Volume of the n-sphere

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Abstract

We derive the volume of an n-sphere, using argumentation given by Weisstein [3].

We use the geometers' nomenclature for n-sphere, n referring to the number of the underlying dimension [3]. Let $V_n(R)$ be the volume of an n-sphere of radius R. Let S_n be the surface area of the unit n-sphere. Consider a small patch of area on the surface of the unit n-sphere. We can approximate such a patch with an $(n-1)$-cube; let the cube have edge-length l. Then, the cube's "area" is l^{n-1}. Now, consider the same patch projected onto the n-sphere of radius R. It's "area" is now $(lR)^{n-1}$, or R^{n-1} times the area of the cube on the unit n-sphere. Thus, the surface area of an n-sphere of radius R is $S_n R^{n-}$; the volume of an n-sphere of radius R is

$$V_n(R) = \int_0^R S_n r^{n-1} dr = \frac{S_n R^n}{n}. \tag{1}$$

Consider the distribution defined by $P(\vec{x}) = \exp(-\|\vec{x}\|_2^2)$, where $\|\vec{x}\|_2^2 = \sum_i x_i^2$ is the squared L_2-norm of $\vec{x}$. This density is proportional to the exponentiated negative Euclidean distance from the origin, or radius. The normalization constant, Z, can either be written in terms of Euclidean coordinates, or Polar coordinates,

$$Z = \int_{\mathbb{R}^n} \exp(-\|\vec{x}\|_2^2) d\vec{x} = \int_0^\infty \exp(-r^2) S_n r^{n-1} dr. \tag{2}$$

Note that the Euclidean version can be written as a 1-D integral raised to the n^{th} power, $\sqrt[n]{Z} = \int_{-\infty}^{\infty} \exp(-x^2)dx$. Also, note that the Gamma function [2] can be written in a form similar to the Polar coordinate version,

$$\Gamma(n) = 2\int_0^\infty \exp(-r^2) r^{2n-1} dr. \tag{3}$$

Thus, $2Z = S_n \Gamma(n/2) = 2\Gamma(1/2)^n = 2\pi^{n/2}$. Solving for S_n, we get

$$S_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{4}$$

For integer n, we can write $\Gamma(n/2) = \frac{(n-2)!!\sqrt{\pi}}{2^{(n-1)/2}}$, where $n!!$ is a double factorial [1]. For even n, this simplifies to $\Gamma(n/2) = (n/2-1)!$.

Thus, the volume of a radius R n-sphere is

$$V_n(R) = \begin{cases} \frac{2^{(n+1)/2}\pi^{(n-1)/2}R^n}{n(n-2)!!} & \text{for } n \text{ odd} \\ \frac{2\pi^{n/2}R^n}{n(n/2-1)!} & \text{for } n \text{ even} \end{cases} . \qquad (5)$$

References

[1] E. W. Weisstein. Double factorial. http://mathworld.wolfram.com/DoubleFactorial.html. From MathWorld–A Wolfram Web Resource.

[2] E. W. Weisstein. Gamma function. http://mathworld.wolfram.com/GammaFunction.html. From MathWorld–A Wolfram Web Resource.

[3] E. W. Weisstein. Hypersphere. http://mathworld.wolfram.com/Hypersphere.html. From MathWorld–A Wolfram Web Resource.