Volume of the Stiefel Manifold

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Abstract

In [1], Edelman gives the volume of the Stiefel manifold, $\mathrm{Vol}(V_{m,n})$. We show that this is equal to the product of unit sphere surface areas.

The Stiefel manifold is the set of $Q \in \mathbb{R}^{n\times m}$ such that $Q^T Q = I_p$ [1]. The surface area of the n-sphere of radius 1 is

$$A_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}. \tag{1}$$

Edelman gives the volume of the Stiefel manifold as

$$\mathrm{Vol}(V_{n,m}) = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}, \tag{2}$$

where $\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma\left[a - \frac{i-1}{2}\right]$. Substituting, we get

$$\mathrm{Vol}(V_{n,m}) = \frac{2^m \pi^{\frac{m}{2}(n-(m-1)/2)}}{\prod_{i=1}^m \Gamma(n/2 - \frac{i-1}{2})}. \tag{3}$$

Consider the following product of surface areas:

$$\prod_{i=n-m+1}^{n} A_n = \prod_{i=n-m+1}^{n} \frac{2\pi^{i/2}}{\Gamma\left(\frac{i}{2}\right)} = \frac{2^m \pi^{\frac{m}{4}(2n-m+1)}}{\prod_{i=m}^{1} \Gamma(\frac{1}{2}(n-i+1))}. \tag{4}$$

The last term is obviously equivalent to the right side of equation 3. Hence, the volume of the Stiefel manifold is a product of radius 1 n-sphere surface areas. Thanks to John Barnett for making this observation.

References

[1] A. Edelman. Volumes and integration. http://web.mit.edu/18.325/www/handouts.html, March 2005. 18.325: Finite Random Matrix Theory, Handout #4.