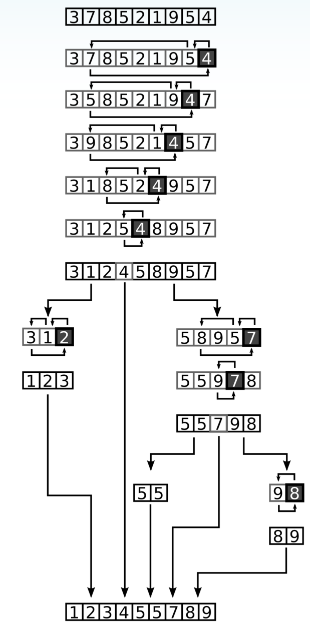
6.506: Algorithm Engineering



Lecture 2
Parallel Algorithms

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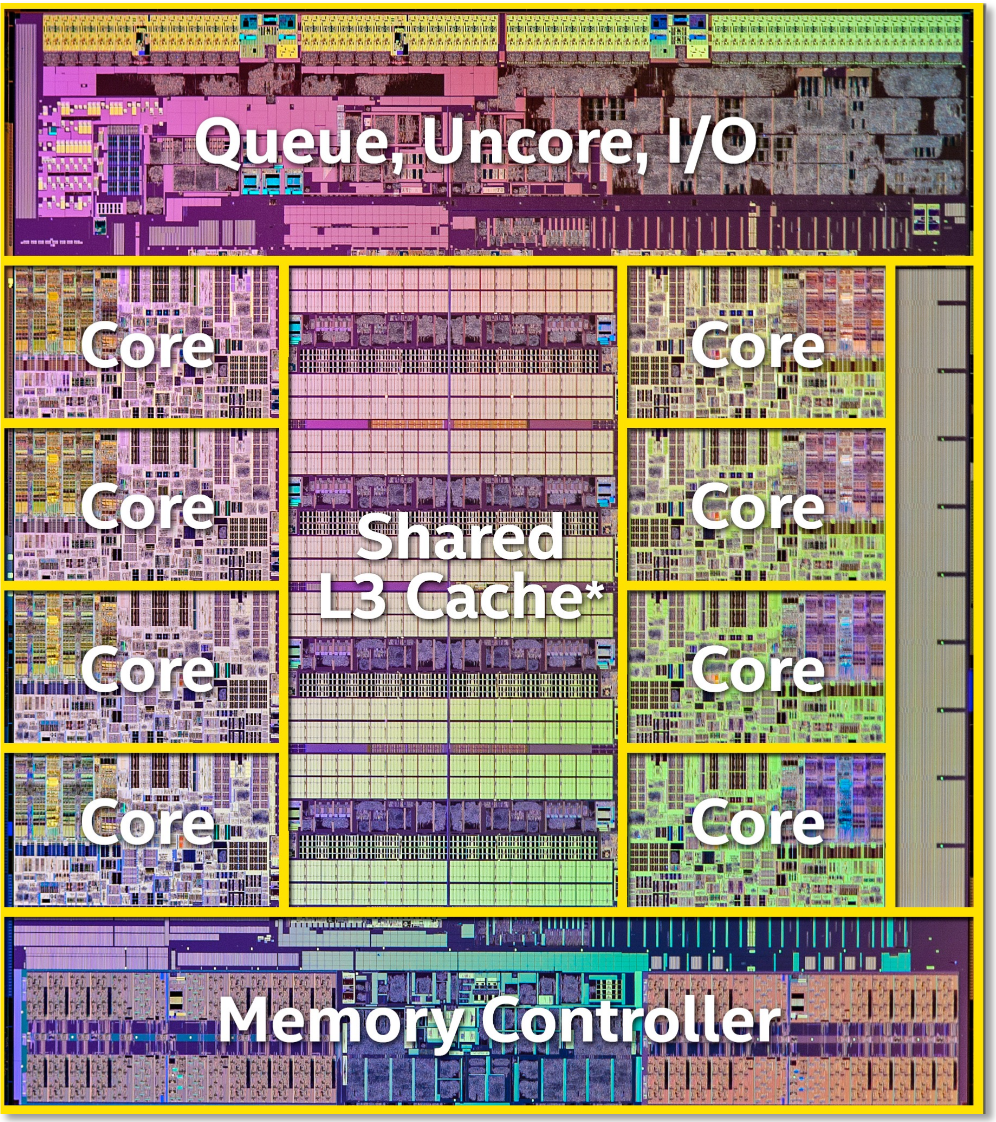
February 8, 2024

Lecture material taken from "Parallel Algorithms" by Guy Blelloch and Bruce Maggs and 6.172, developed by Charles Leiserson and Saman Amarasinghe

Announcement

- Presentation sign-up sheet has been posted
- Problem set will be released on Canvas this week, due on Monday 3/4
- First paper review due Tuesday 10am

Multicore Processors



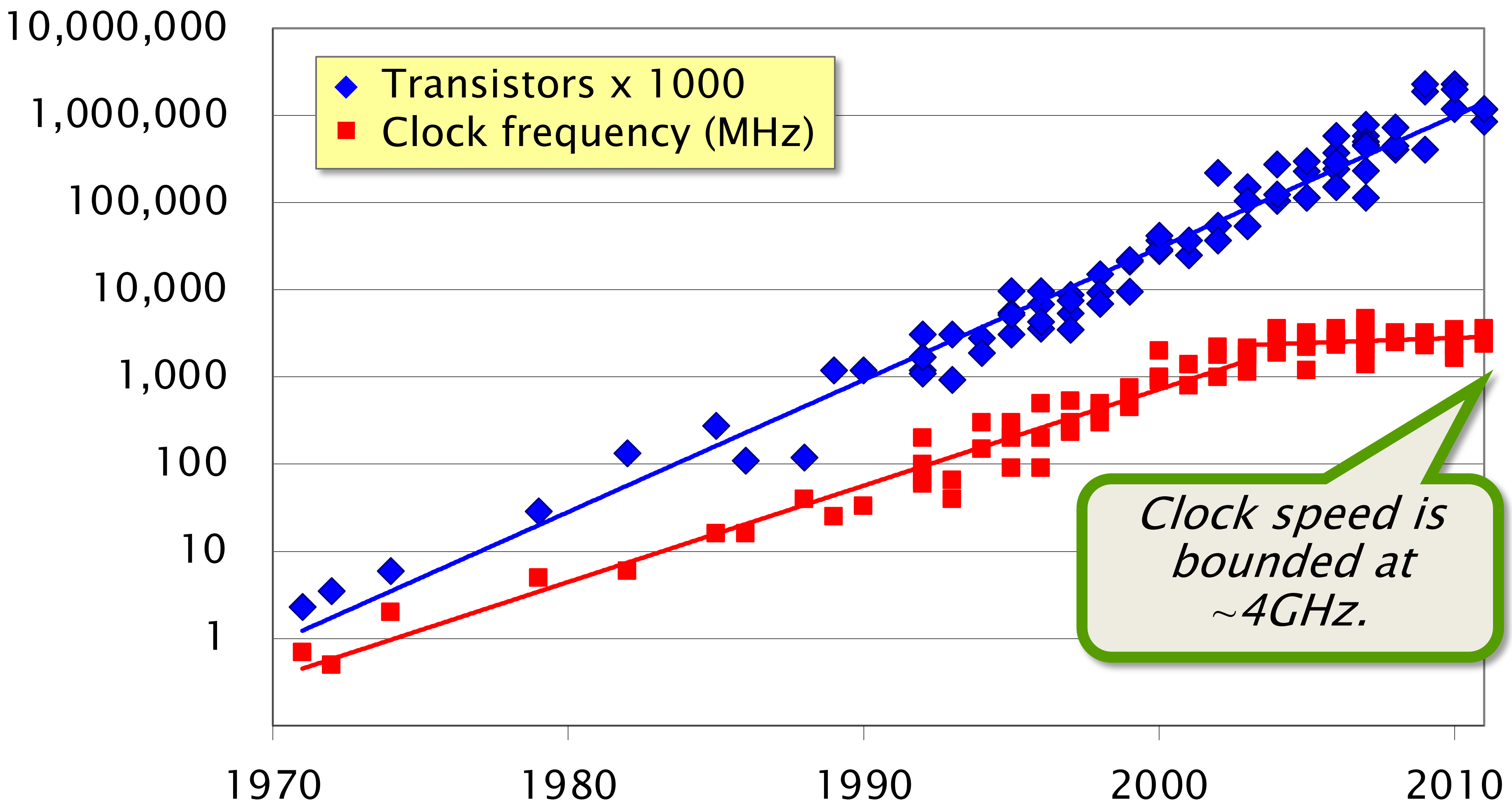
Q Why do semiconductor vendors provide chips with multiple processor cores?

