An $\tilde{O}(n^{3/14})$-coloring for 3-colorable graphs

Avrim Blum[*]
School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

David Karger[†]
Lab for Computer Science
MIT
Cambridge, MA 02139

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Abstract

We show how the results of Karger, Motwani, and Sudan [6] and Blum [3] can be combined in a natural manner to yield an $\tilde{O}(n^{3/14})$-coloring of any n-node 3-colorable graph. This improves on the previous best bound of $\tilde{O}(n^{1/4})$ colors [6].

1 Introduction

A k-coloring of an n-node graph is an assignment of one of k colors to each of the vertices in the graph so that no two adjacent vertices receive the same color. It is well known that the question: "given a graph G, can it be colored with k or fewer colors?" is NP-hard for any $k \geq 3$. In addition, recent results show that coloring 3-colorable graphs with 4 or fewer colors is NP-hard [5] and much stronger approximation hardness results are known for the general chromatic number problem [4]. Wigderson [8] describes a very simple method to color any n-node 3-colorable graph with $O(\sqrt{n})$ colors. This approximation guarantee was improved by Blum [2, 3] to $\tilde{O}(n^{3/8})$ colors and most recently by Karger, Motwani, and Sudan to $\tilde{O}(n^{1/4})$ colors, where the $\tilde{O}$ notation hides logarithmic factors. In this paper we show how ideas used in the latter two results can be combined to produce a $\tilde{O}(n^{3/14})$-coloring of any n-node 3-colorable graph.

2 Preliminaries and Definitions

We will use $N(v)$ to denote the neighborhood of a vertex v, and $N(S)$ to denote the neighborhood of a set of vertices S: that is, $N(S)$ is the union of the neighborhoods of all vertices in S.

[*]Supported in part by NSF National Young Investigator grant CCR-9357793 and a Sloan Foundation Research Fellowship. Email: avrim@cs.cmu.edu. URL: http://www.cs.cmu.edu/~avrim

[†]Supported in part by ARPA Contract N00014-95-1-1246. Email: karger@lcs.mit.edu. URL: http://theory.lcs.mit.edu/~karger

It will be convenient to be able to talk about "making progress" towards a desired approximate coloring. Specifically, we will say that finding one of some set of objects "makes progress towards a $C(n)$-coloring", to mean that if on any 3-colorable n-node graph we can guarantee to find one of those objects, then that can be used by a generic algorithm to produce a $C(n)$-coloring of any n-node 3-colorable graph. For example, it is not hard to see that finding an independent set of size n^α makes progress towards an $O(n^{1-\alpha})$-coloring, because if we can find such a set of vertices, we can give those vertices one color, remove them from the graph, and recurse on the graph remaining. More generally, the following useful lemma is given in [3].

Lemma 1 (From Theorem 3 in [3]) *Finding any of the following objects in an n-vertex 3-colorable graph G makes progress towards an $O(n^\alpha)$-coloring of G.*

1. *An independent or 2-colorable set of size $\Omega(n^{1-\alpha})$.*
2. *An independent set S having a neighborhood of size $O(|S|n^{1-\alpha})$.*
3. *Two vertices that share $\Omega(n^{1-2\alpha})$ neighbors in common.*

(The most interesting part of the above lemma, item (3), follows from noticing that if two vertices u and v belong to different color classes in a legal 3-coloring of the graph, then their shared neighborhood S belongs to a single color class, and therefore *its* neighborhood T is 2-colorable. Thus, if T is not 2-colorable then one can merge u and v; if T is 2-colorable and large, one can apply item (1); and if T is 2-colorable and small, one can apply item (2). A similar argument is given in more detail in the next section.)

3 A simpler result

Before describing the $\tilde{O}(n^{3/14})$-coloring algorithm, we give a simpler argument that yields a $\tilde{O}(n^{2/9})$-coloring. We begin with a theorem from Karger-Motwani-Sudan [6].

Theorem 1 *[6] There is a (randomized) polynomial-time algorithm that will color any 3-colorable graph having maximum degree Δ with $\tilde{O}(\Delta^{1/3})$ colors.*

An examination of the proof of this theorem yields the following corollary:

Corollary 2 *Suppose a 3-colorable graph can have its edges directed so that the maximum outdegree is Δ. Then there is a polynomial-time algorithm to color the graph with $\tilde{O}(\Delta^{1/3})$ colors.*

The goal of our algorithm will be to find an independent set of size $\tilde{\Omega}(n^{7/9})$, or else two vertices that share at least $\tilde{\Omega}(n^{5/9})$ neighbors in common, in our given graph G. Either way we make progress by Lemma 1. We begin by removing from G, one by one, all vertices of degree at most $n^{2/3}$ (in the current G). If in doing so we have removed more than half of the vertices of G, then we are done: we simply apply the algorithm of Corollary 2 to the set of $n/2$ vertices removed, and then choose the largest independent set in the coloring produced (which must have size $\tilde{\Omega}((n/2)/(n^{2/3})^{1/3}) = \tilde{\Omega}(n^{7/9})$). Therefore, we may assume below that we now have a graph G on $n/2$ vertices whose minimum degree is greater than $n^{2/3}$.

