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Near-Isometric Level Set Tracking

Supplementary Material

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1. Proof of Theorem 1

Let $R_t : \mathbb{R}^3 \to \mathbb{R}^3$ be a family of rigid motions of the form $R_t(x) := \mathcal{O}_t x + p_t$ where $\mathcal{O}_t \in \mathrm{SO}(3)$ is an orthogonal matrix with unit determinant and $p_t \in \mathbb{R}^3$ is a vector; neither of these depend on x. The Eulerian velocity vector field of this motion is an *affine* vector field of the form $V_t(x) := A_t x + b_t$ where $A_t := \frac{d\mathcal{O}_t}{dt}\mathcal{O}_t^\top$ and $b_t := \frac{dp_t}{dt} - \frac{d\mathcal{O}_t}{dt}\mathcal{O}_t^\top p_t$. Moreover, by differentiating the identity $\mathcal{O}_t^\top \mathcal{O}_t = Id$, we find that A_t is antisymmetric. The Jacobian matrix of V_t is thus constant and equal to A_t, therefore satisfying $DV_t + [DV_t]^\top = 0$.

For the converse, let $\partial_i V^j$ denote the partial derivatives of the components of V. Then the Killing equation implies that $\partial_1 V^1 = \partial_2 V^2 = \partial_3 V^3 = 0$ as well as $\partial_2 V^1 + \partial_1 V^2 = \partial_3 V^1 + \partial_1 V^3 = \partial_2 V^3 + \partial_3 V^2 = 0$. By taking a second derivative, observe that

$$0 = \partial_2\big(\partial_2 V^1 + \partial_1 V^2\big) = \partial_2\partial_2 V^1 + \partial_1\big(\partial_2 V^2\big)$$

so that $\partial_2\partial_2 V^1 = 0$ since $\partial_2 V^2 = 0$. In the same way, we find $\partial_3\partial_3 V^1 = 0$. Finally, observe that

$$\begin{aligned} 0 &= \partial_3\big(\partial_2 V^1 + \partial_1 V^2\big) + \partial_2\big(\partial_3 V^1 + \partial_1 V^3\big) \\ &= 2\partial_2\partial_3 V^1 + \partial_1\big(\partial_2 V^3 + \partial_3 V^2\big) \end{aligned}$$

so that $\partial_2\partial_3 V^1$ since $\partial_2 V^3 + \partial_3 V^2 = 0$. Thus we have learned that V^1 is an affine function of x^2 and x^3 alone. Similarly, we find that V^2 is an affine function of x^1 and x^3, and V^3 is an affine function of x^1 and x^2. Writing $V^1 := a_{12}x^2 + a_{13}x^3 + c_2$ and so on, we can now substitute this form for V into the Killing equation to find additional constraints on the a- and c-coefficients. In this way, we find that the c-coefficients are unconstrained and the a-coefficients are antisymmetric. This establishes the first part of the lemma

Next, we study the mapping $x \mapsto \mathcal{O}_t(x)$ which solves the ODE (2) with a family V_t satisfying the Killing equation (which we know exists thanks to the assumed smoothness of V_t in t). To show that $\mathcal{O}_t$ is a rigid motion, we show that the derivative matrix $D\mathcal{O}_t$ preserves the inner products of vectors as follows. If $a, b \in \mathbb{R}^3$, then

$$\begin{aligned} \frac{\partial}{\partial t}\big(D\mathcal{O}_t a \cdot D\mathcal{O}_t b\big) &= \sum_{ijk} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{O}_t^k}{\partial x^i}\frac{\partial \mathcal{O}_t^k}{\partial x^j}\right) a^i a^j \\ &= \sum_{ijk}\left(\frac{\partial V_t^k \circ \mathcal{O}_t}{\partial x^i}\frac{\partial \mathcal{O}_t^k}{\partial x^j} + \frac{\partial V_t^k \circ \mathcal{O}_t}{\partial x^j}\frac{\partial \mathcal{O}_t^k}{\partial x^i}\right) a^i a^j \\ &= \sum_{ijk\ell}\left(\frac{\partial V_t^\ell}{\partial x^k} + \frac{\partial V_t^k}{\partial x^\ell}\right)\frac{\partial \mathcal{O}_t^k}{\partial x^i}\frac{\partial \mathcal{O}_t^\ell}{\partial x^j} a^i a^j \\ &= 0 \end{aligned}$$

by the Killing equation. Thus $D\mathcal{O}_t a \cdot D\mathcal{O}_t b$ is constant. □

