An Experimental Analysis of a Compact Graph Representation

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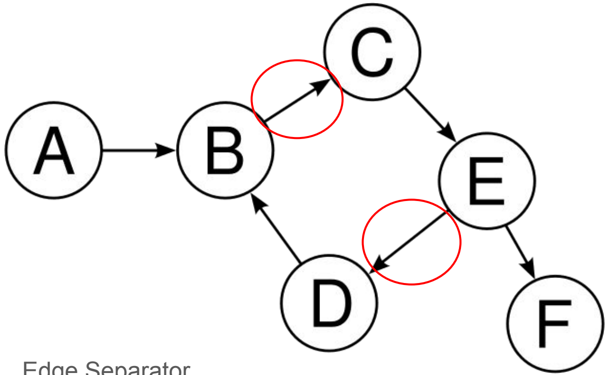
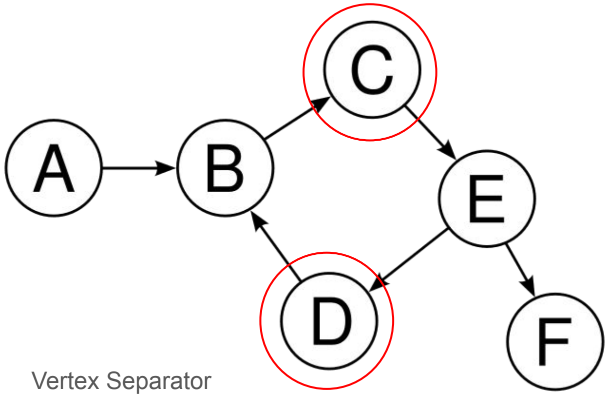
Graph Separators

- An **edge-separator** is a set of edges that, when removed, partitions a graph into two almost equal sized parts
- A **vertex separator** is a set of vertices that when removed (along with its incident edges) partitions a graph into two almost equal parts.
- The **minimum separator** for a graph is the separator that minimizes the number of edges/vertices removed.



Edge Separator



Vertex Separator

Why are separators important?

- A graph has good separators if it and its subgraphs have minimum separators that are significantly better than expected for a random graph of its size.
- Many real world graphs have good separators because they are based on communities or embedded in low dimensional space.
- We can use separators to reorder vertices in graph representations so that we can exploit locality and improve cache misses.

Graph Compression With Separators

Algorithm:

1. Generate an edge separator tree for the graph.
2. Label the vertices in-order across the leaves.
3. Use an adjacency table to represent the relabled graph.

1. Edge Separator Tree

- Each node contains a subgraph of G and a separator for that subgraph
- The children of a node contain the two components of the graph induced by the separator.
- Bottom-up algorithm: start with a complete graph and repeatedly collapse edges until a single vertex remains.
- Heuristic for collapsing edges: $w(E_{AB}) / (s(A)s(B))$
  - $w(E_{AB})$ is the number of edges between multivertices A and B
  - $s(A)$ is the number of original vertices contained in multivertex A
- Child flipping optimization: Another heuristic that decides which child is left and which is right

2. Labeling

- Each leaf node of the edge separator tree represents a single vertex in the graph
- Assign labels to leaves in order to encapsulate the “order” based on locality

3. Adjacency Table

- For each vertex, store the neighbors in a difference-encoded adjacency list
  - If a vertex v has neighbors v_1, v_2, … , v_d, in sorted order, then the data structure encodes the differences $v_1 - v$, $v_2 - v_1$, … $v_d - v_{d-1}$ contiguously in memory
- Differences are encoded using any **logarithmic code**, a prefix code which uses O(log d) bits to encode a difference of size s.
  - Also store a sign bit
- Concatenate adjacency lists together in order to form adjacency table.
  - At the start of each list, also store a code for number of entries in the list

3. Adjacency Table cont

- We need an **indexing structure** to efficiently index into our adjacency table
- **Semi-direct-16**: structure that stores the start locations for sixteen vertices in five 32-bit words.
  - The first word contains the offset to vertex 0.
  - The second word contains three ten-bit offsets from vertex 0 for vertex 4, 8 and 12.
  - The next three words contain twelve eight-bit offsets from respective encoded vertex.
  - If offsets do not fit, we store a pointer to vertex.

