Testing for Concise Representations

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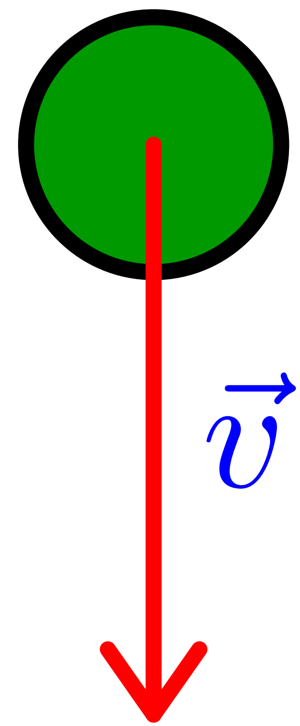
Joint work with **Ilias Diakonikolas, Homin Lee, Kevin Matulef, Ronitt Rubinfeld, Rocco Servedio, Andrew Wan**

Physics Review

Free fall:



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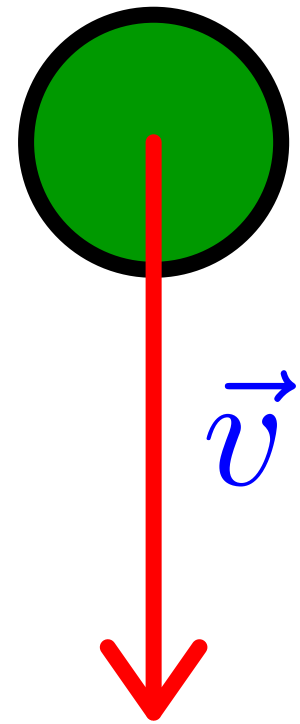

All objects fall at constant speed.

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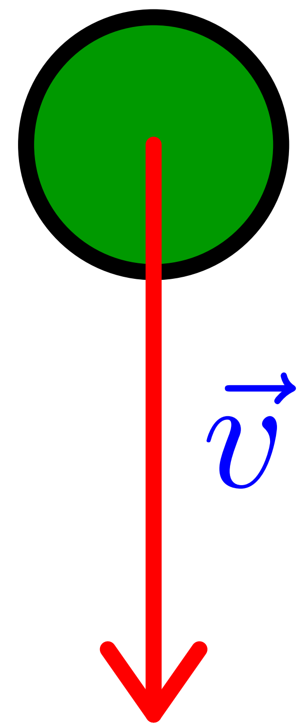

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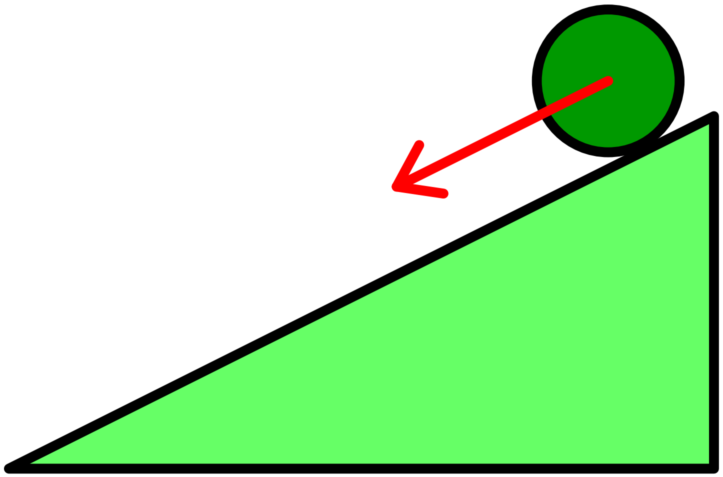
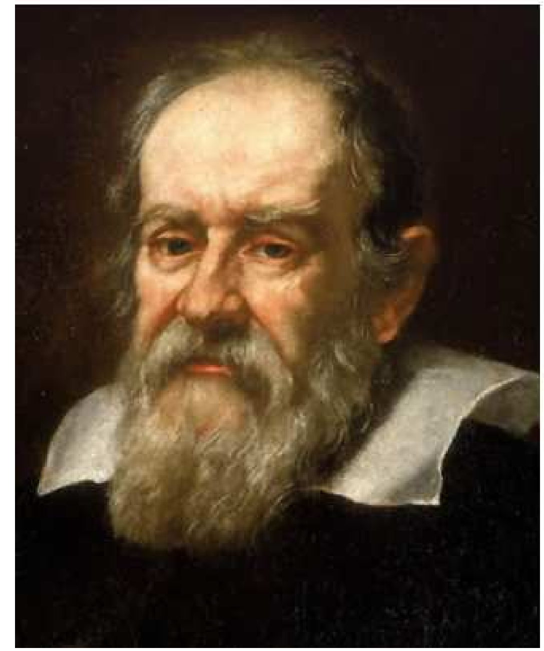
Aristotle