A Because of Moore's Law and the end of the scaling of clock frequency.

Intel Haswell-E

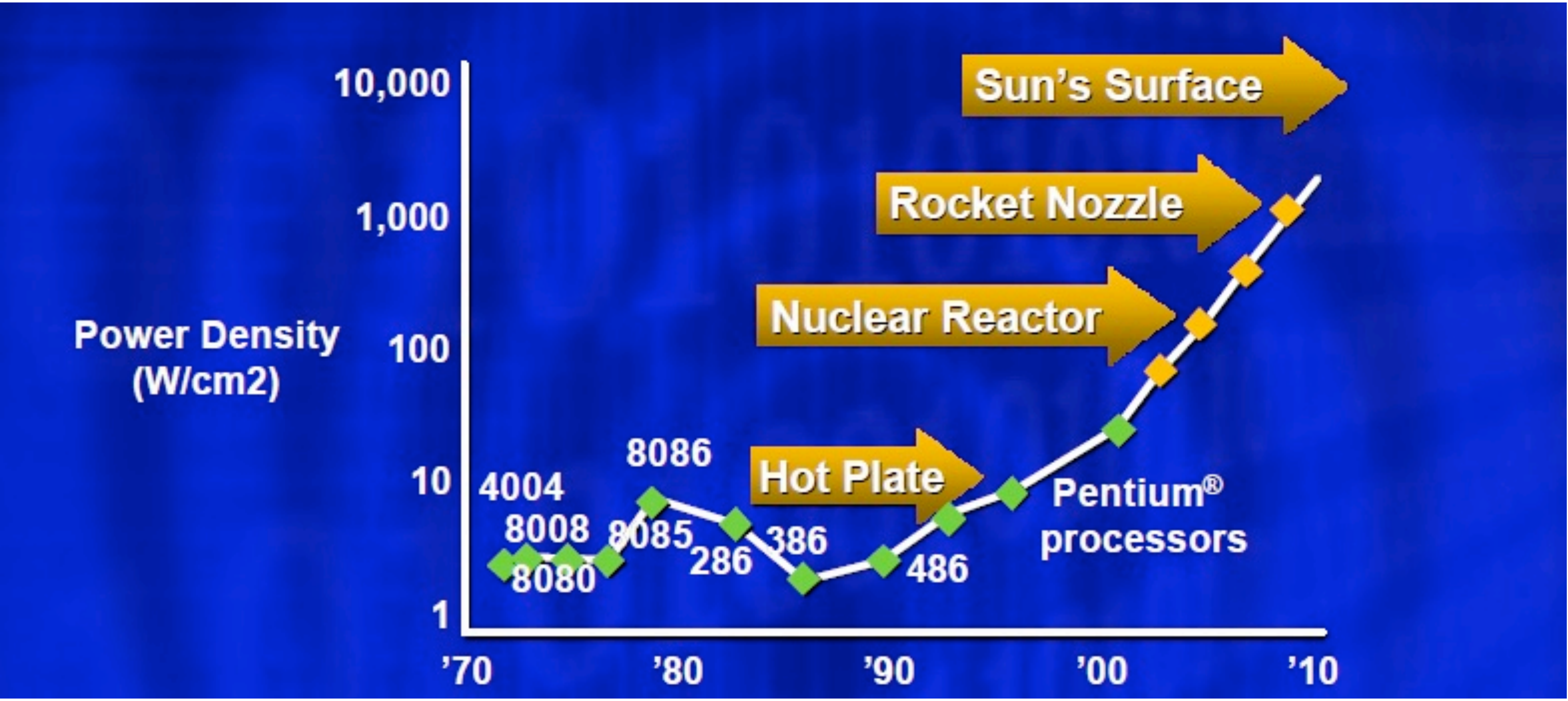
Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Technology Scaling



Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Power Density

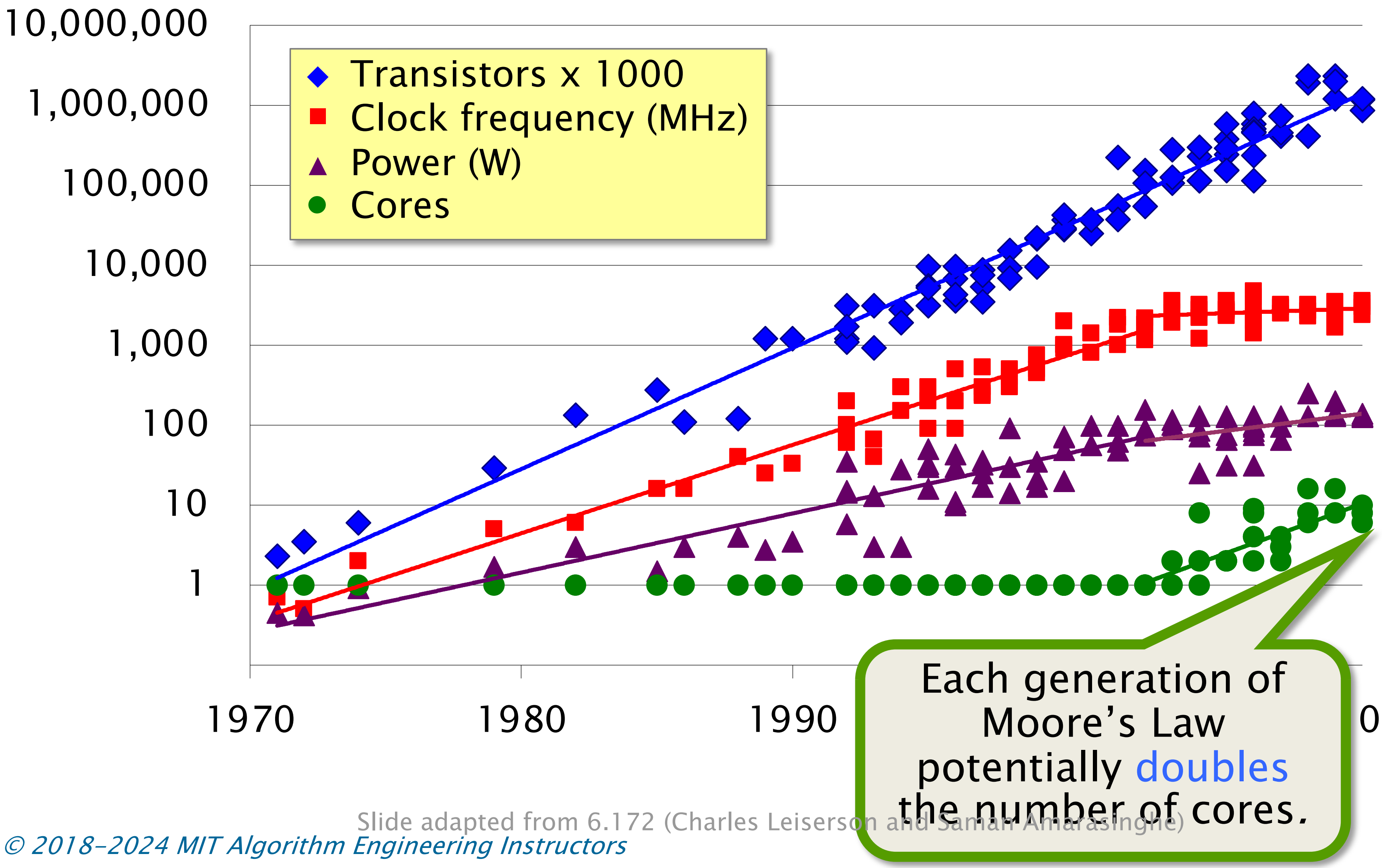


Source: Patrick Gelsinger, *Intel Developer's Forum*, Intel Corporation, 2004.

