The Physics of Flying Discs

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December 13, 2002

Abstract

I hope to clearly explain the physics of flying discs (commonly called Frisbees). After discussing the basic physics behind the problem, we will explore a few interesting details using ideas from class.

1 Introduction

Have you ever wondered how a Frisbee flies? The first time that I pondered the problem was when I was still a child. Unfortunately, none of the people I asked knew the answer. Someone must have suggested that the spinning of the disc somehow causes it to hover, for that is what I have believed ever since then. Something along the lines of how UFOs fly, I thought to my self. I dismissed the problem as something I was unable to fully understand with the small amount of knowledge I had at the time. My physics knowledge, I dare say, has increased somewhat since my childhood. After spending a semester taking a class on Fluid Mechanics, I decided it was high time I solved the problem in detail. Aided by a short Scientific American article [1], I found to my surprise (and slight disappointment) that the basic explanation is really quite simple.

The history of the modern Frisbee [2] dates back to 1871 when William Russell Frisbie bought a bakery in Bridgeport, Connecticut and renamed it the Frisbie Pie Company. According to the popular theory of the origin of the Frisbee, the pies were popular at nearby Yale University, where students would toss the empty pie tins around campus. Fred Morrison was the man behind the first production of plastic flying discs, and was awarded the "flying disc" patent in 1958. The plastic flying discs were not popular until the Wham-O company decided to trademark the name

"Frisbee"; one of the heads of the company supposedly heard some Ivy League students (presumably at Harvard) using the terms "frisbie" and "frisbie-ing" to describe the sport of throwing the Frisbie pie tins. The first Frisbees were produced in 1959 during America's UFO craze.

In order to explain the physics of flying discs, we will first review some basics that we will need in our discussions. The two major ideas are (using terms from popular science) "Gyroscopic Inertia" and the "Bernoulli Principle". In Section 3 we will put these ideas together to describe how Frisbees fly. Following an argument of Shapiro [4], we will then be able to explore the effect of air turbulence on the flight of a flying disc. Finally, some variations on the basic design will be analyzed to illustrate and affirm the principles governing the problem.

2 A Review of Some Basics

The two basic ideas that we will need are "Gyroscopic Inertia" and the "Bernoulli Principle". We are somewhat familiar with both of these concepts, but I plan on expanding on some aspects which may be interesting. "Gyroscopic Inertia" is just a fancy-sounding term for what scientists call angular momentum. We will delve into some aspects of angular momentum that are not usually covered in an introductory physics class. The "Bernoulli Principle" is a result we studied in our fluid mechanics class. We will review the idea and discuss some misconceptions about the workings of aerodynamic lift.

2.1 Gyroscopic Inertia

"Gyroscopic Inertia" refers to the resistance one encounters when trying to change the axis of rotation of a rotating body. We will show that this simply equals the magnitude of the angular momentum. Let us recall how angular momentum is defined. In rotational mechanics, there is always a special reference point about which objects are considered to rotate. In the following discussion, we use the origin as our reference point. If a point particle at distance $\mathbf{x}$ from the origin has linear momentum $\mathbf{p}$, then its angular momentum is defined as

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}\,. \tag{1}$$

Rotational mechanics becomes more interesting when we consider the case of a solid body. Suppose we have a solid body rotating about the origin with angular velocity

$\boldsymbol{\omega}$. This means that at a point $\mathbf{x}$, the body is moving with velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$. A differential mass dm of the body thus contributes a differential angular momentum of

$$d\mathbf{L} = dm\ \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x})\,. \tag{2}$$

We can expand the right-hand side using index notation, as follows.

$$\begin{aligned}
\mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x}) &= \epsilon_{ijl} x_j \epsilon_{lmk} \omega_m x_k \\
&= (\epsilon_{ijl}\epsilon_{lmk})\, x_j x_k \omega_m \\
&= (\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm})\, x_j x_k \omega_m \\
&= x_k x_k \omega_i - x_i x_j \omega_j \\
&= \delta_{ij} x_k x_k \omega_j - x_i x_j \omega_j\,.
\end{aligned} \tag{3}$$

