Pushouts, Pullbacks and Their Properties

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Abstract

Graph rewriting has numerous applications, such as software engineering and biology techniques. This technique is theoretically based on pushouts and pullbacks, which are involved with given categories. This paper deals with the definition of pushout and pullback, and their properties.

1 Introduction

Descriptions of biological systems are largely involved with graph rewriting techniques, and it has been used shortly after its invention. Formally, a graph rewriting system usually consists of a set of graph rewriting rules of the form $L \to R$, with L being called pattern graph and R being called replacement graph. A graph rewriting rule is applied to L, by searching for an occurrence of the pattern graph and by replacing the found occurrence by an instance of the replacement graph, and finally we get R. [2]

There are several approaches to deal with graph rewritings, and the most well-known technique is the algebraic approach, which is based on category theory. The algebraic approach is divided into some substructures - one is called the double-pushout approaches(DPO) and the other is called the single-pushout approach(SPO).

This paper deals with the pushouts and pullbacks, the basic of above two approaches, and will not cover two approaches in detail. We first give some basic preliminaries in order to understand what pushouts and pullbacks are and definitions of them. Then we give some properties to understand the essense of two categorical structures. We note that contents of this paper is largely based on [1].

Overview In section 2, we first give some preliminaries for understanding pushouts and pullbacks, including the basics of category theories. And we give formal definitions and universal properties of pushouts and pullbacks in section 3. We also give some properties of pushouts and pullbacks in section 4, for a better understanding of them.

2 Preliminaries

In this section, we give some preliminaries for understanding pushouts and pullbacks. We assume that readers have a general knowledge of following concepts:

- R-modules, module homomorphisms and isomorphisms
- Kernels and cokernels of modules
- Exact sequences and short exact sequences

2.1 Categories

First of all, we define basic notions of category.

Definition 1 (category)**.** *A* **category** $\mathcal{C}$ *consists of three ingredients: a class* $\mathrm{obj}(\mathcal{C})$ *of* **objects***, a set of* **morphisms** $\mathrm{Hom}(A, B)$ *for every ordered pair* (A, B) *of objects, and* **composition** $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$*, denoted by*

$$(f, g) \mapsto gf,$$

for every ordered triple A, B, C *of objects. (We write* $f : A \to B$ *instead of* $f \in \mathrm{Hom}(A, B)$*.) These ingredients are subject to the following axioms:*

- *the Hom sets are pairwise disjoint; that is, each* $f \in \mathrm{Hom}(A, B)$ *has a unique* **domain** A *and a unique* **target** B*;*
- *for each object* A*, there is an* **identity morphism** $1_A \in \mathrm{Hom}(A, A)$ *such that* $f1_A = f$ *and* $1_A f = f$ *for all* $f : A \mapsto B$*;*
- *composition is associative: given morphisms*

  $$f : A \to B, g : B \to C, h : C \to D,$$

  we get $h(gf) = (hg)f$*.*

We now introduce ${}_R\mathbf{Mod}$, which is the category of all right R-modules. This category is the basis of pushouts and pullbacks, which will be defined later.

Definition 2 (left R-modules)**.** *The category* ${}_R\mathbf{Mod}$ *of all* **left** R**-modules** *(where* R *is a ring) has as its objects all* **left** R**-modules***, as its morphisms all* R*-homomorphisms, and as its composition the usual composition of functions. We denote the sets* $\mathrm{Hom}(A, B)$ *in* ${}_R\mathbf{Mod}$ *by*

$$\mathrm{Hom}_R(A, B).$$

We conclude this subsection by defining subcategory, in order to define isomorphisms, functors, and universal properties in later subsections.

Definition 3 (subcategory)**.** *A category* $\mathcal{S}$ *is a* **subcategory** *of a category* $\mathcal{C}$ *if they satisfy followings:*

- $\mathrm{obj}(\mathcal{S}) \subseteq \mathrm{obj}(\mathcal{C})$

- $\mathrm{Hom}_{\mathcal{S}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$ *for all* $A, B \in \mathcal{S}$.

If $\mathrm{Hom}_{\mathcal{S}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$, $\mathcal{S}$ *is called a* **full subcategory** *of a category* $\mathcal{C}$.

2.2 Functors

In this section, we give the definition of functors of categories.