A Physicist Discovering the World





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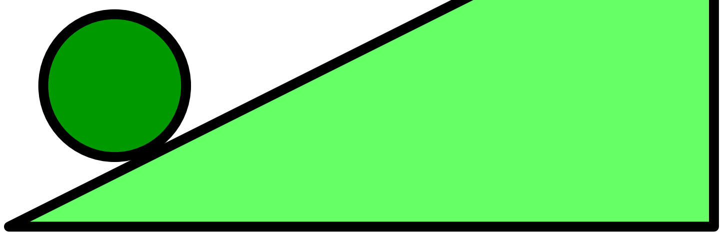

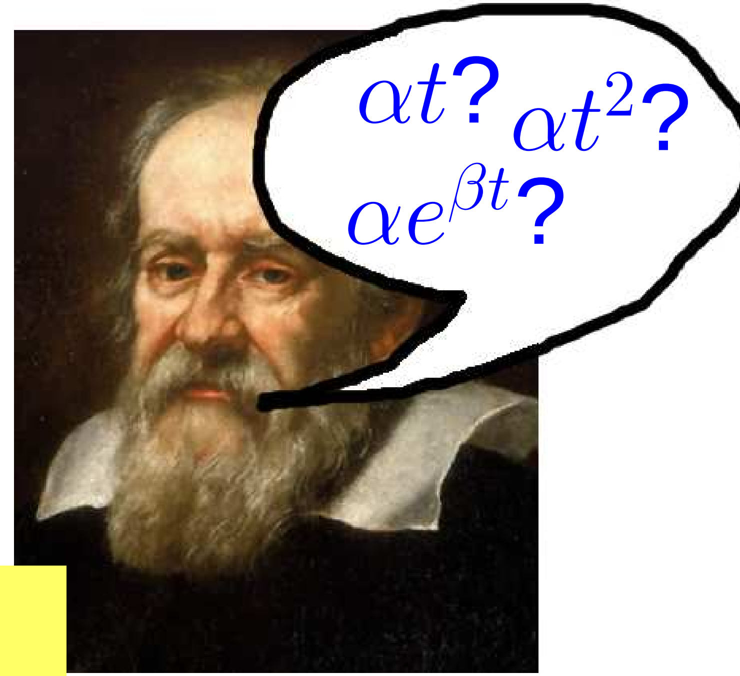

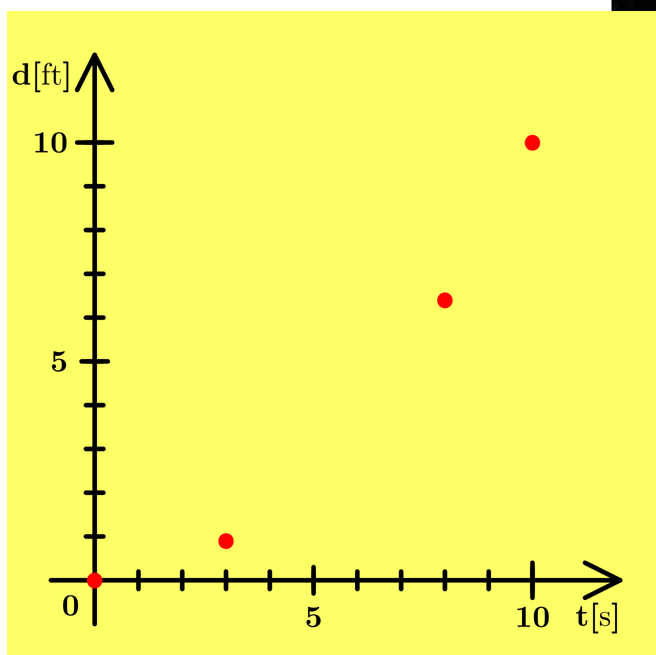

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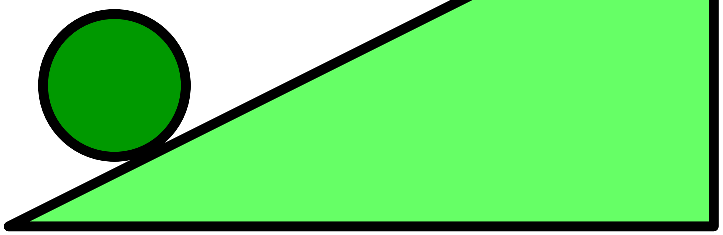