2. Derivation of PDE Form

To derive the first-order optimality conditions satisfied by the minimizer of (5), we form the Lagrangian

$$\mathcal{L}(V, \lambda) := \frac{1}{2}\int_{\mathcal{U}} \|P(V)\|^2 + \int_{\mathcal{U}} \lambda\left(\frac{\partial F_t}{\partial t} + \nabla F_t \cdot V\right) \tag{1}$$

where $\lambda : \mathcal{U} \to \mathbb{R}$ is the Lagrange multiplier function. Since the minimizing pair (V, λ) is a critical point of $\mathcal{L}$, then for any variation δV of V we have $\frac{d}{d\varepsilon}\mathcal{L}(V + \varepsilon\delta V, \lambda)\big|_{\varepsilon=0} = 0$. Expanding this expression provides the *weak form* of the optimality conditions:

$$0 = \int_{\mathcal{U}}\left(\mathrm{Tr}\big(P(V)[P(\delta V)]^\top\big) + \lambda\nabla F_t \cdot \delta V\right). \tag{2}$$

If we then integrate by parts, we find

$$0 = \int_{\mathcal{U}}\left(P^* P(V) + \lambda\nabla F_t\right)\cdot\delta V + \int_{\partial\mathcal{U}} N_{\partial\mathcal{U}} \cdot P(V)\cdot\delta V, \tag{3}$$

where P^* : *Symmetric matrix fields* $\to$ *vector fields* is the adjoint operator of P. Also, $N_{\partial\mathcal{U}}$ is the unit normal vector of $\partial\mathcal{U}$. Since Equation (3) is true for all variations δV, we conclude that the integrands appearing there must vanish.

3. Proof of Theorem 4

If $\Omega_t = R_t(\Omega)$ for some rigid motion R_t its level set function satisfies $F_t := F \circ R_t^{-1}$ where F is a level set function for the reference geometry. As we know, the Eulerian velocity $V_t(x) := \frac{dR_t}{dt} \circ R_t^{-1}(x)$

is a Killing vector field satisfying $P(V_t) = 0$. Let V_t have components $[V_t(x)]^i = \sum_{jk} \frac{d[R_t]^i_j}{dt} [R_t]^k_j x^j$. Then,

$$
\begin{aligned}
&\sum_i [V_t(x)]^i \frac{\partial F \circ R_t(x)}{\partial x^i} + \frac{\partial F \circ R_t(x)}{\partial t} \\
&\quad = \sum_k \frac{\partial F}{\partial x^k} \circ R_t(x) \left([R_t]^k_i \sum_{jk} \frac{d[R_t]^i_j}{dt} [R_t]^k_j + \frac{d[R_t]^k_j}{dt} \right) x^j \,.
\end{aligned}
$$

The term in brackets vanishes because the linear part of R_t is an orthogonal matrix. Thus V satisfies the constraints as well. Therefore $(V_t, \lambda) = (0, 0)$ is the solution of the PDE.

4. Discrete Optimality Conditions

We obtain the discrete optimality equations by substituting the reduced forms of V and δV into (2). That is,

$$
\begin{aligned}
V &:= \sum_i \sum_{s=1}^{3} \sum_{s'=1}^{2} (a_{is'} z_{iss'} + w_{is}) \xi_i e_s \\
V' &:= \sum_{t=1}^{3} z_{jtt'} \xi_j e_t \quad \forall\, j \text{ and } \forall\, t' = 1, 2 \,.
\end{aligned}
\tag{4}
$$

Since the variation V' above is orthogonal to ∇F_t by construction, the Lagrange multiplier term in (2) vanishes, leaving

$$
\begin{aligned}
0 &= \int_{U_\varepsilon} \mathrm{Tr}\big(P(V)[P(V')]^\top\big) \\
&= 2 \sum_i \sum_{s,t,u,v=1}^{3} (\delta_{sv}\delta_{tu} + \delta_{st}\delta_{uv}) v_{is} z_{jtt'} \sum_T \int_T \frac{\partial \xi_i}{\partial x^u} \frac{\partial \xi_j}{\partial x^v}
\end{aligned}
\tag{5}
$$

after expanding in terms of the partial derivatives of the components of V and V'. Here δ_{ij} is the Kronecker delta v_{is} are the components of V in the expansions (5).