Definition 4 (functor)**.** *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are categories, then a* **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ *consists of the following functions:*

$$\begin{aligned} F : \mathrm{obj}(\mathcal{C}) &\to \mathrm{obj}(\mathcal{D}) \\ F : \mathrm{Hom}_{\mathcal{C}}(A,B) &\to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),\ (A, B \in \mathcal{C}) \end{aligned}$$

satisfying

- $F(1_A) = 1_A$ *for all* $A \in \mathcal{C}$
- $F(gf) = F(g)F(f)$ *for all* $f \in \mathrm{Hom}_{\mathcal{C}}(A,B), g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$.

If $F(gf) = F(f)F(g)$ *for all* $f \in \mathrm{Hom}_{\mathcal{C}}(A,B), g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$*, then* F *is called a* **contravariant functor**.

Definition 5 (isomorphism of functors)**.** *A morphism* $f : A \to B$ *in a category* $\mathcal{C}$*is an* **isomorphism** *if there exists a morphism* $g : B \to A$ *in* $\mathcal{C}$*with* $gf = 1_A$ *and* $fg = 1_B$.
The morphism g *is called the* **inverse** *of* f.

2.3 Universal Objects

In this section, we give the definitions of initial and terminal objects. Existence of universal object is one of significant properties of a given category.

Definition 6 (universal object)**.** $U \in \mathcal{C}$ *is called the* **initial object***(universally repelling object) in* $\mathcal{C}$ *if* $|\mathrm{Hom}_{\mathcal{C}}(U,A)| = 1$ *for all* $A \in \mathcal{C}$. $U \in \mathcal{C}$ *is called the* **terminal object** *(universally attracting object) in* $\mathcal{C}$ *if* $|\mathrm{Hom}_{\mathcal{C}}(A,U)| = 1$ *for all* $A \in \mathcal{C}$. *Both initial object and terminal object are called the* **universal object**.

Following lemma explains the uniqueness of universal object in a given category. Note that uniqueness does not mean existence - universal object may not exist in some categories.

Lemma 7. *If the initial object exists, then it is unique up to isomorphism. Similarly, if the terminal object exists, then it is unique up to isomorphism.*

Proof. Let U, V be two initial object in $\mathcal{C}$. Then since $|\mathrm{Hom}_{\mathcal{C}}(U,V)| = 1$ and $|\mathrm{Hom}_{\mathcal{C}}(V,U)| = 1$, there exist morphisms $f : U \to V$ and $g : V \to U$, and they are the unique morphisms between U and V. Then it indicates that $fg = 1_U$ and $gf = 1_V$ since $|\mathrm{Hom}_{\mathcal{C}}(U,U)| = 1$ and $|\mathrm{Hom}_{\mathcal{C}}(V,V)| = 1$.

Now we can conclude that U and V are isomorphic in $\mathcal{C}$. Thus the initial object is unique up to isomorphism. For a terminal object, we can prove it similarly. $\square$

3 Pushouts and Pullbacks

In this section, we define pushouts and pullbacks.

Definition 8 (pushout)**.** *Given two morphisms $f : A \to B$ and $g : A \to C$ in a given category $\mathcal{C}$, a* **pushout***(or* **fibered sum***) is a triple (D, α, β) with $\beta g = \alpha f$ that satisfies the following universal property: for every triple (Y, α', β') with $\beta' g = \alpha' f$, there exists a unique morphism $\theta : D \to Y$ making the diagram commute. The pushout is often denoted by $B \cup_A C$.*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\beta"'] \arrow[ddr, "\beta'"] & \\ B \arrow[r, "\alpha"] \arrow[drr, "\alpha'"'] & D \arrow[dr, dashed, "\theta"] & \\ & & Y \end{tikzcd}$$

Figure 1: Pushout

Figure 1 represents the diagram of pushout and its universal property. For pullbacks, we just need to know following definition.

Definition 9 (pullback)**.** **pullback** *is the dual notion of a pushout.*

We close this section after proving that the pushout exists in ${}_R\mathbf{Mod}$.

Theorem 10. *The pushout of two maps $f : A \to B$ and $g : A \to C$ in ${}_R\mathbf{Mod}$ exists.*

Proof. Let $S = \{(f(a), -g(a)) \in B \oplus C : a \in A\}$, then it is easy to see that S is a submodule of $B \oplus C$. Now let $D = (B \oplus C)/S$, $\alpha : B \to D$ by $b \mapsto (b, 0) + S$, $\beta : C \to D$ by $c \mapsto (0, c) + S$. Then with simple calculation, we see that $\beta g = \alpha f$. Thus for all $a \in A$, we get $\alpha f a - \beta g a = (fa, -ga) + S = S$. Now given another triple (X, α', β') with $\beta' g = \alpha' f$, let $\theta : D \to X$ by $(b, c) + S \mapsto \alpha'(b) + \beta'(c)$. Then it is easy to see that θ is unique. $\square$

4 Properties

In this section, we give some properties of pushouts(pullbacks). First example is that we can get the pushout from the inclusion maps.