Projected power density, if clock frequency had continued its trend of scaling 25%–30% per year.

Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Technology Scaling

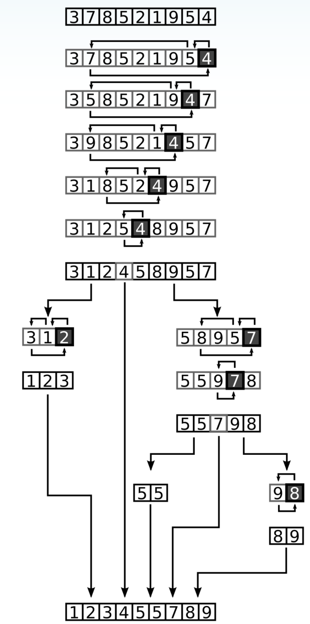


Slide adapted from 6.172 (Charles Leiserson and Saman Amarasinghe)

Parallel Languages

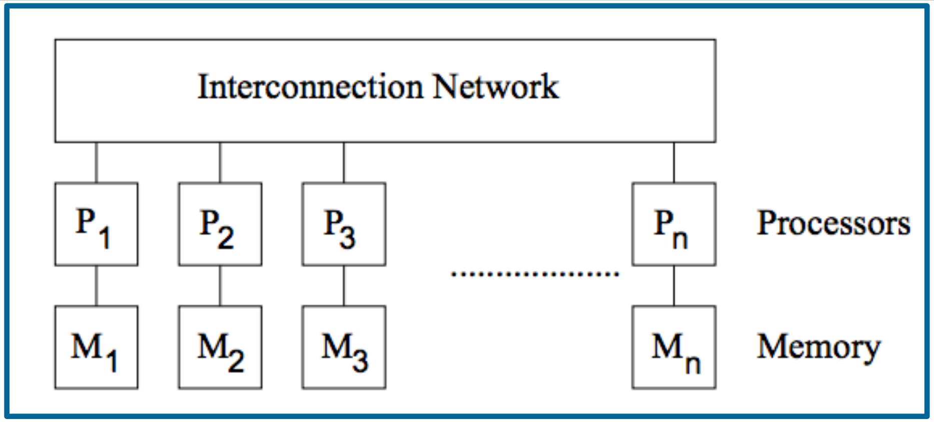
- Pthreads
- Cilk, OpenMP
- Message Passing Interface (MPI)
- CUDA, OpenCL

- Today: Shared-memory parallelism
  - Cilk and OpenMP are extensions of C/C++ that support parallel for-loops, parallel recursive calls, etc.
  - Do not need to worry about assigning tasks to processors as these languages have a runtime scheduler
  - Cilk has a provably efficient runtime scheduler

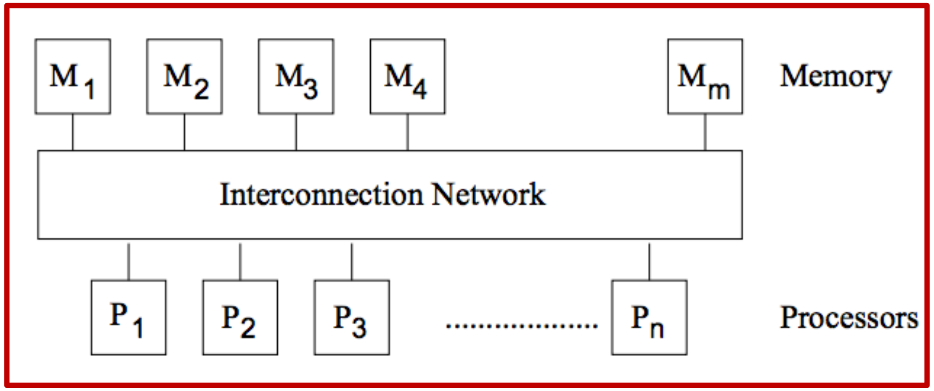


Parallelism Models

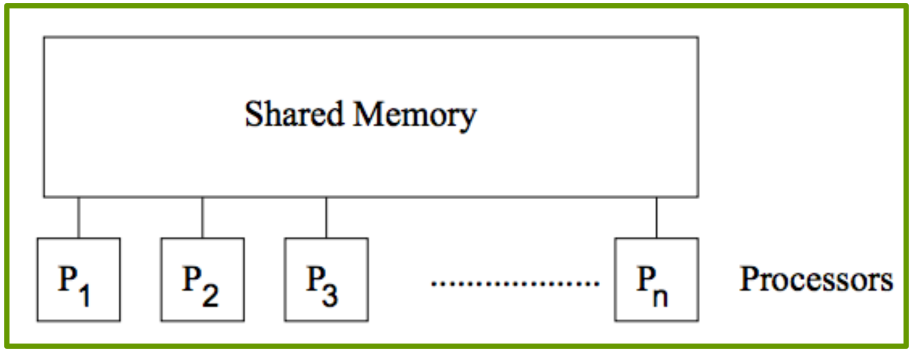
Basic multiprocessor models



Local memory machine



Modular memory machine

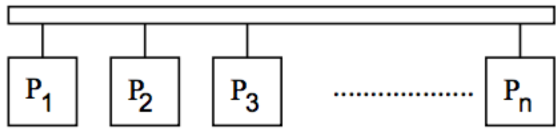


Parallel random-access Machine (PRAM)

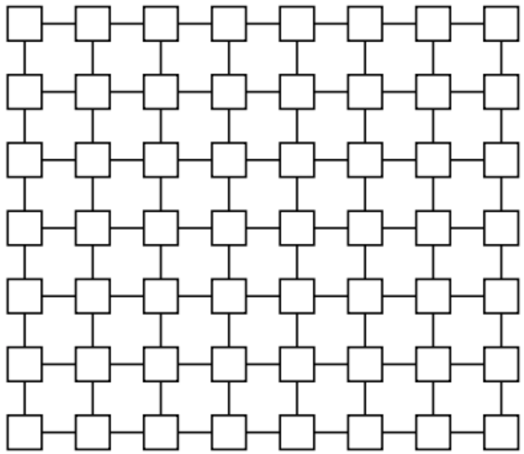
Source: "Parallel Algorithms" by Guy E. Blelloch and Bruce M. Maggs