3. Adjacency Table cont

- Implemented logarithmic codes:
  - Gamma code
    - Unary code for ⌈log d⌉ followed by binary code for d-2^⌈log d⌉ for a total of 1+2⌊log d⌋ bits
  - A k-bit code
    - Each block starts with a continue bit, which indicates if an integer i is greater than 2^(k-1) bits
    - If 0, store binary representation of i-1 in the remainder k-1 bits
    - If 1, store binary representation for (i-1) mod 2^(k-1) in remainder k-1 bits and then ⌊(i-1)/2^(k-1)⌋
    - snip(k=2), nibble(k=4), byte(k=8)
    - Machines are optimized to manipulate bytes and words rather than arbitrary bit sequences

Graph Compression With Separators Lemma

- We say that S satisfies a $f(n)$-edge separator theorem if there are constants $\alpha < 1$ and $\beta > 0$ such that every graph in S with n vertices has a set of at most $\beta f(n)$ edges whose removal separates the graph into components with at most αn vertices each.
- **Lemma**: For a class of graphs satisfying an n^c-edge separator theorem, and labelings based on the separator tree satisfying the bounds of the separator theorem, the adjacency table for any n-vertex member requires **O(n) bits**.

Dynamic Representation

- Memory is managed in blocks of fixed size and each vertex is initialized with one memory block
- Every 1024 vertices have their own pool of contiguous memory blocks.
  - Pool is resized when memory runs out
- A vertex's blocks are connected into a linked list
  - Hashing technique allows for 8 bit pointers but requires that a constant fraction of the blocks remain empty.
  - Test values of pointer value i in the range 0 to 127 until the result of the hash is unused block.
  - If memory pool is at most 80% full, then probability that this technique will fail is at most .80^128 ~ 4*10^(-13).
- Additional temporary structures for storing neighbors are used to improve caching with regards to time locality.

Experimental Setup (Machines, Compilers, Benchmarks)

- Two different machines with 32-bit processors.
  - Pentium 4 performs better than Pentium III when there is strong spatial locality
- Use adjacency lists and adjacency array (static)
  - Test random insertion order for adjacency lists, linear, transpose, and random.
  - Test vertex order for both: randomized and separator
- Test representations using DFS

Graph	DFS	Read		Find	Insert			Space
		Linear	Random	Next	Linear	Random	Transpose	
ListRand	1.000	0.631	0.995	0.508	1.609	17.719	3.391	76.405
ListOrdr	0.710	0.626	0.977	0.516	1.551	17.837	1.632	76.405
LEDARand	3.163	2.649	3.038	2.518	17.543	19.342	17.880	432.636
LEDAOrdr	2.751	2.168	2.878	1.726	11.846	19.365	11.783	432.636
DynSpace	0.626	0.503	0.715	0.433	17.791	22.520	18.423	11.608
DynTime	0.422	0.342	0.531	0.335	13.415	16.926	13.866	17.900
CachedSpace	0.614	0.498	0.723	0.429	2.616	25.380	7.788	13.36
CachedTime	0.430	0.355	0.558	0.360	2.597	20.601	6.569	17.150
ArrayRand	0.729	0.319	0.643	0.298	—	—	—	38.202
ArrayOrdr	0.429	0.319	0.639	0.302	—	—	—	38.202
Byte	0.330	0.262	0.501	0.280	—	—	—	12.501
Nibble	0.488	0.411	0.646	0.387	—	—	—	9.357
Snip	0.684	0.625	0.856	0.538	—	—	—	9.07
Gamma	0.854	0.764	1.016	0.640	—	—	—	9.424

Table 6: Summary of space and normalized times for various operations on the Pentium III.

