The Log-Norm Distribution

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Abstract

The Normal can be thought of as a log-squared-vector-length distribution. We detail this connection and discuss a distribution that can be viewed as a log-vector-length distribution.

1 Introduction

The (unnormalized) standard Normal distribution is

$$\exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2}\right). \tag{1}$$

Discarding the factor of two, we get a distribution such that the unnormalized log-density of a point is its negative squared length. Define

$$\tilde{P}_n(\vec{x}) = \exp(-\|\vec{x}\|_2^2), \tag{2}$$

where $\|\vec{x}\|_2^2 = \sum_i x_i^2$ is the L_2-norm of $\vec{x}$. The normalization constant is $\pi^{-n/2}$, so $P_n(\vec{x}) = \pi^{-n/2}\exp(-\|\vec{x}\|_2^2)$ is a distribution [1].

We can similarly define a distribution on (negative) vector lengths. We call this the "log-norm" distribution because the logarithm of the unnormalized density results in the (negative) L_2-norm of the vector,

$$\tilde{Q}_n(\vec{x}) = \exp(-\|\vec{x}\|_2). \tag{3}$$

Note that if $n = 1$, this is a simple exponential distribution and the normalization constant is 1. If $n = 2$, we can use a polar coordinate transformation to find the normalization constant integral,

$$Z_{Q_2} = \int\!\!\int \exp\left(-\sqrt{x_1^2 + x_2^2}\right) dx_1 dx_2 = \int_0^{2\pi}\!\!\int_0^{\infty} \exp(-r)\, r dr d\theta. \tag{4}$$

The inner polar coordinate integral is constant with respect to θ, so $Z_{Q_2} = 2\pi \int_0^{\infty} \exp(-r)\, r dr$. Integration by parts gives us

$$Z_{Q_2} = 2\pi \int_0^{\infty} e^{-r} dr = 2\pi \tag{5}$$

2 Volume of a Hypersphere

Define the radius R n-sphere as the surface defined by all $\vec{x}$ such that $\sum_i x_i^2 = R^2$. Weisstein [3] gives the content (or n-dimensional volume) as

$$V_n(R) = \frac{S_n R^n}{n}, \tag{6}$$

where S_n is the surface area of the unit (radius) n-sphere. Weisstein gives the surface area of the unit n-sphere as

$$S_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}, \tag{7}$$

which can be rewritten (Weisstein) as

$$S_n = \left\{ \begin{array}{ll} \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{(n-2)!!} & \text{for } n \text{ odd} \\ \frac{2\pi^{n/2}}{(\frac{n}{2}-1)!} & \text{for } n \text{ even} \end{array} \right. , \tag{8}$$

where $n!!$ is a double factorial [2].

3 Integrating a Function of Radius

Here we discuss how to integrate a function in n dimensions which only depends on the radius, or L_2 distance from the origin. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ that only depends on the argument's distance from the origin, $f(\vec{x}) = g\left(\sqrt{\sum_i x_i}\right)$, for some function $g : \mathbb{R} \to \mathbb{R}$. We would like to calculate the integral of the function across the entire Euclidean space; we assume that this integral exists. We compose this integral by summing over shells. A "shell" is a volume of space between two radius value. The volume consumed by the shell defined by radii $r_1 < r_2$ is

$$\begin{aligned} \Delta V_n(r_1, r_2) &= V_n(r_2) - V_n(r_1) && (9) \\ &= \frac{S_n}{n}\left(r_2^n - r_1^n\right) && (10) \\ &= \frac{2S_n}{n}\frac{r_2^{n-1} + r_1^{n-1}}{2}(r_2 - r_1). && (11) \end{aligned}$$

Taking the limit as $r_1 \to r_2$, we get

$$dV_n(r) = \frac{2S_n r^{n-1}}{n} dr. \tag{12}$$

So,

$$\int_{\mathbb{R}^n} f(\vec{x}) d\vec{x} = \frac{2S_n}{n} \int_0^\infty g(r) r^{n-1} dr. \tag{13}$$

3.1 The Log-Norm Distribution

For the Log-Norm distribution, $g(r) = e^{-r}$ and this integral is the normalization constant of the distribution. We can use integration by parts to find the normalizer explicitly,

$$Z_{Q_n} = \frac{2S_n}{n}\int_0^\infty e^{-r} r^{n-1} dr = \frac{2S_n}{n}(n-1)!\int_0^\infty e^{-r} dr \tag{14}$$

$$= \frac{2S_n}{n}(n-1)! \tag{15}$$

$$= \begin{cases} \frac{2(n-1)!}{n}\frac{2^{(n+1)/2}\pi^{(n-1)/2}}{(n-2)!!} & \text{for } n \text{ odd} \\ \frac{2(n-1)!}{n}\frac{2\pi^{n/2}}{(\frac{n}{2}-1)!} & \text{for } n \text{ even} \end{cases} \tag{16}$$

References

[1] J. D. M. Rennie. The Gaussian-like normalization constant. http://people.csail.mit.edu/∼jrennie/writing, November 2005.

[2] E. W. Weisstein. Double factorial. http://mathworld.wolfram.com/DoubleFactorial.html. From MathWorld–A Wolfram Web Resource.

[3] E. W. Weisstein. Hypersphere. http://mathworld.wolfram.com/Hypersphere.html. From MathWorld–A Wolfram Web Resource.