Now we integrate Equation (2) and obtain

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}\,, \tag{4}$$

where $\mathbf{I}$ is called the inertia tensor [3], and is given by the following integral over the entire mass of the body:

$$I_{ij} = \int dm\ (\delta_{ij} x_k x_k - x_i x_j)\,. \tag{5}$$

The inertia tensor is dependent both on the mass distribution and the orientation of the body. Equation (4) is the generalization of the more familiar equation $\mathbf{L} = I\boldsymbol{\omega}$, where I is a scalar value called the moment of inertia about the axis of rotation. The inertia tensor contains information about the moments of inertia, as well as cross terms called the products of inertia. We now use this generalized picture to understand "gyroscopic inertia".

The torque $\boldsymbol{\tau}$ on a particle is defined as $\mathbf{x} \times \mathbf{F}$, where $\mathbf{x}$ is the position of the particle with respect to the origin and $\mathbf{F}$ is the force acting on it. The effect of the torque is the time rate of change of the angular momentum. In other words,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}\,. \tag{6}$$

For a rotating body, the angular momentum $\mathbf{L}$ is given by Equation (4). Thus the torque can be written as

$$\boldsymbol{\tau} = \mathbf{I} \cdot \frac{d\boldsymbol{\omega}}{dt} + \frac{d\mathbf{I}}{dt} \cdot \boldsymbol{\omega}\,. \tag{7}$$

Here we have used the product rule. By applying a torque, the angular velocity $\boldsymbol{\omega}$ may change. But we must realize that the inertia tensor $\mathbf{I}$ may also change. The first term corresponds to the torque resulting in a change in the angular velocity. If the object is initially stationary, it will start spinning along the torque vector. If the object is already spinning, and the torque is applied along the axis of rotation, the object will spin up or spin down. The second term corresponds to a change in the orientation of the object with respect to the coordinate system. In order to understand "gyroscopic inertia", however, we must consider a special case.

Consider an object spinning rapidly along an axis for which the moment of inertia is a maximum, and consider a torque applied *perpendicular* to the angular momentum. According to the first term in Equation (7), the angular velocity might change. But since the torque is perpendicular to the axis of rotation, the *magnitude* of the angular velocity does not change (in the limit that the initial angular velocity is high). Let's suppose that the change in the direction of the angular velocity vector changes the orientation between it and the inertia tensor. The object would then be rotating about a different axis of rotation with a different (and thus smaller) moment of inertia. The rotation kinetic energy (given by $T = \frac{1}{2}I\omega^2$) would be smaller. Invoking conservation of energy, we see this cannot happen. The only possibility is that the object continues to spin about the same axis, such that the magnitude of $\mathbf{I} \cdot \boldsymbol{\omega}$ is constant. Again we note that this only holds when there is a large angular velocity. In this case we may write

$$\mathbf{L} = I\omega\hat{\boldsymbol{\omega}}\,, \tag{8}$$

where I is the particular (maximum) moment of inertia of the problem, ω is the magnitude of the angular velocity, and $\hat{\boldsymbol{\omega}}$ is the axis of rotation. From Equation (5) we see that the moment of inertia about a particular axis can be calculated as

$$I = \int dm\; r^2\,, \tag{9}$$

where r is the distance to the axis of rotation and the integral is over the mass of the body. Since the axis of rotation is the only changing component of the angular momentum, we may write the torque as

$$\boldsymbol{\tau} = L\frac{d\hat{\boldsymbol{\omega}}}{dt}\,, \tag{10}$$

where L is the magnitude of the angular momentum. Since it is the ratio between torque and the rate of change of the rotation axis, we may also call it the "gyroscopic inertia", the quantity we set out to find. Note, however, that the direction in which the axis of rotation changes is not intuitive. The axis of rotation will not rotate according to the applied torque, as one might expect, but will move in the direction of the torque vector. This explains the phenomenon of precession (e.g. a spinning top acted on by gravity).