Lemma 2 *Every vertex in G shares at least $n^{1/3}$ neighbors with some other vertex.*

Proof: Let v be some vertex in G. v has at least $1 + n^{2/3}$ neighbors, each with at least $n^{2/3}$ neighbors in addition to v. Thus, there are at least $n^{4/3}$ length-two paths from v that terminate in vertices besides v. Therefore at least $n^{1/3}$ of them must terminate at some specific other vertex u. It follows that u and v have at least $n^{1/3}$ neighbors in common. ∎

We now proceed as follows, using an argument similar to that for item (3) of Lemma 1. Let u and v be two vertices whose shared neighborhood $S = N(u) \cap N(v)$ has size at least $n^{1/3}$, and let T be the neighborhood of S. Notice that if u and v have different colors in the optimum coloring, then S is monochromatic and therefore T may be 2-colored. Thus, if we find that T is *not* 2-colorable, we know u and v must have the same color in the optimum coloring and so we can merge them and start over on a graph of one less vertex. Also, if we find that T *is* 2-colorable and $|T| \geq n^{7/9}$, then we can make progress towards an $O(n^{2/9})$-coloring by Lemma 1. Therefore, we may assume below that $|T| < n^{7/9}$.

We now show that there must be two vertices in S that share at least $\frac{1}{4}n^{5/9}$ neighbors in common. Let S' be an arbitrary subset of S of size $2n^{1/9}$ and suppose for contradiction that no two vertices of S' share $\frac{1}{4}n^{5/9}$ neighbors. Since each vertex of S' has degree at least $n^{2/3}$ it follows that each vertex of S' has at least $n^{2/3} - 2n^{1/9}(\frac{1}{4}n^{5/9}) = \frac{1}{2}n^{2/3}$ neighbors *not* in common with *any* other vertex of S'. Therefore, $|N(S')| \geq 2n^{1/9} \cdot \frac{1}{2}n^{2/3} = n^{7/9}$. But this contradicts our assumption that $|T| < n^{7/9}$. Therefore, there must be two vertices in S' that share at least $\frac{1}{4}n^{5/9}$ neighbors in common, so we can make progress by Lemma 1.

4 The $\tilde{O}(n^{3/14})$-coloring

We will need the following definitions from [3]. Let $d(v)$ denote the degree of vertex v. If S and T are sets of vertices, let $d_T(v) = |N(v) \cap T|$, and let $D_T(S) = \sum_{v \in S} d_T(v)$. In other words, $D_T(S)$ is the number of edges between S and T, double-counting for edges whose endpoints are in both sets. Write $D_V(S)$, which is just the sum of degrees of vertices in S, as $D(S)$. Define (as before) $N(S) = \bigcup_{v \in S} N(v)$.

Let $\delta = \delta(n) = \frac{1}{5 \log n}$. We define $I_j = \{v \in V \mid d(v) \in [(1+\delta)^j, (1+\delta)^{j+1})\}$ for $j = 0, 1, 2, \ldots$. That is, we divide the set of vertices of degree at least 1 into bins I_j so that in each bin, the ratio of the degrees of any two vertices is less than $(1+\delta)$. For a set of vertices S, let $N_i(S) = \{v \in N(S) \mid d_S(v) \in [(1+\delta)^i, (1+\delta)^{i+1})\}$ for $i = 0, 1, 2, \ldots$. In other words, $N_i(S)$ is the subset of vertices in $N(S)$ that are hit by at least $(1+\delta)^i$ and less that $(1+\delta)^{i+1}$ edges from S.

We now present slightly simplified versions of three theorems from [3]. Let R (red) be a color class in a correct 3-coloring of G such that $D(R) \geq \frac{1}{3}D(V)$, or equivalently, $D(R) \geq \frac{1}{2}D(V-R)$. Notice that $D_R(V-R) = D(R)$ so $D_R(V-R) \geq \frac{1}{2}D(V-R)$; or in words: our definition of "red" is such that on average, a non-red vertex has at least half its neighbors colored red.

Theorem 3 ([3]) *Given a 3-colorable n-vertex graph $G = (V, E)$, there is some $v \in R$ and some $j \in \{0, 1, \ldots, \log_{1+\delta} n\}$ such that the set $S = N(v) \cap I_j$ satisfies the following:*

$$\begin{array}{ll} 1. & |S| \;\geq\; \delta^2 D(R)/(|R| \log_{1+\delta} n), \\ 2. & D_R(S) \;\geq\; \frac{1}{2}(1-3\delta)D(S). \end{array}$$

In other words, for some $v \in R$, the set $S = N(v) \cap I_j$ is reasonably large and almost half of the edges incident to it have their other endpoint in R.