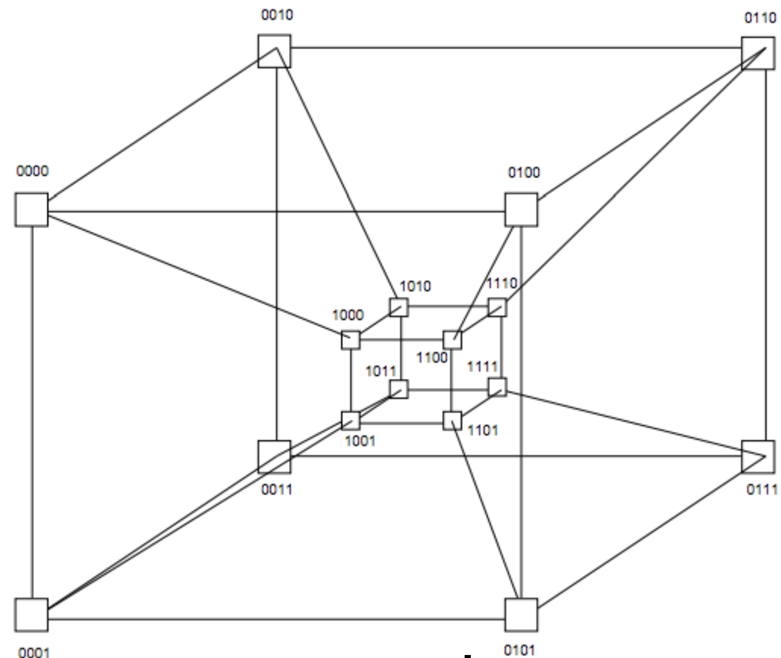
Network topology

Bus

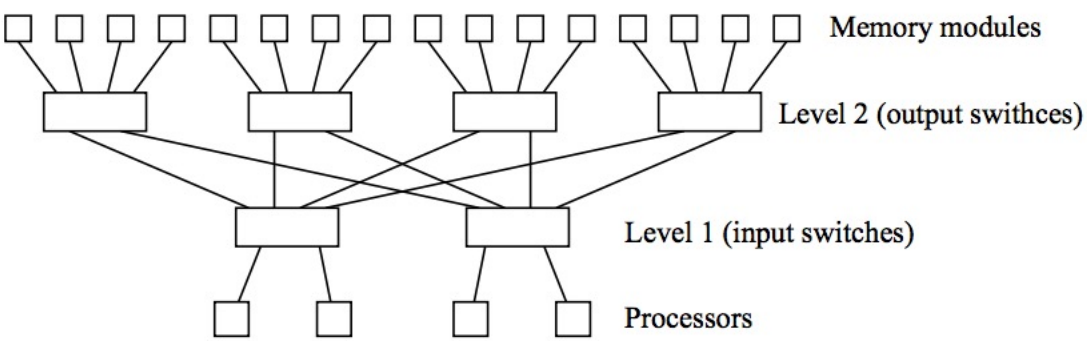


Mesh

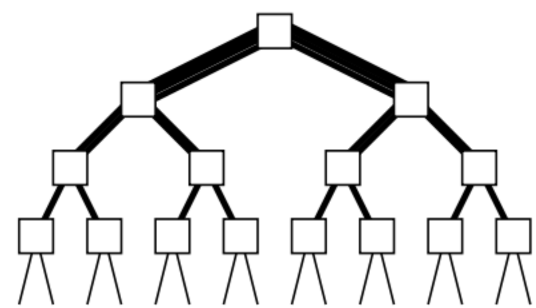




Hypercube



2-level multistage network



Fat tree

Source: "Parallel Algorithms" by Guy E. Blelloch and Bruce M. Maggs

Network topology

- Algorithms for specific topologies can be complicated
  - May not perform well on other networks
- Alternative: use a model that summarizes latency and bandwidth of network
  - Postal model
  - Bulk-Synchronous Parallel (BSP) model
  - LogP model

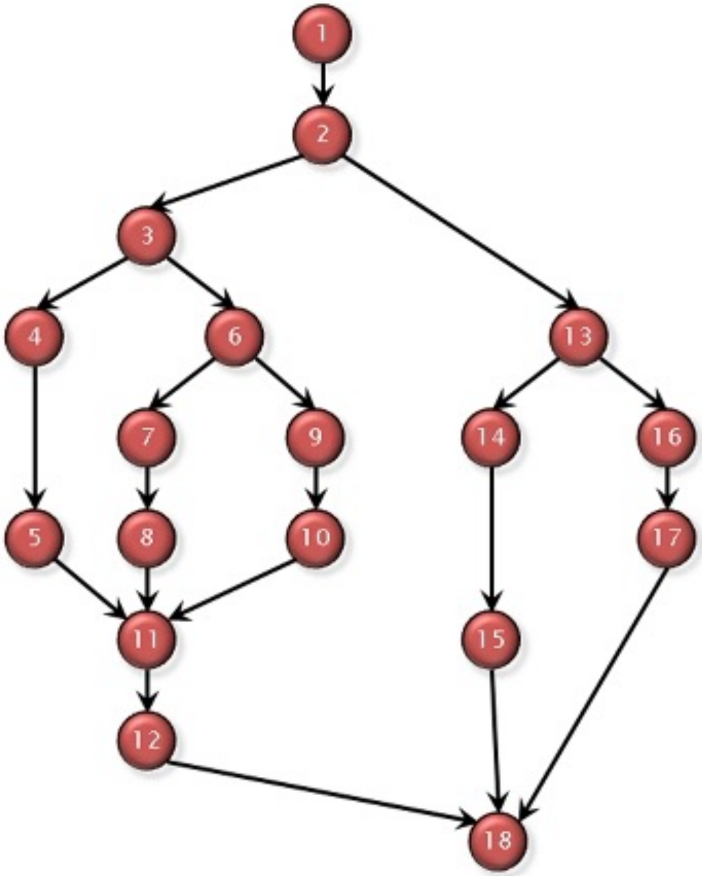
PRAM Model

- All processors can perform same local instructions as in the RAM model
- All processors operate in lock-step
- Implicit synchronization between steps
- Models for concurrent access
  - Exclusive-read exclusive-write (EREW)
  - Concurrent-read concurrent-write (CRCW)
    - How to resolve concurrent writes: arbitrary value, value from lowest-ID processor, logical OR of values, sum of values
  - Concurrent-read exclusive-write (CREW)
  - Queue-read queue-write (QRQW)
    - Allows concurrent access in time proportional to the maximal number of concurrent accesses

Work-Span model

- Similar to PRAM but does not require lock-step or processor allocation

Computation graph

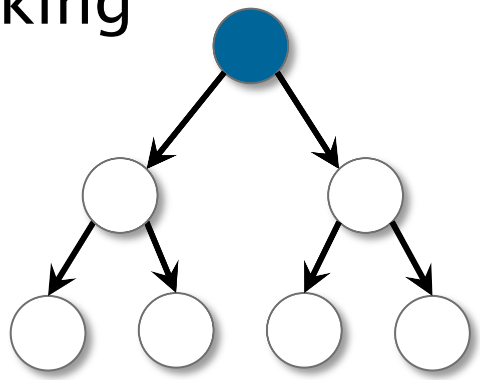


- Work = number of vertices in graph (number of operations)
- Span (Depth) = longest directed path in graph (dependence length)
- Parallelism = Work / Span
- A work-efficient parallel algorithm has work that asymptotically matches the best sequential algorithm for the problem

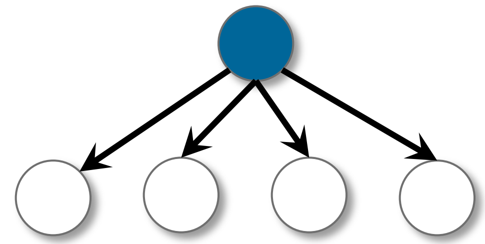
Goal: work-efficient and low (polylogarithmic) span parallel algorithms

Work-Span model

- Spawning/forking tasks
  - Model can support either binary forking or arbitrary forking



Binary forking



Arbitrary forking

  - Cilk uses binary forking, as seen in 6.172
  - Converting between the two changes work by at most a constant factor and span by at most a logarithmic factor
    - Keep this in mind when reading textbooks/papers on parallel algorithms
  - We will assume arbitrary forking unless specified

Work-Span model

- State what operations are supported
  - Concurrent reads/writes?
  - Resolving concurrent writes

Scheduling

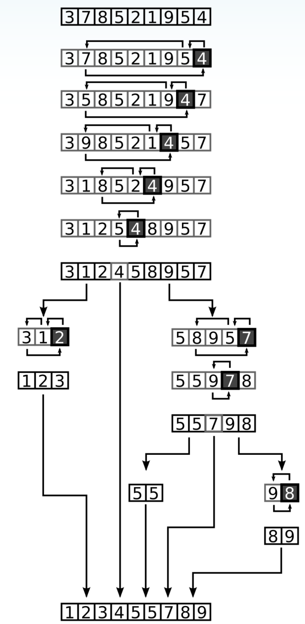
- For a computation with work W and span S, on P processors a greedy scheduler achieves

Running time $\leq$ W/P + S

- For a computation with work W and span S, on P processors Cilk's work-stealing scheduler achieves

Expected running time $\leq$ W/P + O(S)

- Work-efficiency is important since P and S are usually small



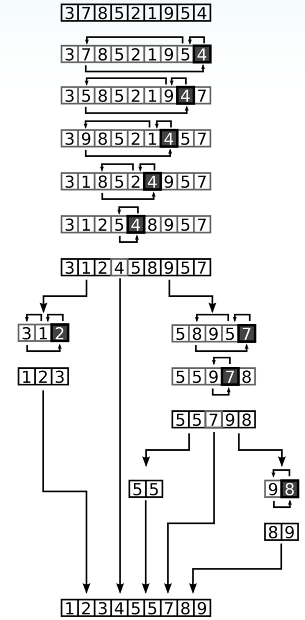
PARALLEL SUM

Parallel Sum

- Definition: Given a sequence $A=[x_0, x_1, \ldots, x_{n-1}]$, return $x_0+x_1+\ldots+x_{n-2}+x_{n-1}$

