Laplacian Operators

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“Laplace-Beltrami: The Swiss Army Knife of Geometry Processing”
SGP 2014 tutorial
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My Background

Similar themes in ML!

Geometry processing
Motivation: Interpolation Problem

Predict $f: \Omega \rightarrow \mathbb{R}$ from values on the boundary $\partial \Omega$. 

\[
\begin{align*}
f &= -1 \\
f &= 1
\end{align*}
\]
Desired: Smooth Functions

No discontinuities allowed!
Desired: Smooth Functions

Don’t want large variation over small distance
Desired: Smooth Functions
Dirichlet Energy

\[ E(f) \equiv \int_{\Omega} \| \nabla f \|^2 \, dA \]
Dirichlet Energy

\[ E(f) \equiv \int_{\Omega} \| \nabla f \|^2 dA \]

- Nonnegative
- Zero for constant functions
- Measures smoothness
Dirichlet Problem

\[ \min_f E(f) \equiv \int_\Omega \| \nabla f \|^2 \, dA \]

s.t. \( f \big|_{\partial \Omega} \) given

Set derivative to zero

\[ \Delta f(x) = 0 \quad \forall x \in \Omega \setminus \partial \Omega \]

\[ f(x) = f_0(x) \quad \forall x \in \partial \Omega \]
Laplacian Operator

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Laplacian Operator

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Intuition

Temperature at steady state

\[ \Delta f(x) = 0 \]

\[ \Delta f(x) = 0 \quad \forall x \in \Omega \setminus \partial \Omega \]

\[ f(x) = f_0(x) \quad \forall x \in \partial \Omega \]
Dirichlet on Other Domains

\[ E(f) \equiv \int_{\Omega} \| \nabla f \|^2 \, dA \]
Dirichlet on Other Domains

\[ E(f) \equiv \int_{\Omega} \| \nabla f \|^2 \, dA \]
Pattern for Finding Laplacian

$$\min_f \quad E(f) \equiv \int_\Omega \| \nabla f \|^2 \, dA$$

s.t. \quad f |_{\partial \Omega} \text{ given}

Set derivative to zero

$$\Delta f(x) = 0 \quad \forall x \in \Omega \setminus \partial \Omega$$

$$f(x) = f_0(x) \quad \forall x \in \partial \Omega$$
Related Equations

Heat equation

\( \frac{\partial f}{\partial t} = \Delta f \)

Gradient descent on Dirichlet energy
Related Equations

\[ \Delta f = g \]

\[ \min_f \int_{\Omega} \| \nabla f - v \|^2 \, dA \]

\[ \implies \Delta f = \nabla \cdot v \]

Poisson equation
Algebraic Properties

- **Linearity:** \[ \Delta (f(x) + \alpha g(x)) = \Delta f(x) + \alpha \Delta g(x) \]
- **Constants in kernel:** \[ \Delta \alpha = 0 \]

For functions that vanish on \( \partial M \):

- **Self-adjoint:** \[ \int_M f \Delta g \, dA = - \int_M \langle \nabla f, \nabla g \rangle \, dA = \int_M g \Delta f \, dA \]
- **Negative:** \[ \int_M f \Delta f \, dA \leq 0 \]

(Intuition: \( \Delta \approx \) an \( \infty \)-dimensional negative-semidefinite matrix)
Harmonic Functions

$$\Delta f(x) = 0$$
Smooth and analytic

Mean value property:

\[ f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y) \, dA \]

Maximum principle: No local maxima or minima
(can have saddles)
for a curve $\gamma(u) = (x[u], y[u]) : \mathbb{R} \to \mathbb{R}^2$

- total Dirichlet energy $\int \| \nabla x \|^2 + \| \nabla y \|^2$ is arc length
- $\Delta \gamma = (\Delta x, \Delta y)$ is gradient of arc length
- $\Delta \gamma$ is the curvature normal $\kappa \hat{n}$
- minimal curves are harmonic (straight lines)
for a surface $r(u,v) = (x[u,v], y[u,v], z[u,v]) : \mathbb{R} \to \mathbb{R}^3$