2.2 The Bernoulli Principle

The Bernoulli Principle is an idea which is poorly understood by many people. In this discussion, we will review the derivation of the Bernoulli Theorem and try to understand the reasons for the common misconceptions.

Recall the Navier-Stokes equations:

$$\rho\,(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p + \mathbf{f} + \mu\nabla^2\mathbf{u}\,, \tag{11}$$

$$\nabla\cdot\mathbf{u} = 0\,. \tag{12}$$

In these equations, ρ is the density of the fluid, $\mathbf{u}$ is the velocity of the fluid, p is the pressure, $\mathbf{f}$ is a body force, and μ is the dynamic viscosity of the fluid. For Bernoulli's Theorem to hold, we consider a steady state problem in the limit of high Reynolds number in which viscosity is negligible. Equation (11) becomes

$$\rho\,\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{f}\,. \tag{13}$$

Furthermore, let us suppose that the body force is conservative (e.g. gravity) so that $\mathbf{f} = -\nabla\psi$ for some scalar field ψ. Also, note the following:

$$\begin{aligned}
\mathbf{u}\times(\nabla\times\mathbf{u}) &= \epsilon_{ijl}u_j\epsilon_{lmk}\partial_m u_k \\
&= (\epsilon_{ijl}\epsilon_{lmk})\,u_j\partial_m u_k \\
&= (\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm})\,u_j\partial_m u_k \\
&= u_k\partial_i u_k - u_j\partial_j u_i \\
&= \tfrac{1}{2}\partial_i(u_k u_k) - u_j\partial_j u_i \\
&= \nabla(\tfrac{1}{2}u^2) - \mathbf{u}\cdot\nabla\mathbf{u}\,.
\end{aligned} \tag{14}$$

Making the appropriate substitutions, Equation (13) becomes

$$\nabla(\tfrac{1}{2}\rho u^2) - \rho\;\mathbf{u}\times(\nabla\times\mathbf{u}) = -\nabla(p+\psi)\,. \tag{15}$$

Taking the dot product with $\mathbf{u}$ gives

$$\mathbf{u} \cdot \nabla(\tfrac{1}{2}\rho u^2 + p + \psi) = 0\,. \tag{16}$$

The expression $\mathbf{u} \cdot \nabla$ is the gradient projected onto the direction of the flow. We have obtained Bernoulli's Theorem, which states that the quantity

$$H \equiv \tfrac{1}{2}\rho u^2 + p + \psi \tag{17}$$

is constant along streamlines. For systems in which the body forces are negligible or irrelevant, this means that regions of higher velocity are regions of low pressure, and vice versa. This is usually what is called the Bernoulli Principle.

As an application of the Bernoulli Principle, we will now discuss how an airplane wing generates lift. The most common explanation proceeds as follows: the airplane's wing is curved on the top surface and relatively flat on the bottom surface. When air travels over the top of the wing, it has to travel a larger distance than if the air were to travel along the bottom of the wing. The air thus travels faster on top of the wing. Now Bernoulli's Principle is invoked: since the air travels faster on top of the wing, the pressure is greater below the wing than above the wing. This creates lift.

This is the explanation most often given to schoolchildren and the general masses. It is a terrible explanation! Just because it has a larger distance to travel, the air on the top of the wing does not necessarily have to travel faster. Invoking the Bernoulli Principle, unfamiliar to most people, is also not very convincing. There a couple alternative explanations I have heard that could fare better.

The first asks the question: "Why does a fluid follow the contour of a (smooth) surface?" If a surface is "in the way" of the stream (e.g. a concave surface), a region of slightly higher pressure forms and diverts the fluid away. If the surface curves away, then the fluid cannot continue straight, since a vacuum would form in the neglected space; what actually happens is that a region of slightly lower pressure is formed, and the fluid is pulled towards the surface. In an airplane wing, the top of the wing is curved away, and the bottom of the wing is not; thus the lower pressure on top of the wing creates lift.