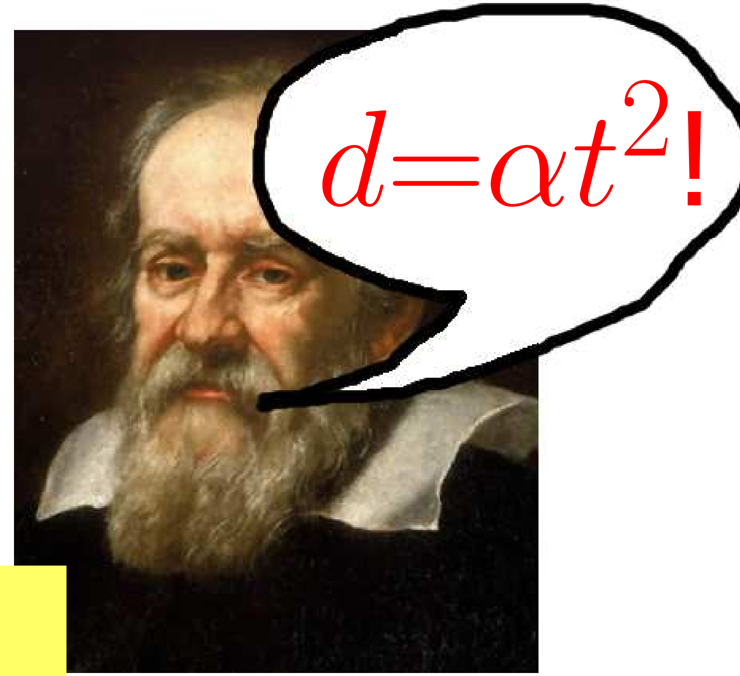

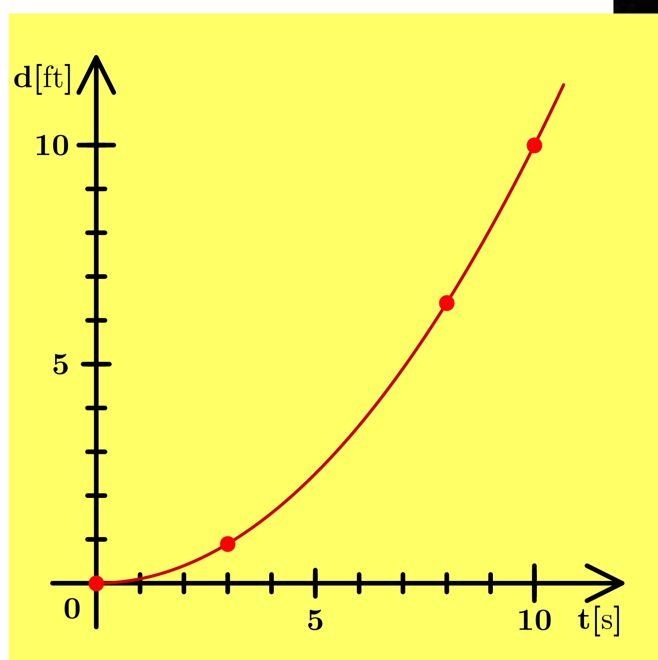

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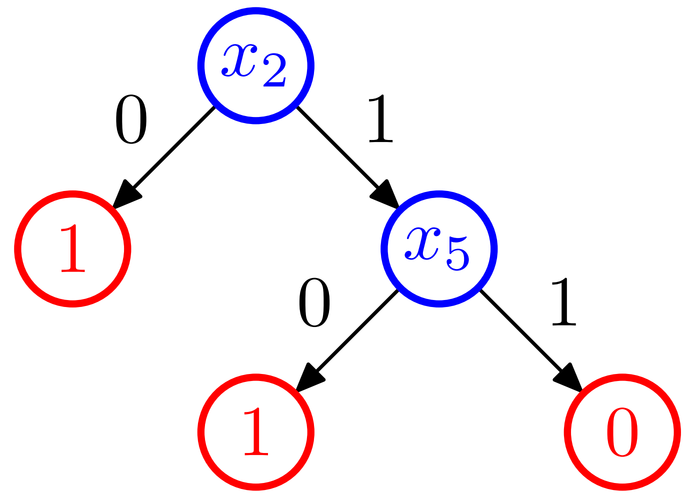

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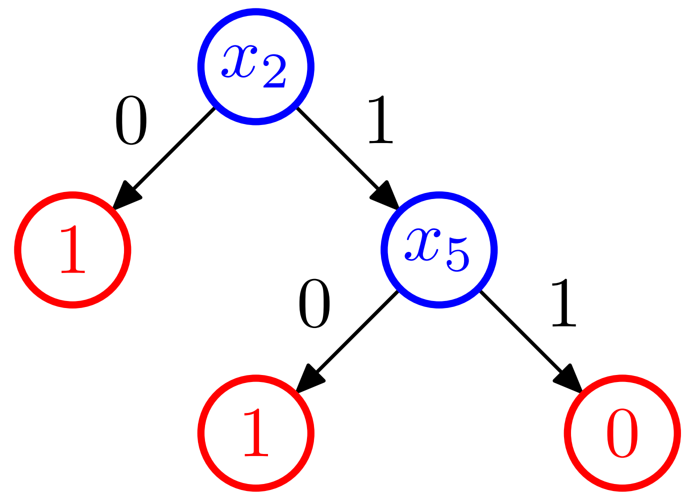

- This could help you choose the right representation, if you wanted to learn the function.

The Model

- Query access to a function $f : \Omega^n \to X$.

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- Primary objective: minimize the number of queries

Plan of the Talk

- Basic definitions
- Testing vs. learning
- Previous results and our results
- Review of the junta test
- Our techniques:
  - Junta test + learning
  - Classes of functions close to juntas
  - Non-Boolean ranges
- Open questions

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- direct democracy $\approx$ majority

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- Implicit learning: Learn the structure of f, but not which variables it depends on.

Selected Previous Results (1/2)

- Parnas, Ron, Samorodnitsky 2001:
  is a Boolean function $f : \{0,1\}^n \to \{0,1\}$:
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    $$f(x_1, \ldots, x_{2007}) = x_2 \wedge \neg x_4 \wedge x_{10}$$

  - an s-term monotone DNFs? $\tilde{O}(s^2/\epsilon)$ queries

    $$f(x_1, \ldots, x_{2007}) = (x_2 \wedge x_3 \wedge x_{20} \wedge x_{37}) \vee x_{21} \vee (x_2 \wedge x_5)$$

Selected Previous Results (2/2)

- Fischer, Kindler, Ron, Safra, Samorodnitsky 2002
  - testing J-juntas $f : \Omega^n \rightarrow \{0,1\}$
    - non-adaptive one-sided test: $\tilde{O}(J^4/\epsilon)$
    - adaptive one-sided test: $\tilde{O}(J^3/\epsilon)$
    - non-adaptive two-sided test: $\tilde{O}(J^2/\epsilon)$
  - $\tilde{\Omega}(\sqrt{J})$ lower bound for non-adaptive testing juntas

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- Chockler, Gutfreund 2004
  - $\Omega(J)$ lower bound for testing juntas

Our Results (1/2)