The next theorem considers the neighbors of S and states that for some i, the set $T = N_i(S)$ is nearly half red in the correct 3-coloring. This is useful because given a set of vertices T which contains an independent set of size at least $\frac{1}{2}(1 - \frac{1}{\log n})|T|$, one can *find* an independent set of size at least $|T|/\log |T|$ using an algorithm of Bar-Yehuda and Even [1] and (independently) Monien and Speckenmeyer [7]. Thus, if T is sufficiently large we can make progress towards our desired coloring.

Theorem 4 ([3]) *Given an n-vertex 3-colorable graph $G = (V, E)$, and $\lambda \in [0, 1]$:*

For any set S such that $D_R(S) \geq \lambda D(S)$, there must exist some $i < \log_{1+\delta} n$ such that the set $T = N_i(S)$ satisfies the following:

$$\begin{array}{ll} 1. & D_T(S) \;\geq\; \delta D_R(S)/\log_{1+\delta} n, \\ 2. & |T \cap R|/|T| \;\geq\; (1-2\delta)\lambda. \end{array}$$

The final theorem from [3] that we will use implies that if the set T in the above theorem is much smaller than what one would expect given the number of edges between S and T, then one can make progress towards the desired coloring.

Theorem 5 ([3]) *Given sets of vertices S and T in an n-vertex 3-colorable graph G, such that*

1. *S is 2-colored under some legal 3-coloring of G,*
2. *$D_T(S) = \Omega(|S| n^{1-2\alpha} \log^2 n)$, and*
3. *$[D_T(S)]^3 = \Omega\left(\left[|S| + \max_{v \in S} d_T(v)\right] \times \left[|S||T|^2 n^{1-2\alpha} \log n + |T||S|^2 n^{2-4\alpha}\right]\right)$,*

we can make progress towards an $O(n^\alpha)$-coloring of G.

We now combine these theorems with the results of KMS [6] to achieve an $\tilde{O}(n^{3/14})$-coloring of any n-vertex 3-colorable graph.

Theorem 6 *There is a polynomial time algorithm to color any n-vertex 3-colorable graph with $\tilde{O}(n^{3/14})$ colors.*

Proof:

As in the algorithm of the previous section, we begin by removing all low-degree vertices from G, where "low degree" now means degree at most $n^{9/14}$. If we are able to remove more than $n/2$ vertices we make progress using Corollary 2 as before. Thus, we may assume from now on that all vertices in G have degree at least $n^{9/14}$.

Theorems 3 and 4 imply that for some vertex v and some indices $i, j \in \{0, \ldots, \log_{1+\delta} n\}$, the sets $S = N(v) \cap I_j$ and $T = N_i(S)$ satisfy the following properties.

$$\begin{aligned} |T \cap R|/|T| &\geq (1-2\delta)(1-3\delta)\frac{1}{2} \\ &\geq \frac{1}{2}(1 - \frac{1}{\log n}) \end{aligned}$$

and

$$\begin{aligned} D_T(S) &= \Omega(\delta D(S)/\log_{1+\delta} n) \\ &= \Omega(\delta n^{9/14}|S|/\log_{1+\delta} n). \text{ (because of the minimum degree)} \end{aligned} \tag{1}$$

(In the algorithm, we simply try all choices of v, i, j.)

Since T has an independent set of size at least $\frac{1}{2}|T|(1-\frac{1}{\log n})$, a vertex-cover approximation algorithm of Bar-Yehuda and Even [1] and Monien and Speckenmeyer [7] (or see Lemma 11 in [3]) will find in T an independent set of size $|T|/\log|T|$. Therefore, we have made progress towards our desired coloring if $|T| = \Omega(n^{11/14})$.

If on the other hand $|T| = o(n^{11/14})$, we will make progress using Theorem 5 as follows. First, S satisfies property (1) of the theorem since it is part of the neighborhood of some node. Second, property (2) is satisfied easily by Equation 1 above (with $\tilde{\Omega}(n^{1/14})$ to spare).

The key constraint is property (3), which is what results in the specific exponent of 3/14. Define $\hat{d}$ to be the average degree of vertices in S. In fact, by definition of S, all vertices in S have nearly the same degree, so $\max_{v \in S} d_T(v) \leq \max_{v \in S} d(v) = O(\hat{d})$. By Equation 1 we have that $D_T(S) = \tilde{\Omega}(D(S)) = \tilde{\Omega}(\hat{d}|S|)$. Thus, we can reduce property (3) to showing that:

$$[\hat{d}|S|]^3 = \tilde{\Omega}\left((|S| + \hat{d})(|S|n^{30/14} + |S|^2 n^{27/14})\right).$$

This can now easily be verified by using the fact that $|S|$ and $\hat{d}$ are both $\tilde{\Omega}(n^{9/14})$. (For $|S|$, this fact follows from item 1 of Theorem 3.) ■

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