What is the span?
$S(n) = S(n/2)+O(1)$
$S(1) = O(1)$
→ $S(n) = O(\log n)$

What is the work?
$W(n) = W(n/2)+O(n)$
$W(1) = O(1)$
→ $W(n) = O(n)$

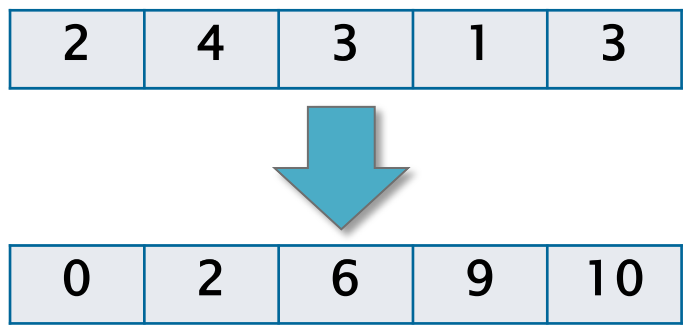

3 7 8 5 2 1 9 5 4
3 7 8 5 2 1 9 5 4
3 5 8 5 2 1 9 4 7
3 9 8 5 2 1 4 5 7
3 1 8 5 2 4 9 5 7
3 1 2 5 4 8 9 5 7
3 1 2 4 5 8 9 5 7
3 1 2
5 8 9 5 7
1 2 3
5 5 9 7 8
5 5 7 9 8
5 5
9 8
8 9
1 2 3 4 5 5 7 8 9

PREFIX SUM

Prefix Sum

- Definition: Given a sequence $A=[x_0, x_1, \ldots, x_{n-1}]$, return a sequence where each location stores the sum of everything before it in A, $[0, x_0, x_0+x_1, \ldots, x_0+x_1+\ldots+x_{n-2}]$, as well as the total sum $x_0+x_1+\ldots+x_{n-2}+x_{n-1}$

- Example:



Total sum = 13

- Can be generalized to any associative binary operator (e.g., $\times$, min, max)

Sequential Prefix Sum

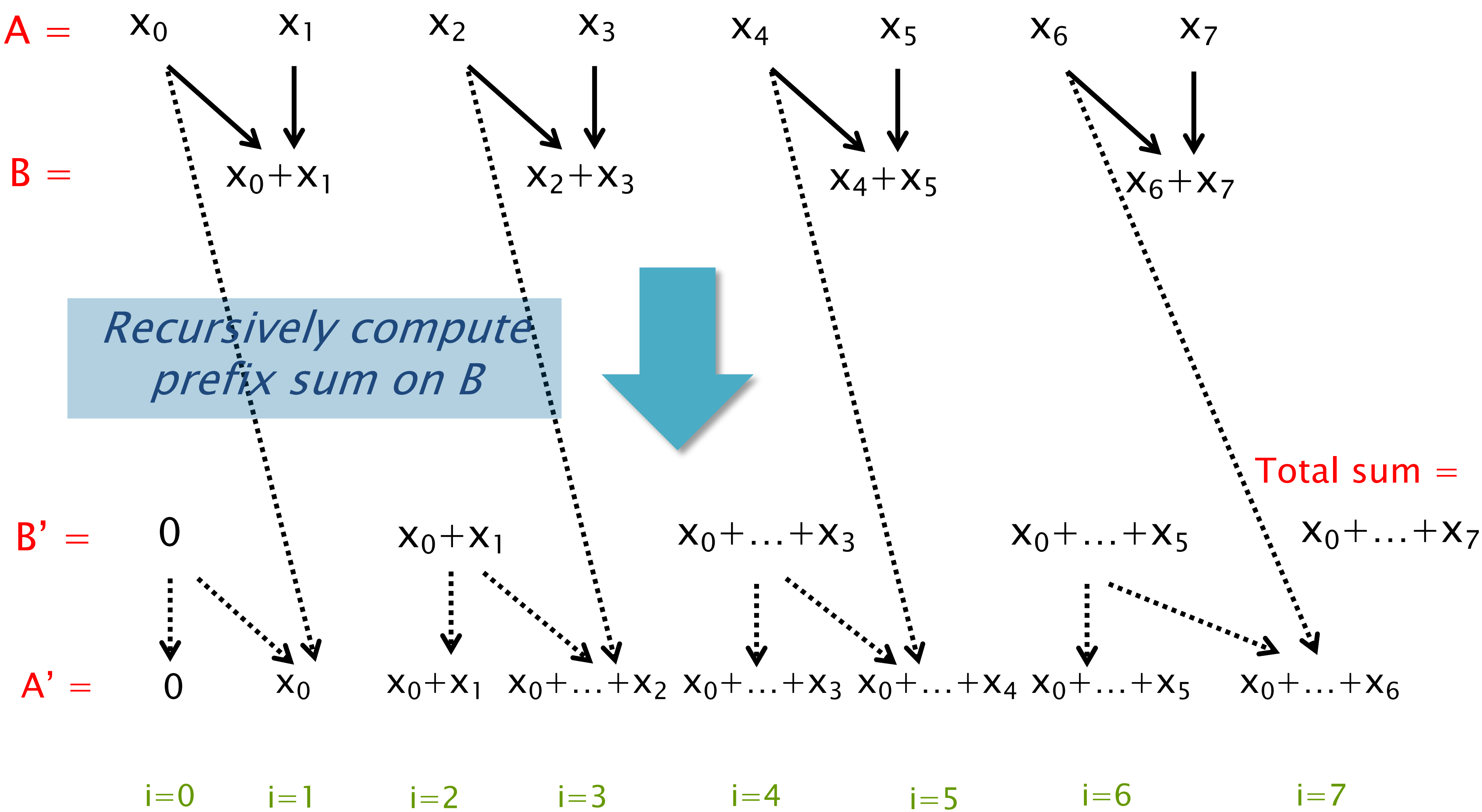
Input: array A of length n

Output: array A' and total sum

```
cumulativeSum = 0;
for i=0 to n-1:
   A'[i] = cumulativeSum;
   cumulativeSum += A[i];
return A' and cumulativeSum
```

- What is the work of this algorithm?
  - O(n)
- Can we execute iterations in parallel?
  - Loop carried dependence: value of cumulativeSum depends on previous iterations

Parallel Prefix Sum



for even values of i: A'[i] = B'[i/2]

for odd values of i: A'[i] = B'[(i−1)/2]+A[i−1]

Total sum = $x_0+\ldots+x_7$

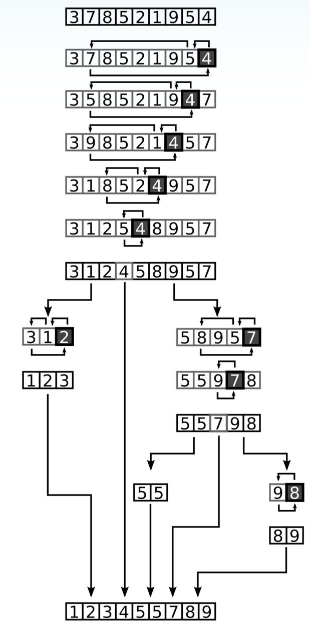
Parallel Prefix Sum

Input: array A of length n (assume n is a power of 2)
Output: array A' and total sum

```
PrefixSum(A, n):
  if n == 1: return ([0], A[0])
  for i=0 to n/2-1 in parallel:
     B[i] = A[2i] + A[2i+1]
  (B', sum) = PrefixSum(B, n/2)
  for i=0 to n-1 in parallel:
     if (i mod 2) == 0:   A'[i] = B'[i/2]
     else:   A'[i] = B'[(i-1)/2] + A[i-1]
  return (A', sum)
```

