Dirichlet Energy for Analysis and Synthesis of Soft Maps

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— Introduction —
Mappings Between Shapes

Let $M_0$ and $M$ be smooth surfaces discretized as triangle meshes. We consider discrete representations of smooth maps $\phi : M_0 \rightarrow M$.

The maps of interest should satisfy certain properties:

- **Geometric**
  - Bijective
  - Continuous
  - Preserves fine details

- **Semantic**
  - Meaningful
  - Preserves features
  - Satisfies user constraints
Difficulties with Point-to-Point Representations

An obvious discrete representation for a map is a vertex-to-vertex correspondence. This is inherently \textit{combinatorial} and has drawbacks.

- The vast majority of $v_x$-to-$v_x$ maps are in no way desirable.
- Continuity cannot be properly defined and quantified.
- The mesh itself interferes at the smallest scale!

\textbf{Thus:} $v_x$-to-$v_x$ maps involve

- Subsampling.
- Measuring pairwise distances and adjacency relationships.
- This leads to problems!
Continuity

In principle: These problems should be detectable via a failure of continuity somewhere. Continuity should have a regularizing effect.

• Why? Think of a result like the Intermediate Value Theorem.

The problem: Vertex-to-vertex representations are not adequate for quantifying continuity at this infinitesimal scale.

Possible resolution: An alternate representation for smooth maps.

• It should make sense for smooth surfaces yet be easily discretized, and should be convergent under mesh refinement.

• Continuity should make sense both discretely and in the smooth limit, and should be quantifiable.

• We should still be able to incorporate desirable map properties.
Soft Maps

We propose a representation that takes a probabilistic approach.

**Definition:** A soft map from $M_0$ to $M$ is a map $\mu : M_0 \rightarrow \text{Prob}(M)$.

I.e. every point on the source surface $M_0$ maps to a probability distribution of potential matches on the target surface $M$.

- **Interpretation:**
  
  $$\mu_x = \left[ \begin{array}{c} \text{Probability that } y \in M \\ \text{corresponds to } x \in M_0 \end{array} \right]$$

- **Recall SGP 2012.**
  
  (Then: approximation by histograms. Now: the limit as the bin size $\rightarrow 0$.)
Advantages of Soft Maps

- They can be defined via scalar functions on $M_0 \times M$.
  - Each $\mu_x$ has a positive density that integrates to one.

- They generalize point-to-point maps $\phi : M_0 \rightarrow M$.
  - The associated density is sharply peaked at $\phi(x)$.

- They permit blurring and superposition.

The “ideal” soft map is a convex combination of a small number (associated with symmetries) of blurred point-to-point maps.
— Soft Map Energies —
Quantifying Continuity

Recall: Dirichlet energies quantify the “degree of continuity” of mappings between domains in many different contexts.

Examples:

- For $f : M_0 \rightarrow \mathbb{R}$,
  $$\mathcal{E}_D(f) := \int_{M_0} \|\nabla_0 f(x)\|^2 \, dx$$

- For $\phi : M_0 \rightarrow M$,
  $$\mathcal{E}_D(\phi) := \int_{M_0} \|\nabla_0 \phi(x)\|^2 \, dx$$
Quantifying Continuity

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  \]

- For \( \phi : M_0 \rightarrow M \)
  \[
  \mathcal{E}_D(\phi) := \int_{M_0} \| \nabla_0 \phi(x) \|^2 \, dx
  \]

A generalization: These are all instances of a general framework for maps \( \phi : (M_0, dist_0) \rightarrow (M, dist) \) between any metric spaces:

\[
\mathcal{E}_D(\phi) := \int_{M_0} \left( \lim_{\varepsilon \rightarrow 0} \frac{1}{\text{Area}(B_\varepsilon(x))} \int_{B_\varepsilon(x)} \frac{\text{dist}^2(\phi(x), \phi(x'))}{\text{dist}_0^2(x, x')} \, dx' \right) \, dx
\]
The Wasserstein Metric on $\text{Prob}(M)$

**Key idea:** We can view $\text{Prob}(M)$ as a metric space.

- The theory of **Optimal Transportation** gives us a metric on $\text{Prob}(M)$ called the **Wasserstein metric** with **quadratic cost**.

Let $\mu, \nu \in \text{Prob}(M)$.

The distance $W_2(\mu, \nu)$ is the cost of the optimal way of transporting mass from the distribution $\mu$ to the distribution $\nu$.

The transportation cost for an individual particle from $y$ to $y'$ is $\text{dist}^2(y, y')$.

- This is also called the **Earth Mover’s Distance** (with quadratic cost).

**Consequence:** We can define a Dirichlet energy for soft maps.
The Dirichlet Energy of a Soft Map

**Definition:**

Let $\mu : M_0 \to \text{Prob}(M)$ be a soft map.