- Generic tester for many classes of Boolean functions:
  - s-term DNFs (open problem in [PRS]): $\tilde{O}(s^4/\epsilon^2)$
  - size-s decisions trees, size-s branching programs, size-s Boolean formulas, s-sparse polynomials over $\mathbb{F}_2$: $\tilde{O}(s^4/\epsilon^2)$
  - size-s Boolean circuits: $\tilde{O}(s^6/\epsilon^2)$
  - decision lists: $\tilde{O}(1/\epsilon^2)$
  - functions with Fourier degree $\leq d$: $\tilde{O}(2^{6d}/\epsilon^2)$

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  - also generalizes the generic algorithm:
    - s-sparse polynomials over field $\mathbb{F}$: $\tilde{O}((s|\mathbb{F}|)^4/\epsilon^2)$
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    - size-s algebraic circuits, and size-s algebraic computation trees over $\mathbb{F}$: $\tilde{O}(s^4 \log^4 |\mathbb{F}|/\epsilon^2)$
- Lower bounds:
  - s-sparse polynomials over $\mathbb{F}_2$: $\tilde{\Omega}(\sqrt{s})$
  - functions with Fourier degree $\leq d$: $\tilde{\Omega}(\sqrt{d})$
  - s-sparse polynomials over field $\mathbb{F}$: $\tilde{\Omega}(\sqrt{s})$ for $|\mathbb{F}| = O(1)$

The (Simplest) Junta Test [FKRSS]

- Assign input variables to $O(J^2)$ buckets at random.

$$x_2\, x_7 \;\Big|\; x_8 \;\Big|\; x_1\, x_5\, x_6 \;\Big|\; x_3\, x_4$$

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  - If the value of the function on the two assignments is different, mark the bucket.

$$
\begin{array}{l}
\begin{array}{cc|c|ccc|cc}
x_2 & x_7 & x_8 & x_1 & x_5 & x_6 & x_3 & x_4 \\
\end{array} \\
f(\; 1 \;\; 0 \;|\; 1 \;|\; 0 \;\; 1 \;\; 1 \;|\; 0 \;\; 0 \;) = 1 \\
f(\; 1 \;\; 0 \;|\; 1 \;|\; 1 \;\; 0 \;\; 0 \;|\; 0 \;\; 0 \;) = 0
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$$

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- If more than J buckets marked, output **NO**. Otherwise, output **YES**.

$$x_2\, x_7 \;\Big|\; x_8 \;\Big|\; x_1\, x_5\, x_6 \;\Big|\; x_3\, x_4$$

Variation [FKRSS]

- The variation of $f : \Omega^n \to \{0,1\}$ on a subset of variables I is

$$\mathrm{Vr}_f(I) = \mathbb{E}_{\substack{\text{random}\\\text{assignment}\\\text{to vars}\\\text{not in } I}} \mathbb{V}_{\substack{\text{random}\\\text{assignment}\\\text{to vars}\\\text{in } I}} f\left(\begin{array}{c}\text{concatenation of}\\\text{assignments}\end{array}\right)$$

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- Measures sensitivity of f to the values of variables in I.
- Plays a central role in the proof that the junta test works.

Testing Classes of Juntas

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$$x_2\, x_7 \;\Big|\; x_8 \;\Big|\; x_1\, x_5\, x_6 \;\Big|\; x_3\, x_4$$

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  - Collect sufficiently many samples for the function restricted to the relevant variables (see next slide).

Implicit Learning

$x_2\, x_7 \mid x_8 \mid x_1\, x_5\, x_6 \mid x_3\, x_4$

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  - Test them against all restrictions of functions in C to relevant variables. If at least one restriction survives, output **YES**. Otherwise, output **NO**.

Implicit Learning

$x_2\, x_7 \;|\; x_8 \;|\; x_1\, x_5\, x_6 \;|\; x_3\, x_4$

Collecting a Sample

- Pick a random assignment $\mathbf{x} = (x_1, \ldots, x_n)$. Need to read off the hidden relevant variables.

$$\begin{array}{rc|c|c|cl} & x_2\;\; x_7 & x_8 & x_1\;\; x_5\;\; x_6 & x_3\;\; x_4 & \\ f(& 0\;\;\; 0 & 1 & 1\;\;\; 0\;\;\; 1 & 0\;\;\; 0 &) = 0 \end{array}$$