What is the span?
$S(n) = S(n/2)+O(1)$
$S(1) = O(1)$
→ $S(n) = O(\log n)$

What is the work?
$W(n) = W(n/2)+O(n)$
$W(1) = O(1)$
→ $W(n) = O(n)$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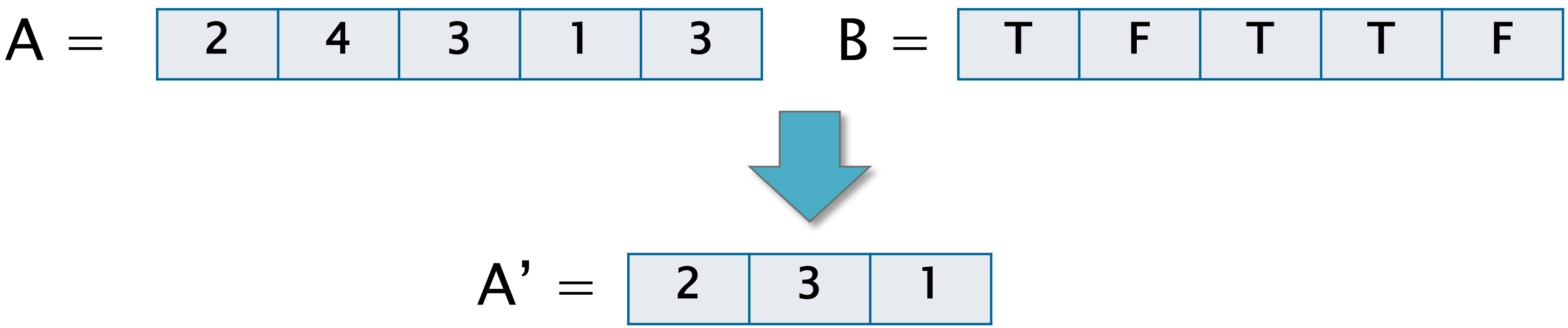
378521954
358521947
398521457
318524957
312548957
312458957
312
58957
123
55978
55798
55
98
89
123455789

FILTER

Filter

- Definition: Given a sequence $A=[x_0, x_1, \ldots, x_{n-1}]$ and a Boolean array of flags $B[b_0, b_1, \ldots, b_{n-1}]$, output an array A' containing just the elements A[i] where B[i] = true (maintaining relative order)
- Example:



- Can you implement filter using prefix sum?

Filter Implementation

A =

2	4	3	1	3

B =

T	F	T	T	F

1	0	1	1	0

Prefix sum

B' =

0	1	1	2	3

Total sum = 3

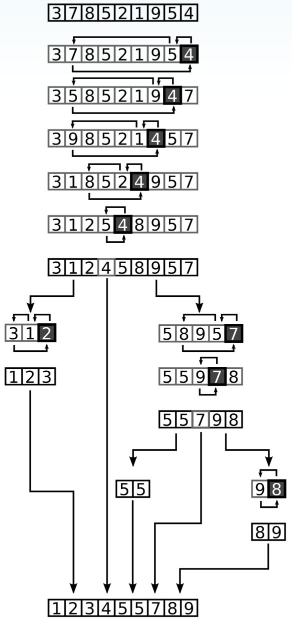
```
//Assume B'[n] = total sum
parallel-for i=0 to n-1:
    if(B'[i] != B'[i+1]):
        A'[B'[i]] = A[i];
```

Allocate array of size 3

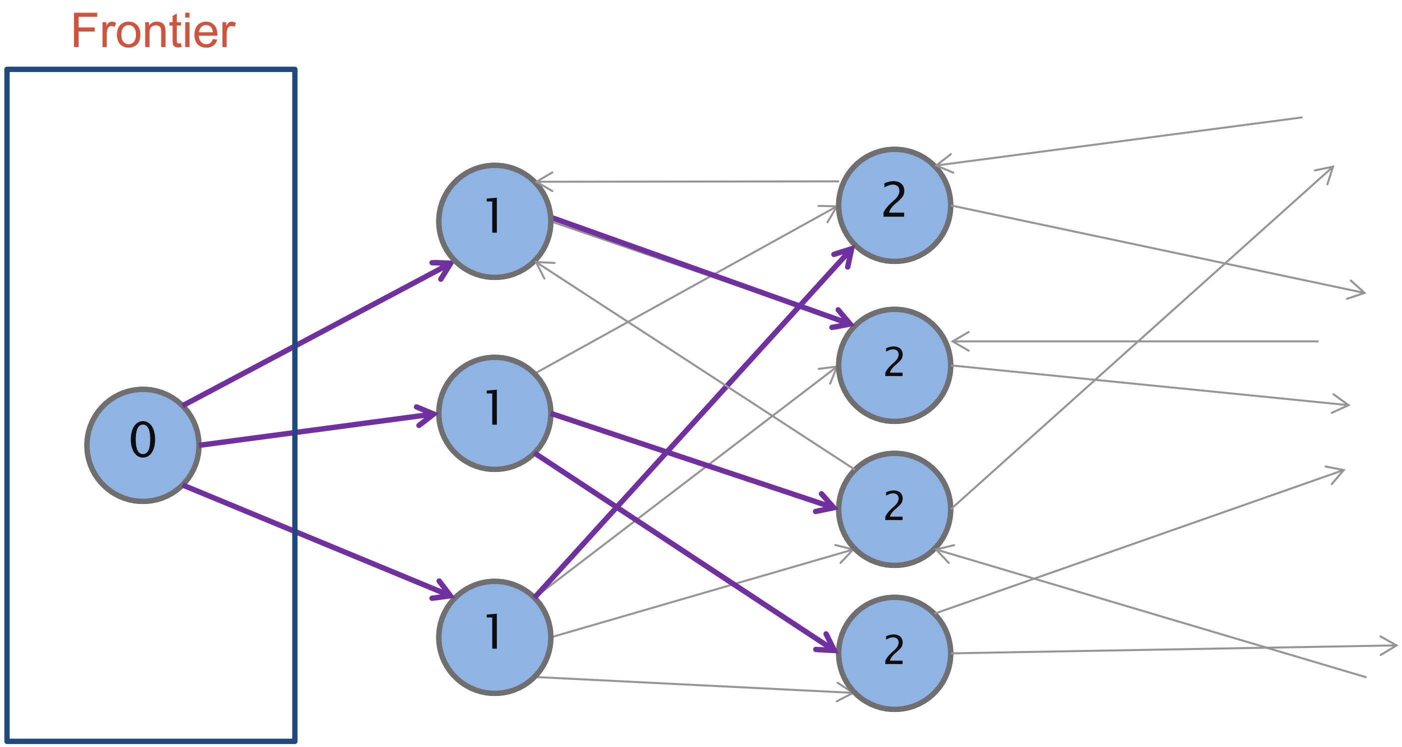
A' =

2	3	1

PARALLEL BREADTH-FIRST SEARCH



Parallel BFS Algorithm



- Can process each frontier in parallel
  - Parallelize over both the vertices and their outgoing edges

Parallel BFS Code

```
BFS(Offsets, Edges, source) {
  parent, frontier, frontierNext, and degrees are array
  parallel_for(int i=0; i<n; i++) parent[i] = -1;
  frontier[0] = source, frontierSize = 1, parent[source] = source;

  while(frontierSize > 0) {
    parallel_for(int i=0; i<frontierSize; i++)
        degrees[i] = Offsets[frontier[i]+1] – Offsets[frontier[i]];
    perform prefix sum on degrees array
    parallel_for(int i=0; i<frontierSize; i++) {
        v = frontier[i], index = degrees[i], d = Offsets[v+1]-Offsets[v];
        for(int j=0; j<d; j++) { //can be parallel
                ngh = Edges[Offsets[v]+j];
                if(parent[ngh] == -1 && compare-and-swap(&parent[ngh], -1, v)) {
                    frontierNext[index+j] = ngh;
                } else { frontierNext[index+j] = -1; }
        }
    }
    filter out "-1" from frontierNext, store in frontier, and update frontierSize to be
        the size of frontier (all done using prefix sum)
  }
}
```

frontierSize = 5

2	4	3	1	3

Prefix sum

0	2	6	9	10

v1 v2 v3 v4 v5

frontier =

frontierSize = 8

24	9	24	9	15	89	25	90	99	4	-1	-1	4

BFS Work-Span Analysis

- Number of iterations <= diameter Δ of graph
- Each iteration takes O(log m) span for prefix sum and filter (assuming inner loop is parallelized)