To evaluate further we need a formula for $\nabla \xi_i$, which is piecewise constant since ξ_i is piecewise linear. Let $T := [x_i, y_1, y_2, y_3]$ be a tetrahedron containing x_i and let $n(i,T)$ be the inward-pointing unit vector normal to the face $[y_1, y_2, y_3]$ and let $A(i,T)$ be its area. Then a straightforward geometric calculation shows that

$$
\nabla \xi_i \big|_T = -\frac{A(i,T)}{3\mathrm{Vol}(T)} n(i,T) \,.
$$

The product of the partial derivatives in (5) is supported on T if and only if x_i, x_j are both vertices of T, so for each i

$$
\sum_T \int_T \frac{\partial \xi_i}{\partial x^u} \frac{\partial \xi_i}{\partial x^v} = \frac{1}{9} \sum_{T \in R(i)} \frac{(A(i,T))^2}{\mathrm{Vol}(T)} n_u(i,T) n_v(i,T), \tag{6a}
$$

where $R(i)$ is the one-ring of tetrahedra containing x_i; and for each pair $i \neq j$ such that $[x_i, x_j]$ is an edge

$$
\sum_T \int_T \frac{\partial \xi_i}{\partial x^u} \frac{\partial \xi_j}{\partial x^v} = \frac{1}{9} \sum_{T \in R(i,j)} \frac{A(i,T)A(j,T)}{\mathrm{Vol}(T)} n_u(i,T) n_v(j,T), \tag{6b}
$$

where $R(i,j)$ is the one-ring of tetrahedra containing $[x_i, x_j]$. We now substitute these expressions into (5) and find

$$
0 = \sum_i \sum_{s,t=1}^{3} K_{ijst} (a_{is'} z_{iss'} + w_{is}) z_{jtt'} \quad \forall\, j \text{ and } \forall\, t' = 1, 2 \,. \tag{7}
$$

where K_{ijst} are the coefficients of the stiffness matrix.

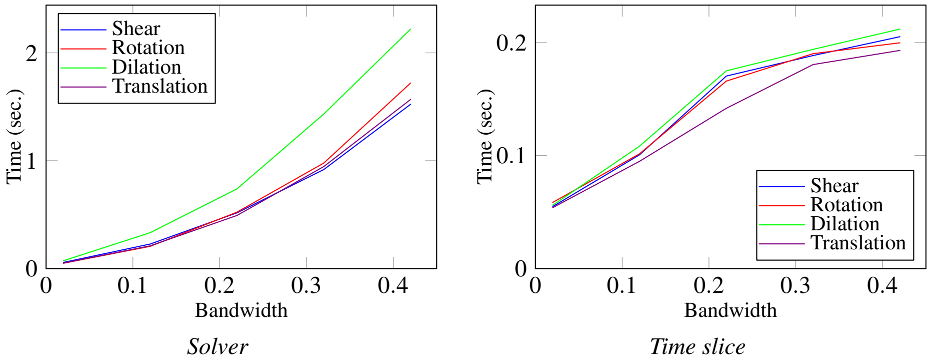

Figure 1: Solver execution time and time slice construction time as a function of bandwidth size with respect to a fixed background grid resolution (equal to $60 \times 60 \times 60$) in 3D. Units are fractions of the diameter of the background grid. We collect data from the ellipse moving according to four different types of motion.

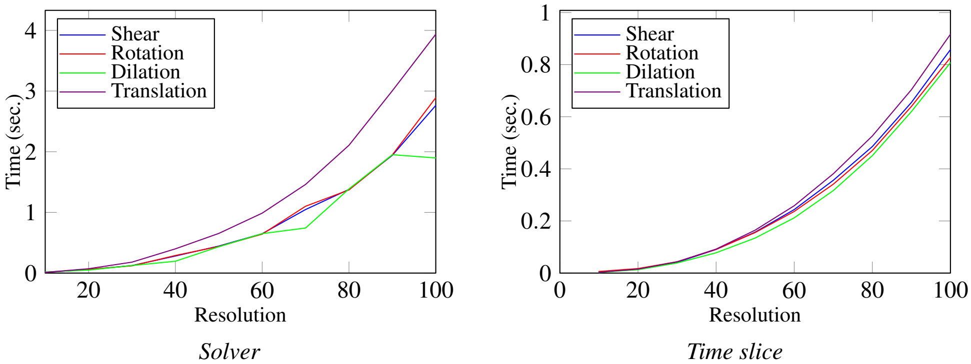

Figure 2: Solver execution time and time slice construction time as a function of grid resolution using a fixed-size narrow band (equal to 0.25) in 3D. We collect data from the ellipse moving according to four different types of motion.