Graph	DFS	Read		Find	Insert			Space
		Linear	Random	Next	Linear	Random	Transpose	
ListRand	1.000	0.099	0.744	0.121	0.571	28.274	3.589	76.405
ListOrdr	0.322	0.096	0.740	0.119	0.711	28.318	0.864	76.405
LEDARand	2.453	1.855	2.876	2.062	16.802	21.808	16.877	432.636
LEDAOrdr	1.119	0.478	2.268	0.519	7.570	20.780	7.657	432.636
DynSpace	0.633	0.440	0.933	0.324	14.666	23.901	15.538	11.608
DynTime	0.367	0.233	0.650	0.222	9.725	15.607	10.183	18.763
CachedSpace	0.622	0.431	0.935	0.324	2.433	28.660	8.975	13.34
CachedTime	0.368	0.240	0.690	0.246	2.234	19.849	6.600	19.073
ArrayRand	0.945	0.095	0.638	0.092	—	—	—	38.202
ArrayOrdr	0.263	0.092	0.641	0.092	—	—	—	38.202
Byte	0.279	0.197	0.693	0.205	—	—	—	12.501
Nibble	0.513	0.399	0.873	0.340	—	—	—	9.357
Snip	0.635	0.562	1.044	0.447	—	—	—	9.07
Gamma	0.825	0.710	1.188	0.521	—	—	—	9.424

Table 5: Summary of space and normalized times for various operations on the Pentium 4.

Experimental Setup (Graphs)

Graph	Vtxs	Edges	Max Degree	Source
auto	448695	6629222	37	3D mesh [35]
feocean	143437	819186	6	3D mesh [35]
m14b	214765	3358036	40	3D mesh [35]
ibm17	185495	4471432	150	circuit [1]
ibm18	210613	4443720	173	circuit [1]
CA	1971281	5533214	12	street map [34]
PA	1090920	3083796	9	street map [34]
googleI	916428	5105039	6326	web links [10]
googleO	916428	5105039	456	web links [10]
lucent	112969	363278	423	routers [25]
scan	228298	640336	1937	routers [25]

Table 1: Properties of the graphs used in our experiments.

Results (Static vs adjacency array)

	Array			Our Structure									
	Rand	Sep		Byte		Nibble		Snip		Gamma		DiffByte	
Graph	T_1	T/T_1	Space	T/T_1	Space	T/T_1	Space	T/T_1	Space	T/T_1	Space	T/T_1	Space
auto	0.268s	0.313	34.17	0.294	10.25	0.585	7.42	0.776	6.99	1.063	7.18	0.399	12.33
feocean	0.048s	0.312	37.60	0.312	12.79	0.604	10.86	0.791	11.12	1.0	11.97	0.374	13.28
m14b	0.103s	0.388	34.05	0.349	10.01	0.728	7.10	0.970	6.55	1.320	6.68	0.504	11.97
ibm17	0.095s	0.536	33.33	0.536	10.19	1.115	7.72	1.400	7.58	1.968	7.70	0.747	12.85
ibm18	0.113s	0.398	33.52	0.442	10.24	0.867	7.53	1.070	7.18	1.469	7.17	0.548	12.16
CA	0.920s	0.126	43.40	0.146	14.77	0.243	10.65	0.293	10.55	0.333	11.25	0.167	14.81
PA	0.487s	0.137	43.32	0.156	14.76	0.258	10.65	0.310	10.60	0.355	11.28	0.178	14.80
lucent	0.030s	0.266	41.95	0.3	14.53	0.5	11.05	0.566	10.79	0.700	11.48	0.333	14.96
scan	0.067s	0.208	43.41	0.253	15.46	0.402	11.84	0.477	11.61	0.552	12.14	0.298	16.46
googleI	0.367s	0.226	37.74	0.258	11.93	0.405	8.39	0.452	7.37	0.539	7.19	0.302	13.39
googleO	0.363s	0.250	37.74	0.278	12.59	0.460	9.72	0.556	9.43	0.702	9.63	0.327	13.28
Avg		0.287	38.202	0.302	12.501	0.561	9.357	0.696	9.07	0.909	9.424	0.380	13.662

Table 2: Performance of our **static** algorithms compared to performance of an adjacency array representation. Space is in bits per edge; time is for a DFS, normalized to the first column, which is given in seconds.