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 & & & \\
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	$x_2\ x_7$	x_8	$x_1\ x_5\ x_6$	$x_3\ x_4$	
$f($	$0\ \ 0$	1	$1\ \ 0\ \ 1$	$0\ \ 0$	$) = 0$
			$x_1\ \ \ \ x_6$		
			x_5		
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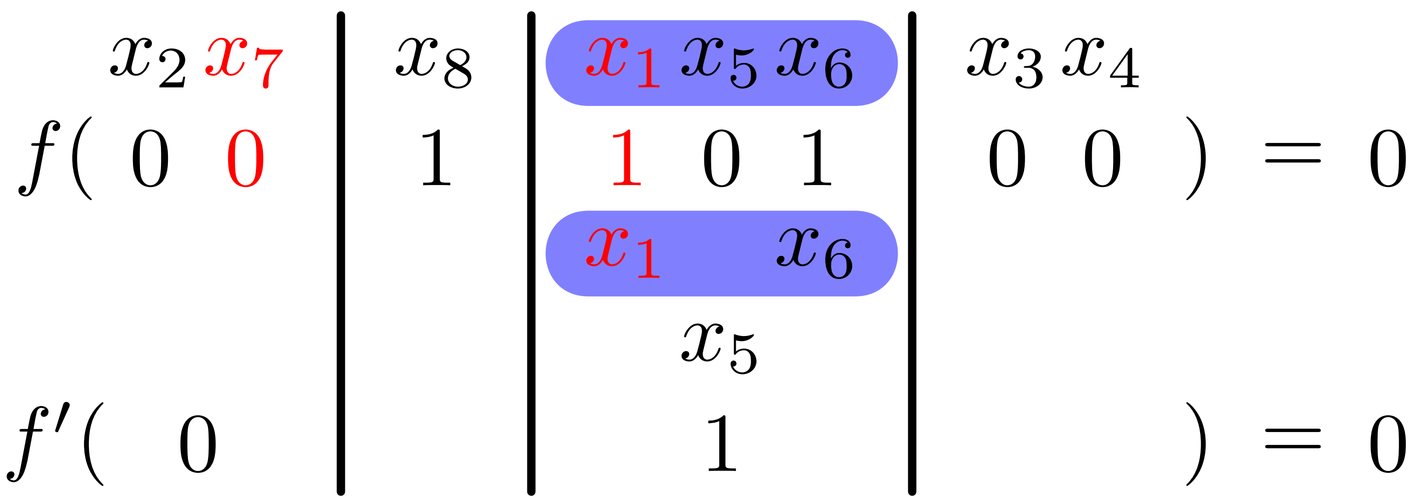

Approximation by Juntas

- s-term DNFs are not a class of small juntas.

$$f(x_1, \ldots, x_{1000}) = (x_1 \wedge \neg x_2) \vee (x_1 \wedge \neg x_4 \wedge \neg x_5) \vee (x_6 \wedge x_7 \wedge \ldots \wedge x_{1000})$$

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- If a DNF term $x_1 \wedge x_2 \wedge \ldots \wedge x_k$ is long, it becomes almost irrelevant. The probability of difference in a random assignment is $\leq 2^k$.

$$f(x_1, \ldots, x_{1000}) \approx (x_1 \wedge \neg x_2) \vee (x_1 \wedge \neg x_4 \wedge \neg x_5)$$

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- Can drop such a term for large k's. Each of our arguments to f is random.
- Suffices to focus on DNFs that are J-juntas for sufficiently large J.
- If want to stay τ-close, suffices to take $J = s \log(s/\tau)$.

Non-Boolean Ranges ($f : \Omega^n \rightarrow X$)

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- We prove that the junta test works with only a constant factor overhead.
- Our testing techniques can be applied as well.
- Need new tools to prove that. The variation only works for the Boolean range.

Non-Boolean Ranges ($f : \Omega^n \to X$)

- Maybe mapping to $\{0,1\}$? Won't lose much? Which mapping?

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- Need both! But they are within a constant factor.
- Can make all the proofs work.

Open Questions

- We gave a generic algorithm. Can improve the query complexity for any of the considered classes of functions?
- Can improve the exponential running time by, for instance, replacing the exponential implicit learning step with a more efficient algorithm?

Questions?