- total Dirichlet energy is surface area
- $\Delta r = (\Delta x, \Delta y, \Delta z)$ is gradient of surface area
- $\Delta r$ is the mean curvature normal $2H\hat{n}$
- minimal surfaces are harmonic!
Laplacian is Intrinsic
The Laplacian spectrum can be represented as:

$$\Delta \phi_k(x) = \lambda_k \phi_k(x)$$

with the eigenvalues ordered as:

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

Intrinsic Fourier basis
Boundary Conditions

- can specify $\nabla f \cdot \hat{n}$ on boundary instead of $f$:
  \[
  \Delta f(x) = 0 \quad x \in \Omega \\
  f(x) = f_0(x) \quad x \in \partial \Omega_D \quad \text{(Dirichlet bdry)} \\
  \nabla f \cdot \hat{n} = g_0(x) \quad x \in \partial \Omega_N \quad \text{(Neumann bdry)}
  \]

- usually: $g_0 = 0$ (**natural bdry conds**)
- physical interpretation: free boundary through which heat cannot flow
Laplacian matrices should be:
- Sparse
- Positive (semi-)definite

Typical solvers
- Direct: LDLT
- Iterative: Conjugate gradients
Constructing Laplacian Operators

Per-vertex functions on a graph

\[ E(f) = \sum_{e=(i,j)} (f(v_i) - f(v_j))^2 \]
Constructing Laplacian Operators

Given pairwise similarity measure

\[ E(x) = \sum_{i,j} w_{ij} (x_i - x_j)^2 \]
Integration by Parts

\[
\int_{\Omega} f \Delta g \, dA = \text{boundary terms} - \int_{\Omega} \nabla f \cdot \nabla g \, dA
\]
$L^2$ Dual of a Function

$f : \Omega \rightarrow \mathbb{R}$

$\Downarrow$

$\mathcal{L}_f : L^2(\Omega) \rightarrow \mathbb{R}$

$\mathcal{L}_f [g] \equiv \int_{\Omega} fg \, dA$

“Test function”
Observation

Can recover function from dual
Use Laplacian without evaluating it!

Dual of Laplacian

$$\mathcal{L}_{\Delta f} [g] = \int_{\Omega} g \Delta f \, dA$$

$$= \text{const.} - \int_{\Omega} \nabla f \cdot \nabla g \, dA$$

One derivative is enough
Choose one of each:

- Function space
- Test functions

Often the same!
First Order Finite Elements

One “hat function” per vertex

\[ f(x) = \sum_i a_i h_i(x) \rightarrow \vec{a} \]
Weak Solutions

\[ \int_{\Omega} \phi \Delta f \, dA = \int_{\Omega} \phi g \, dA \quad \forall \text{test functions } \phi \]
\[ \int_{\Omega} h_i \Delta f \, dA = \int_{\Omega} h_i g \, dA \quad \forall \text{hat functions } h_i \]

\[ \int_{\Omega} h_i \Delta f \, dA = - \int_{\Omega} \nabla h_i \cdot \nabla f \, dA \]

\[ = - \int_{\Omega} \nabla h_i \cdot \left( \sum_j a_j h_j \right) \, dA \]

\[ = - \sum_j a_j \int_{\Omega} \nabla h_i \cdot \nabla h_j \, dA \]

\[ \equiv \sum_j L_{ij} a_j \quad \text{(linear system!)} \]
Poisson Equation with FEM

\[ \Delta f = g \quad \rightarrow \quad L f = A g \]

Must integrate to zero

Determined up to constant
(Generalized) Eigenhomomers

\[ L\phi = \lambda A\phi \]
Applications to Geometry
Most obvious:

Graph Laplacian

- Geometric structure for data points
- Use case: semi-supervised learning
But:
Graph Laplacian is a weak notion of geometry.

- How do you construct the graph?
- How do you understand distances?
Connections to Machine Learning

Laplacians (and their inverses) come from "kernel matrices."

- Ingredients: Gradients and inner products
- PDE in high-dimensional point clouds?
- Can we learn Laplacians?
- Can we determine intrinsic dimensionality?
- Can you hear the shape of a dataset?
Laplacian Operators

Headed out tomorrow!