The second explanation is even further from the classic description. Given that a fluid follows the contour of a surface, we take a look at where the air is diverted. Air traveling under the wing is diverted downwards, since the tail-end of the wing is curved downwards. Air traveling over the wing is also diverted downwards, for the same reason.[1] We see that the wing pushes the air down. Thus the air must be pushing the wing up.

The airplane wing is a good example of how lift can be generated by air flow. Regardless of the explanation, we see that the convex part of a surface corresponds to the region of low pressure. We will now apply this idea to Frisbee flight.

3 How Frisbees Fly

We finally discuss how the Frisbee actually flies. We combine the ideas in the previous section to form the basic explanation for Frisbee flight. We will then discuss how turbulence plays a role.

3.1 The Basic Idea

A Frisbee is basically a plastic disc that is curled down at the edges. The point to note is that the cross section of a Frisbee roughly approximates the cross section of an airplane wing. Since the top surface is convex, the pressure on top of the disc will be less than the pressure below the disc. Thus the Frisbee will experience lift. But a Frisbee that isn't spinning will not fly very far. Because of how the Frisbee is shaped, the pressure distribution will not be symmetric with respect to the front and back of the Frisbee.[2] For example, if the point of highest pressure (on the bottom side of the disc) and the point of lowest pressure (on the top surface) are near the back of the disc, there will be a sizable torque. The disc will flip over and fall to the ground without having flown very far.

A Frisbee, of course, is usually spun. The moment of inertia is greatest when the axis of rotation is the axis of symmetry. Thus we know from Section 2.1 that a Frisbee rotating about its axis of symmetry tends to remain rotating about its axis of symmetry. Now that the Frisbee is stable, it won't flip over when flying through the air. The curved shape will provide the disc with lift, and it will fly a considerable distance.

Equation (10) allows us to make a couple observations about Frisbee flight. First of all, the higher the rate of spin ω, the more stable the disc; a Frisbee can never

[1] The reason that the surface is curved is to keep the wing streamlined to avoid turbulence. We will discuss streamlining in a later section.

[2] This could also depend on the angle of attack of the Frisbee.

be spun too fast. The second observation is more interesting. Suppose the torque we discussed earlier is strong enough to affect the Frisbee. If we are the thrower of the Frisbee, the torque vector points to the left. If we suppose that the disc is thrown backhanded with the right arm, it spins clockwise viewed from the top; the rotation vector points down. According to Equation (10), the rotation vector will move towards the torque vector. Thus the disc will bank to the right. Moreover, this happens at a constant rate. If the Frisbee is spun counterclockwise, the rotation vector would initially point up, and the disc would bank to the left. This is actually what is observed for some flying discs. From this we can conclude that the torque is indeed as we supposed; the high and low pressure points must be situated towards the back of the disc.

3.2 The Effect of Turbulence

So far we have seen a somewhat simplified description of the problem. There is one major difference between the cross section of an airplane wing and the cross section of a Frisbee. The airplane wing is streamlined, whereas the Frisbee is far from it. This means that turbulent flow is much more prevalent in a Frisbee than in an airplane wing.

In high Reynolds number flow, the fluid velocity far from the surface is negligibly affected by viscosity. This is what we assumed in our discussion on Bernoulli's Theorem (Section 2.2). The velocity of the fluid is zero at both the head and the tail; it is a maximum somewhere in between. This means that the pressure is highest at the head and tail, and minimum at some point in between. There is, however, a thin boundary layer near the surface, in which the velocity drops to zero at the surface [4]. This viscous boundary layer is the only source of drag for a streamlined object such as an airplane wing. Since the pressure is the same at the head and the tail, there is very little pressure contribution to the drag. This picture changes dramatically for a object which is not streamlined.

