Some Hypersphere Volumes

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Abstract

We calculate the volume of the 1-, 2-, 3-, and 4-spheres and give an inductive procedure for calculating the volume of the n-sphere.

We use the geometers' nomenclature for n-sphere, n referring to the number of the underlying dimension [1].

We proceed by induction. The volume of the 1-sphere with radius R is

$$V_1(R) = \int_{-R}^{R} dx = 2R. \tag{1}$$

The volume of the 2-sphere is

$$\begin{aligned}
V_2(R) &= \int_{-R}^{R} V_1\left(\sqrt{R^2 - x^2}\right) dx & (2)\\
&= 2\int_{-R}^{R} \sqrt{R^2 - x^2} dx & (3)\\
&= \left[x\sqrt{R^2 - x^2} + R^2 \sin^{-1}\frac{x}{R}\right]_{-R}^{R} & (4)\\
&= R^2\frac{\pi}{2} - R^2\left(-\frac{\pi}{2}\right) = \pi R^2. & (5)
\end{aligned}$$

The volume of the 3-sphere is

$$\begin{aligned}
V_3(R) &= \int_{-R}^{R} V_2\left(\sqrt{R^2 - x^2}\right) dx & (6)\\
&= \pi\int_{-R}^{R} \left(R^2 - x^2\right) dx & (7)\\
&= \pi\left[xR^2 - \frac{1}{3}x^3\right]_{-R}^{R} & (8)\\
&= \frac{4}{3}\pi R^3 & (9)
\end{aligned}$$

The volume of the 4-sphere is

$$V_4(R) = \int_{-R}^{R} V_3\left(\sqrt{R^2 - x^2}\right) dx \tag{10}$$

$$= \frac{4}{3}\pi \int_{-R}^{R} \left(R^2 - x^2\right)^{3/2} dx \tag{11}$$

$$= \frac{4}{3}\pi \left[-\frac{x}{8}(2x^2 - 5R^2)\sqrt{R^2 - x^2} + \frac{3R^4}{8}\sin^{-1}\frac{x}{R}\right]_{-R}^{R} \tag{12}$$

$$= \frac{4}{3}\pi \left(\frac{3R^4}{8}\frac{\pi}{2} - \frac{3R^4}{8}\left(-\frac{\pi}{2}\right)\right) = \frac{R^4\pi^2}{2} \tag{13}$$

Assume we know(for $n \geq 4$) that the volume of an $(n-1)$-sphere to be $V_{n-1}(R) = \frac{S_n R^n}{n}$, where S_n is the surface area of the unit n-sphere. Then, the volume of an n-sphere is

$$V_n(R) = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2 - x^2}\right) dx \tag{14}$$

$$= \frac{S_n}{n} \int_{-R}^{R} \left(R^2 - x^2\right)^{n/2} dx \tag{15}$$

References

[1] E. W. Weisstein. Hypersphere. http://mathworld.wolfram.com/Hypersphere.html. From MathWorld–A Wolfram Web Resource.