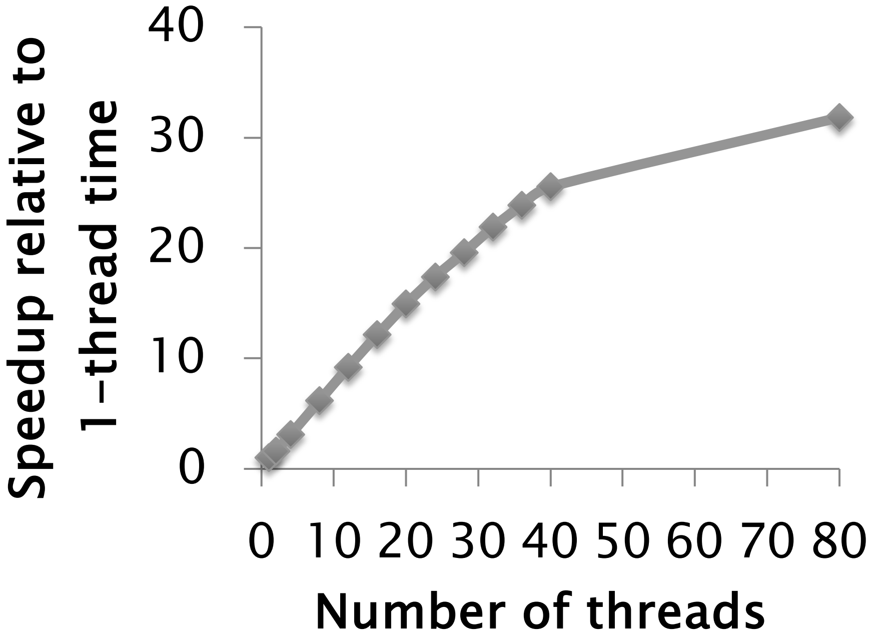
Span = O(Δ log m)

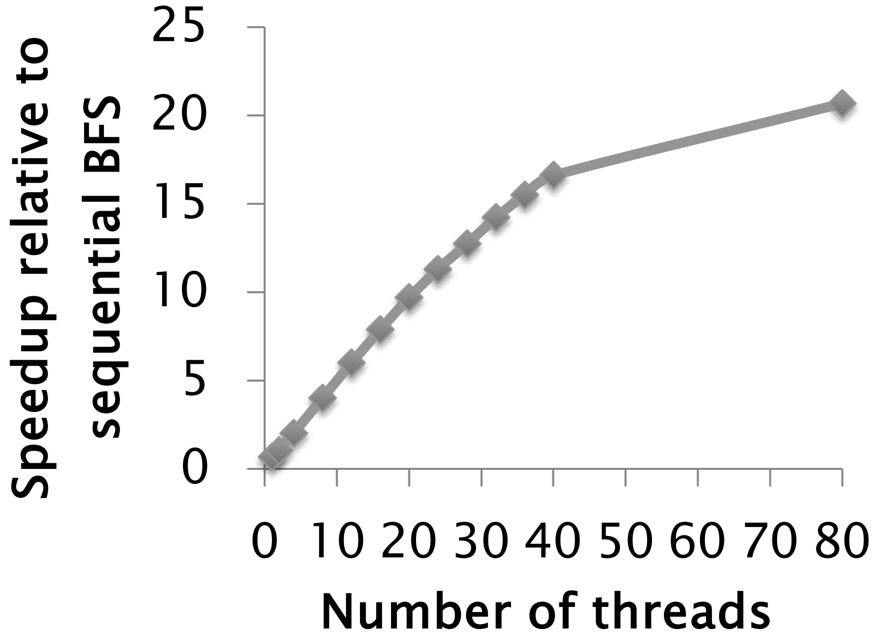
- Sum of frontier sizes = n
- Each edge traversed once -> m total visits
- Work of prefix sum on each iteration is proportional to frontier size -> Θ(n) total
- Work of filter on each iteration is proportional to number of edges traversed -> Θ(m) total

Work = Θ(n+m)

Performance of Parallel BFS

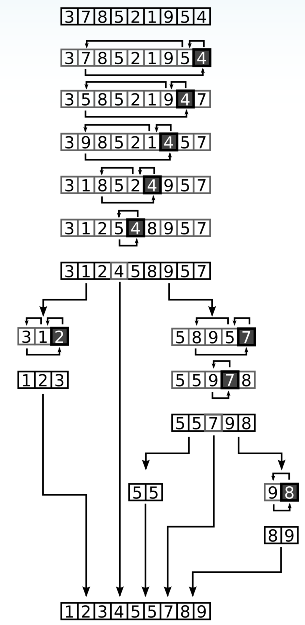
- Random graph with $n=10^7$ and $m=10^8$
  - 10 edges per vertex
- 40-core machine with 2-way hyperthreading





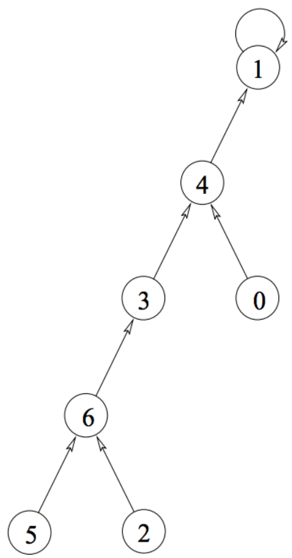
- 31.8x speedup on 40 cores with hyperthreading
- Sequential BFS is 54% faster than parallel BFS on 1 thread

POINTER JUMPING AND LIST RANKING

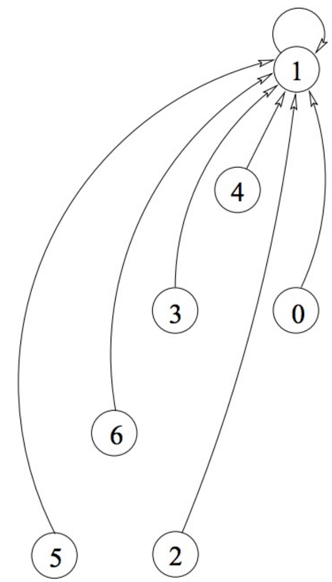


Pointer Jumping

- Have every node in linked list or rooted tree point to the end (root)



(a) The input tree $P = [4, 1, 6, 4, 1, 6, 3]$.



(b) (c) The final tree $P = [1, 1, 1, 1, 1, 1, 1]$. iteration

```
for j=0 to ceil(log n)-1:
    parallel-for i=0 to n-1:
        temp[i] = P[P[i]];
    parallel-for i=0 to n-1:
        P[i] = temp[i];
```

What is the work and span?

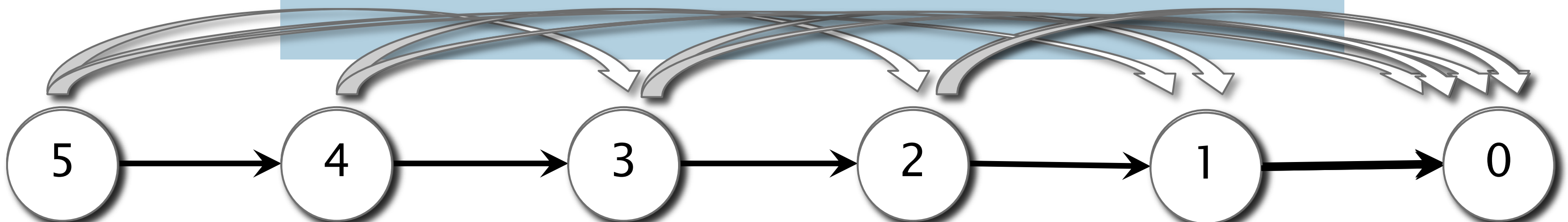
W = O(n log n)
S = O(log n)

List Ranking

- Have every node in linked list determine its distance to the end

```
parallel-for i=0 to n-1:
    if P[i] == i then rank[i] = 0
    else rank[i] = 1

for j=0 to ceil(log n)-1:
    temp, temp2;
    parallel-for i=0 to n-1:
        temp[i] = rank[P[i]];
        temp2[i] = P[P[i]];
    parallel-for i=0 to n-1:
        rank[i] = rank[i] + temp[i];
        P[i] = temp2[i];
```



Work-Span Analysis

```
parallel-for i=0 to n-1:
    if P[i] == i then rank[i] = 0
    else rank[i] = 1

for j=0 to ceil(log n)-1:
    temp, temp2;
    parallel-for i=0 to n-1:
        temp = rank[P[i]];
        temp2 = P[P[i]];
    parallel-for i=0 to n-1:
        rank[i] = rank[i] + temp;
        P[i] = temp2;
```

What is the work and span?