A blunt (i.e. not streamlined) object causes a turbulent wake through boundary-layer separation.[3] Here we follow the logic of Shapiro [4] in describing how boundary-layer separation is formed. Fluid in the boundary layer is acted upon by a number of forces. One of these is the pressure gradient. Fluid particles in the boundary layer gain momentum going from the point of maximum pressure at the head to the

[3] In fact, the definition of a streamlined object is one that lacks boundary-layer separation.

point of minimum pressure. But the particles are also losing momentum at some rate due to viscosity. During the latter half of the journey, the pressure gradient is in the opposite direction. The boundary layer must travel from the region of lowest pressure to the region of highest pressure at the tail. At this rate, the boundary layer will not make it to the tail! Fortunately, the flow outside of the boundary layer also acts on the boundary layer. For a streamlined object, enough momentum is imparted by the outside fluid that the fluid in the boundary layer reaches the tail. If the object is not streamlined, however, the force due to the outside flow is not strong enough; at some point, the fluid in the boundary layer loses all of its momentum and stalls. The flow along the surface beyond this point moves in the *opposite* direction, and causes the separation of the boundary layer. The overall result is a large-scale turbulent wake. Intuitively, producing such a large energetic wake means that the drag on the object is a lot larger. Moreover, a faster fluid speed causes an earlier boundary-layer separation, a larger wake, and a larger drag.

A Frisbee is far from being streamlined. How does it deviate from a streamlined airplane wing? The top surface is relatively similar. The "head" is curled like an airplane wing. But the important difference is the shape of the "tail". In a streamlined shape, the tail tapers off slowly. This is important, since this is the region in which momentum must be transferred from the outside flow to the boundary layer. The surface must not suddenly curve away. But this is exactly what happens in a Frisbee! The boundary layer is bound to separate when the surface curves away at the end. This, however, is not nearly as bad as the shape of the bottom surface. Boundary layer separation most likely happens right at the front edge of the disc. The entire bottom of the disc will thus contain turbulent flow, causing some contribution to the drag.

As the fluid speed is increased, at some point the drag on an object will actually drop. This is due to a transition from a laminar boundary flow to a turbulent boundary flow. In turbulent boundary flow, there is more mixing between the boundary layer and the outside flow. More momentum is transferred from the outside to the boundary, and the boundary-layer separation is delayed. The decrease in pressure drag from the smaller wake more than makes up for the increased viscous drag caused by the turbulence in the boundary layer. According to Daish [5], this transition occurs around a Reynolds number of 200,000 for a smooth sphere. The significant drop in drag can, however, be *induced* at Reynolds numbers below this critical number by a roughness of the surface. This is the reason why golf balls have dimples and why tennis balls are fuzzy. In both cases, the Reynolds numbers are lower than the critical value but not significantly so.

For a Frisbee with a characteristic size of $a = 20\,\text{cm}$ flying in air at a speed of $U = 10\,\text{m/s}$, the Reynolds number is about

$$\frac{Ua}{\nu} = \frac{(1000\,\text{cm/s})(20\,\text{cm})}{(0.15\,\text{cm}^2/\text{s})} \approx 130,000\,.$$

It is true that a Frisbee is not a sphere, but we see that the Reynolds number is in the right range that a roughened surface might reduce drag. In fact, the top surface of the curved areas of some Frisbees do have ridges. The purpose of these ridges is probably[4] to induce a turbulent boundary layer and reduce the drag on the disc.

4 Frisbee Design

Now that we understand the physics behind Frisbee flight, we may ponder the possibilities of variations on the basic design.

4.1 The Freebie Frisbee

Flying discs can be produced relatively cheaply because they are made out of plastic. It seems, however, that some discs can be made even more cheaply if they have a constant thickness and sharp bends. I am referring to the cheap Frisbees given to people for free by some corporations and organizations. We will use our understanding of flying discs to find what is better about the more solid and smoothly curved discs (e.g. the Frisbee produced by the Wham-O company) that one must actually purchase to obtain.

