The MMMF Objective: Primal and Dual

Jason D. M. Rennie
jrennie@gmail.com

February 4, 2005

First, some notation and basic equivalences:

- $|A|_2$ is the spectral norm of A
- $|A|_{\mathrm{tr}}$ is the trace norm of A
- $X \bullet Q = \sum_{ij} X_{ij} Q_{ij}$
- $|Q|_2 = \max_{|X|_{\mathrm{tr}} \le 1} X \bullet Q$
- $|X|_{\mathrm{tr}} = \max_{|Q|_2 \le 1} X \bullet Q$

The dual objective is

$$\min_Q |Q * Y|_2, \quad \text{s.t. } 0 \le Q_{ij} \le c \ \forall i,j \quad \text{and} \sum Q_{ij} = t. \tag{1}$$

We follow a series of steps to rewrite this in the primal form.

$$P = \min_{Q \ge 0} \max_{|\tilde{X}|_{\mathrm{tr}} \le 1, \alpha_{ij} \ge 0} \tilde{X} \bullet (Q * Y) + \sum \alpha_{ij}(Q_{ij} - c) + v\left(\sum Q_{ij} - t\right) \tag{2}$$

$$= \max_{|\tilde{X}|_{\mathrm{tr}} \le 1, \alpha_{ij} \ge 0} \min_{Q \ge 0} -c \sum \alpha_{ij} - tv + (\tilde{X} * Y + \alpha + v) \bullet Q \tag{3}$$

Note that if $\tilde{X} * Y + \alpha + v \not\geq 0$, then the minimization objective will be $-\infty$. So, we can safely require that $\tilde{X} * Y + \alpha + v \ge 0$, which forces $Q = 0$.

$$P = \max_{|\tilde{X}|_{\mathrm{tr}} \le 1, \alpha_{ij} \ge 0, \tilde{X} * Y + \alpha + v \ge 0} -c \sum \alpha_{ij} - tv \tag{4}$$

Note that one of the constraints involving α_{ij} must be strict, $\alpha_{ij} = \max(0, -\tilde{X}_{ij} Y_{ij} - v)$. Hence, we can rewrite the problem as

$$P = \max_{|\tilde{X}|_{\mathrm{tr}} \le 1} -c \sum \max(0, -\tilde{X}_{ij} Y_{ij} - v) - tv \tag{5}$$

$$= \max_{|\tilde{X}|_{\mathrm{tr}} \le 1} c \sum \min(0, \tilde{X}_{ij} Y_{ij} + v) - tv \tag{6}$$

Note that the margin is the negation of v, $m = -v$. We make the substitution, $X = m\tilde{X}$. The trace norm will be strict, so we can substitute $m = 1/|X|_{\mathrm{tr}}$.

$$P = -c \min \frac{1}{|X|_{\mathrm{tr}}} \left(\sum \max(0, 1 - X_{ij} Y_{ij}) - \frac{t}{c} \right) \tag{7}$$