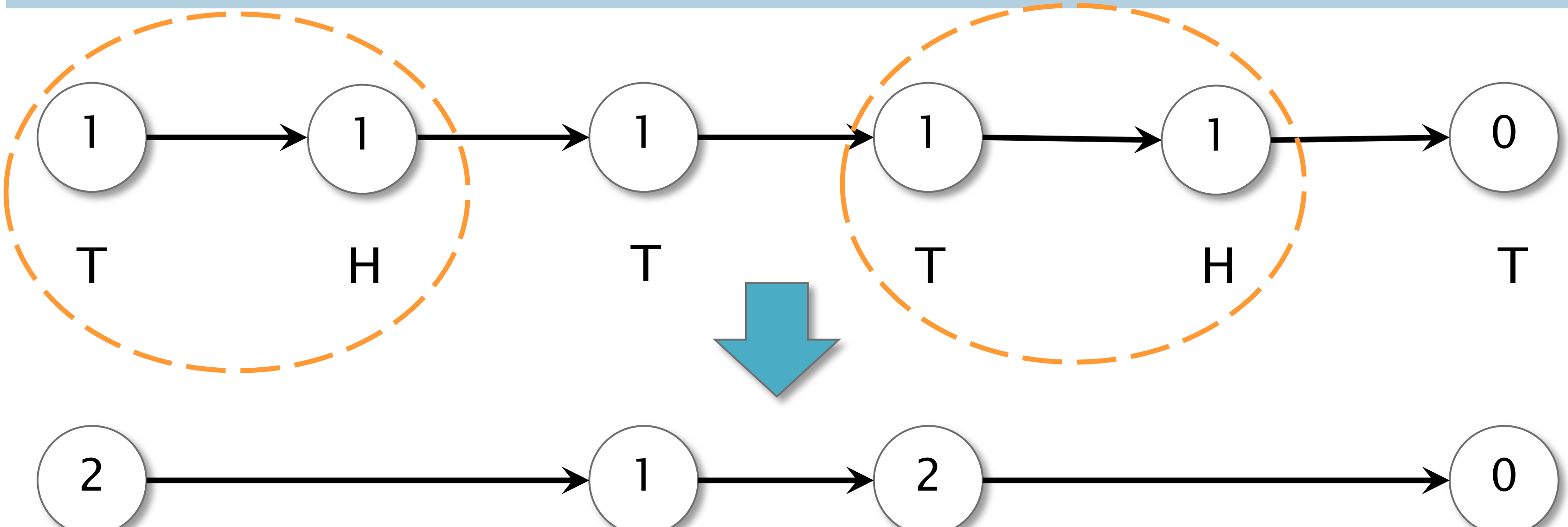
$W = O(n \log n)$
$S = O(\log n)$

Sequential algorithm only requires O(n) work

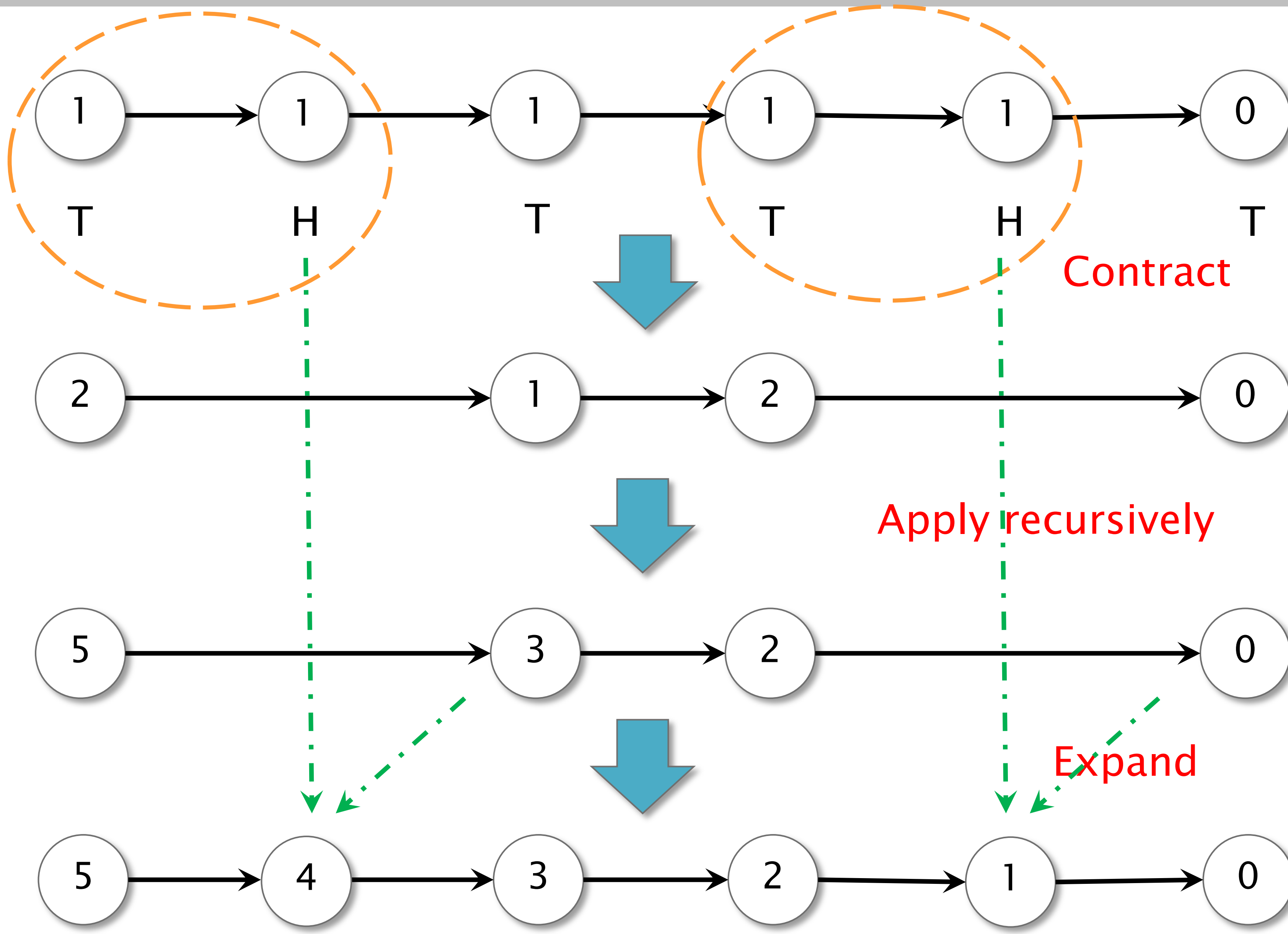
Work-Efficient List Ranking

ListRanking(list P)

1. If list has two or fewer nodes, then return //base case
2. Every node flips a fair coin
3. For each vertex u (except the last vertex), if u flipped Tails and P[u] flipped Heads then u will be paired with P[u]
   A. rank[u] = rank[u]+rank[P[u]]
   B. P[u] = P[P[u]]
4. Recursively call ListRanking on smaller list
5. Insert contracted nodes v back into list with rank[v] = rank[v] + rank[P[v]]



Work-Efficient List Ranking



Work-Span Analysis

- Number of pairs per round is $(n-1)/4$ in expectation
  - For all nodes u except for the last node, probability of u flipping Tails and P[u] flipping Heads is 1/4
  - Linearity of expectations gives $(n-1)/4$ pairs overall
- Each round takes linear work and O(1) span
- Expected work: $W(n) \leq W(7n/8) + O(n)$
- Expected span: $S(n) \leq S(7n/8) + O(1)$

$$W = O(n)$$
$$S = O(\log n)$$

- Can show span with high probability with Chernoff bound