One of the differences between the cheap discs and an official Frisbee is that the official versions are heavier and have a different mass distribution. Most of the mass of the Frisbee is located at its edges. According to the definition of the moment of inertia (Equation (9)), this means that the official Frisbee has a larger moment of inertia. Thus at the same rotation rate, the official Frisbee will have a much higher angular momentum. The higher "gyroscopic inertia" will keep the disc much more stable.

Another obvious improvement of the official design is the lack of hard edges. The disc is closer to being streamlined. Its top surface is one continuous curve, whereas

[4]One may also argue that the ridges were simply added for a better grip. . .

the surface of the cheap frisbee has several kinks that may induce boundary-layer separation. The ridged edges of the official design should also delay the boundary-layer separation and minimize the turbulent wake. This smooth flow should also yield a more predictable lift. The underside of both discs are likely to induce turbulent flow, but one expects that the smoother design of the official Frisbee would yield a smaller amount of turbulence. In any case, the official Frisbee is more stable and produces less overall drag, allowing it to fly further than the cheap disc.

4.2 The Long-Distance "Disc"

Frisbees are usually thrown from person to person. The discs are thus designed to have a limiting velocity so that they may be more easily caught. This limiting factor, of course, is the pressure drag from the turbulent flow generated by the disc. What if, however, all we wanted from a disc was that it fly long distances? There are a few factors that we should consider in searching for the best design. The design must minimize the formation of turbulence. It also must still generate enough lift. And we must still have a high moment of inertia for stability.

Generating lift requires a convex upper surface, but this curved surface is also the cause of the turbulence. Keeping the lift seems to require keeping the turbulence. Fortunately, there are a couple of ideas that help fix this problem. The first and most important is to realize that when the disc is traveling at higher speeds, the surface does not need to be as curved to produce the same amount of lift. This is similar to the reason why airplane wings are so complicated. At takeoff and landing, an airplane's wings should provide a large amount of lift. But when cruising at a much higher speed, much of the lift is unnecessary. This is why the wings of a large airplane actually change shape by extending and contracting to produce more and less lift, respectively. We can decrease the curvature of the disc to reduce the drag, and the higher speed should allow the new slimmer design to generate enough lift.

We wish the surface to be as flat as possible to produce the smallest amount of drag. The second way of minimizing the need for lift is to remove some mass from the disc. But this must be done in such a way to keep the moment of inertia large. One of the successful implementations I have seen is a doughnut-shaped disc made out of rubber.[5] By keeping all of the mass away from the center, the moment of inertia remains large, but by removing much of the mass from the center, the lift

[5] Of course, it isn't technically a disc anymore; it is more of a ring.

requirements are lower. The cross section of this new object is essentially two separate surfaces (the sections forward and behind the center). Each surface cross section is a miniature version of the cross section of a Frisbee, but much more streamlined. Since the surface need not produce very much lift, the top surface is only slightly curved. The bottom surface is not caved in but flat; it thus produces no turbulent flow.[6] This object thus experiences very little drag compared to a Frisbee and can travel very large distances. We have thus found a good design for a long distance flying "disc".

5 Conclusions

The basic physics of Frisbee flight turns out to be quite a bit simpler than one might initially expect. It consists of two concepts which are more or less familiar to most people. The first is the gyroscopic stability of rotating objects. Using some new tools such as the inertia tensor, we argued that the "gyroscopic inertia" was in fact simply the magnitude of the angular momentum of the object. Thus the faster a disc rotates, the more stable its flight. The second is aerodynamic lift. We saw a couple arguments as to why lift is produced for air flow around curved surfaces. The combination of gyroscopic stability and aerodynamic lift gives us the simple picture of flying discs. We found, however, that turbulence caused by boundary-layer separation played a large role in the flight of Frisbees. Using these concepts, we were able to discuss some specific details of Frisbee design. As we can see, Fluid mechanics has allowed us to be able to think about Frisbees and understand the physics to quite a level of detail.

[6]There is a flying object with a very similar cross section: a boomerang. A boomerang also travels at high speeds through the air, and thus only needs a slightly curved surface to generate lift. No, I do not know why a boomerang returns to its thrower.

References

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