Results (Block sizes)

	3		4		8		12		16		20	
Graph	T_1	Space	T/T_1	Space	T/T_1	Space	T/T_1	Space	T/T_1	Space	T/T_1	Space
auto	0.318s	11.60	0.874	10.51	0.723	9.86	0.613	10.36	0.540	9.35	0.534	11.07
feocean	0.044s	14.66	0.863	13.79	0.704	12.97	0.681	17.25	0.727	22.94	0.750	28.63
m14b	0.146s	11.11	0.876	10.07	0.684	9.41	0.630	10.00	0.554	8.92	0.554	10.46
ibm17	0.285s	12.95	0.849	11.59	0.614	10.44	0.529	10.53	0.491	10.95	0.459	11.39
ibm18	0.236s	12.41	0.847	11.14	0.635	10.12	0.563	10.36	0.521	10.97	0.5	11.64
CA	0.212s	10.62	0.943	12.42	0.952	23.52	1.0	35.10	1.018	46.68	1.066	58.26
PA	0.119s	10.69	0.941	12.41	0.949	23.35	1.0	34.85	1.025	46.35	1.058	57.85
lucent	0.018s	13.67	0.888	14.79	0.833	22.55	0.833	31.64	0.833	41.22	0.888	51.09
scan	0.034s	15.23	0.941	16.86	0.852	26.39	0.852	37.06	0.852	48.08	0.882	59.34
googleI	0.230s	11.91	0.895	12.04	0.752	15.71	0.730	20.53	0.730	25.78	0.726	31.21
googleO	0.278s	13.62	0.863	13.28	0.694	15.65	0.658	19.52	0.640	24.24	0.676	29.66
Avg		12.58	0.889	12.62	0.763	16.36	0.735	21.56	0.721	26.86	0.736	32.78

Table 3: Performance of our dynamic algorithm using nibble codes with various block sizes. For each size we give the space needed in bits per edge (assuming enough blocks to leave the secondary hash table 80% full) and the time needed to perform a DFS. Times are normalized to the first column, which is given in seconds. .

Results (Dynamic vs Linked Lists)

	Linked List							Our Structure					
	Random Vtx Order			Sep Vtx Order				Space Opt			Time Opt		
Graph	Rand T_1	Trans T/T_1	Lin T/T_1	Rand T/T_1	Trans T/T_1	Lin T/T_1	Space	Block Size	Time T/T_1	Space	Block Size	Time T/T_1	Space
auto	1.160s	0.512	0.260	0.862	0.196	0.093	68.33	16	0.148	9.35	20	0.087	13.31
feocean	0.136s	0.617	0.389	0.801	0.176	0.147	75.21	8	0.227	12.97	10	0.117	14.71
m14b	0.565s	0.442	0.215	0.884	0.184	0.090	68.09	16	0.143	8.92	20	0.086	13.53
ibm17	0.735s	0.571	0.152	0.904	0.357	0.091	66.66	12	0.205	10.53	20	0.118	14.52
ibm18	0.730s	0.524	0.179	0.890	0.276	0.080	67.03	10	0.190	10.13	20	0.108	14.97
CA	1.240s	0.770	0.705	0.616	0.107	0.101	86.80	3	0.170	10.62	5	0.108	15.65
PA	0.660s	0.780	0.701	0.625	0.112	0.109	86.64	3	0.180	10.69	5	0.115	15.64
lucent	0.063s	0.634	0.492	0.730	0.190	0.142	83.90	3	0.285	13.67	6	0.174	20.49
scan	0.117s	0.735	0.555	0.700	0.188	0.128	86.82	3	0.290	15.23	8	0.170	28.19
googleI	0.975s	0.615	0.376	0.774	0.164	0.096	75.49	4	0.211	12.04	16	0.125	28.78
googleO	0.960s	0.651	0.398	0.786	0.162	0.108	75.49	5	0.231	13.54	16	0.123	26.61
Avg		0.623	0.402	0.779	0.192	0.108	76.405		0.207	11.608		0.121	18.763

Table 4: The performance of our **dynamic** algorithms compared to linked lists. For each graph we give the space- and time-optimal block size. Space is in bits per edge; time is for a DFS, normalized to the first column, which is given in seconds.

Summary

- The paper proposes and implements a compressed graph representation using edge separators.
- We see a significant improvement in space and time performance for the static representation and dynamic representation
- The paper also shows the importance of different orderings in any graph representations