ii)** *for all nodes $i \in V$, we have $\sum_{j \in N(i)} \tau_{ij} < \gamma_i$. If there is a feasible setting of edge weights, then LP decoding succeeds.*

To prove Theorem 1, we apply expansion to a portion of the graph around where errors occur. This allows us to set edge weights τ_{ij} to satisfy the conditions of Proposition 2. The construction goes through as long as the number of errors in the channel is at most $\frac{3\delta-2}{2\delta-1}(\alpha n - 1)$.

III. Conclusions

Our "dual witness" technique applies to any LP decoder, and it would be interesting to see it used in different settings. A full version with proofs can be found in the technical report [5].

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