The Dirichlet energy of $\mu$ is the quantity

$$E_D(\mu) := \int_{M_0} \left( \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(B_\varepsilon(x))} \int_{B_\varepsilon(x)} \frac{W^2_2(\mu_x, \mu_{x'})}{\text{dist}^2_0(x, x')} dx' \right) dx$$

**Key properties:**

- The Dirichlet energy is convex in $\mu$.
- It generalizes the Dirichlet energy for maps. So if $\phi$ is a map and $\mu_\phi$ is the associated soft map then $E_D(\mu_\phi) = E_D(\phi)$.
- The Dirichlet energy of any constant soft map is zero.
Simplification of the Dirichlet Energy

Problem: This form of the Dirichlet energy is difficult to work with.

Theorem: The following simplification holds.

Consider a soft map with smooth positive density $\rho(x, y)$. Then the Dirichlet energy of $\rho$ satisfies

$$\mathcal{E}_D(\rho) = \int_{M_0 \times M} \rho(x, y) \| \nabla Q(x, y) \|^2 dy \, dx.$$ 

The quantity $Q$ is vectorial and lives on $M_0 \times M$.

It is defined as follows. For each $x$ and direction $V$, then $Q(x, y)$ satisfies the linear PDE in the $y$-variables given by

$$\nabla \cdot (\rho \nabla \langle Q, V \rangle) = -\langle \nabla_0 \rho, V \rangle$$
Interpretation of $Q$

We call $Q$ the **transportation potential** of the soft map. We can interpret it in terms of conservative mass flow.

I.e. we view each $\rho(x, \cdot)$ is a swarm of particles. Now:

- Displace $x$ in the direction $V$ to an infinitesimally near $x'$.
- The mass distribution $\rho(x, \cdot)$ changes into $\rho(x', \cdot)$.
- Assume it’s by optimal transport.
- The particle at $y$ has velocity equal to $\nabla Q(x, y) \cdot V$
- The Wasserstein distance relates to the kinetic energy.

\[
\frac{W_2^2(\mu_x, \mu_{x'})}{\text{dist}_0^2(x, x')} \approx \int_M \rho(x, y) \| \nabla Q(x, y) \cdot V \|^2 \, dy
\]
Soft Map Bijectivity

An issue: All constant soft maps all have the same minimal Dirichlet energy equal to zero. Can we tell them apart?

Idea: Measure the equidistribution of probabilistic mass pushed forward from $M_0$ to $M$. Quantify as follows.

- We can interpret the integral $b(y) := \int_{M_0} \rho(x, y)dx$ as the probability that $y$ receives mass from somewhere in $M_0$.

- So if the square integral
  \[ \mathcal{E}_b(\rho) := \int_M \left( \int_{M_0} \rho(x, y) \, dx \right)^2 \, dy \]

  is small, then each $\rho(x, \cdot)$ is as spread out as possible and each point of $M$ receives an equal amount of mass from $M_0$.

- We call $\mathcal{E}_b(\rho)$ the bijectivity energy of $\rho$. 
— Soft Map Analysis and Synthesis —
The two energies and their densities that we have introduced can be used for soft map analysis.

We can study:

- The soft map of a pt-to-pt map.
- Or a soft map coming from shape descriptor differences, of the form
  \[ \rho(x, y) \propto e^{-\frac{(d_1(x) - d_2(y))^2}{\sigma^2}} \]

Energies of various self-maps of the sphere.

Unfavourable stretching in WKS revealed by the Dirichlet energy.
Local Correspondence Extraction

Recall: Choose $x$ and a direction $V$. Let the mass distribution $\rho(x, \cdot)$ change optimally with $x$.

Then the particle at $y$ moves with velocity $\nabla Q(x, y) \cdot V$.

So what: We get a method for extracting point correspondences.

- Choose a path $x(\varepsilon)$ s.t. $x(0) = x_{\text{init}}$ and decide on a point $y_{\text{init}} \in M$ that should correspond to $x_{\text{init}}$.
- Integrate the velocity field $\dot{y} = \nabla Q(x, y) \cdot \dot{x}$
- Get a path $y(\varepsilon)$ in $M$ with initial data $y(0) = y_{\text{init}}$.
- The paths $x(\varepsilon)$ and $y(\varepsilon)$ are now in correspondence.
Generating Soft Maps

**Goal:** Generate soft maps by solving a constrained optimization problem in the space of soft maps. It’s convex!

\[
\text{minimize} \quad \mathcal{E}(\mu) := \mathcal{E}_D(\mu) + \lambda \mathcal{E}_b(\mu)
\]

**And:** To avoid the constant soft map, we must impose constraints.

- E.g. a few points or subsets of \( M_0 \) and \( M \) must correspond.
- This is similar to the harmonic maps problem.

Source, red constraints \( \rightarrow \) Optimal soft map distributions associated to the yellow points.
Conclusion and Future Work

What we have done:

- Introduced a representation for maps that supports a Dirichlet energy for measuring continuity.
- Used this representation for map analysis and synthesis.

What we would like to do next:

- More efficient computation of $Q$.
- Decomposition of $\rho$ into a convex combination of soft maps associated to maps.
- Map extraction at multiple scales.
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