Proposition 11. *If B and C are submodules of a left R-module U, there are inclusions $f : B \cap C \to B$ and $g : B \cap C \to C$. In this case, the pushout D exists in ${}_R\mathbf{Mod}$ and $D = B + C$.*

Proof. By Theorem 10, we just need to show that $(B \oplus C)/S$ is isomorphic to $B + C$, where $S = \{(a, -a) : a \in B \cap C\}$. Let $\phi : (B \oplus C)/S \to B + C$ by $(b, c) + S \mapsto b + c$. If $(b, c) + S = (b', c') + S$, then there exists $a' \in B \cap C$ such that $(b, c) = (b' + a, c' - a)$. Then we get $b + c = b' + a + c' - a = b' + c'$, which means that ϕ is well-defined. For injectivity, if $b + c = b' + c'$ in $B + C$, we define $a' = b' - b$. Then it is easy to prove that $(b, c) + S = (b', c') + S$. For surjectivity, it is obvious since $\phi((b, c) + S) = b + c$. $\square$

Second one states that how cokernels are involved with pushouts. Since the proof of the following proposition is almost similar to the previous proposition, we omit the proof of it.

Proposition 12. *If $f : A \to B$ is a homomorphism in ${}_R\mathbf{Mod}$, then cokernel of f, coker f is the pushout of the following diagram.*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "0"'] & B \\
0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "0"'] & B \arrow[d] \arrow[ddr] & \\
0 \arrow[r] \arrow[drr] & \operatorname{coker} f \arrow[dr, dashed, "\theta"] & \\
& & X
\end{tikzcd}
$$

Figure 2: Pushouts and cokernels

The last property in this paper is about the relationship between pushouts and short exact sequences.

Proposition 13. *Suppose we have following two short exact sequences in ${}_R\mathbf{Mod}$ with an assumption that all diagrams are commutative:*

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "\beta"] & P \arrow[r, "\sigma"] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & B \arrow[r, "j"'] & X \arrow[r, "\tau"'] & A \arrow[r] & 0.
\end{tikzcd}
$$

Figure 3: Pushouts and sequences

Then X is the pushout of B and P with respect to i and β.

Proof. First, from the definition of short exact sequence, it is easy to see that $X = \alpha(P) + j(B)$. And since all diagrams are commutative, we get $j\beta = \alpha i$.
Now we prove the universal property. Suppose G is an R-module with homomorphisms $\alpha' : P \to G$ and $j' : B \to G$ such that $\alpha' i = j'\beta$. Let $\phi : X \to G$ by $\alpha(p) + j(b) \mapsto \alpha'(p) + j'(b)$. To see that ϕ is well-defined, we just need to show that if $\alpha(p) + j(b) = 0$ then $\alpha'(p) + j'(b) = 0$. Let

$\alpha(p) + j(b) = 0$, then $0 = \tau(\alpha(p)) = \sigma(p)$. Thus $p = i(m)$ for some $m \in M$. Thus we get $0 = \alpha(i(m)) + j(b) = j(\beta(m)) + j(b)$. Since j is injective, $\beta(m) + b = 0$. Thus we get

$$\alpha'(p) + j'(b) = \alpha'(i(m)) + j'(-\beta(m)) = (\alpha' i - j'\beta)(m) = 0.$$

Therefore, ϕ is well-defined. Now we get $\phi\alpha = \alpha'$ and $\phi j = j'$ by setting $b = 0$ and $p = 0$ respectively. So the only thing that we need to show is its uniqueness. if $\psi : X \to G$ satisfies the universal property, we get $\psi\alpha = \alpha'$ and $\psi\beta = \beta'$, then

$$\psi(\alpha(p) + j(b)) = \psi(\alpha(p)) + \psi(j(b)) = \alpha'(p) + j'(b) = \phi(\alpha(p) + j(b)).$$

This shows the uniqueness of ϕ. Note that obviously ϕ is a homomorphism due to its well-definedness. $\square$

References

[1] J.J. Rotman. *An Introduction to Homological Algebra*. Universitext - Springer-Verlag. Springer Science+Business Media, LLC, 2009.

[2] Grzegorz Rozenberg, editor. *Handbook of graph grammars and computing by graph transformation: volume I. foundations*. World Scientific Publishing Co., Inc., River Edge, NJ, USA, 1997.