Descents and the Weak Bruhat order

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Abstract

Let D_k be the set of permutations in S_n with k descents and A_k be the set of permutations with k ascents. For permutations of type $\mathcal{A}$, which are the usual symmetric group elements, bijections $\sigma : D_k \to A_k$ satisfying $\sigma(w) \geq w$ in the weak Bruhat ordering are constructed for $k = 1$ and $k = 2$. Such a bijection is also described explicitly for $k = 1$ for permutations of type $\mathcal{B}$. We discuss how this bijection can be applied to solve a conjecture concerning linear extensions of a poset.

1 Introduction

A *linear extension* of a partially ordered set P is a permutation of the elements of P such that whenever $\rho <_P \rho'$, the numerical label of ρ appears to the left of the label of ρ'. The set of all linear extensions of a poset P is denoted by $\mathcal{L}(P)$. A poset is called *naturally labeled* if $123\dots n$ is a linear extension. For every linear extension $\pi \in \mathcal{L}(P)$ let $d(\pi)$ be the number of descents of π. Let H_k be the set of linear extensions with k descents, and $h_k = \#H_k$.

The *weak Bruhat ordering* on the symmetric group is defined as follows: Given $w = w_1w_2\dots w_n$ and $v = v_1v_2\dots v_n$, we say that $w \leq v$ if and only if v can be successively obtained from w by transposing adjacent elements w_i and w_{i+1} such that $w_i < w_{i+1}$.

Theorem 1.1 (Stanley, 1981)**.** *Let P be a finite naturally labeled poset. Let $M = \max\{d(\pi) \mid \pi \in \mathcal{L}(P)\}$. Then the following are equivalent:*

- $h_k = h_{M-k}$ *for* $0 \leq k \leq M$.
- P *is ranked.*

A combinatorial proof of this problem was given in the paper by Farley [Fa1]. However, it is not clear what happens if the poset is not ranked.

Conjecture 1.2 (Hibi, 1991)**.** *For a finite naturally labeled poset P, $h_k \leq h_{M-k}$ and $h_0 \leq h_1 \leq \ldots \leq h_{[\frac{M}{2}]}$ $(0 \leq k \leq M)$.*

Conjecture 1.3. *Let P a naturally labeled poset P on n elements. Then*

$$h_k \geq h_{n-1-k} \; (0 \leq k \leq \frac{n-1}{2})$$

An ideal of a partial order P on the set S is a subset of S such that if $a \in S$ and $b <_P a$, then $b \in S$. For a naturally labeled poset, $\mathcal{L}(P)$ forms an ideal in weak Bruhat ordering. Indeed, assume we have a permutation $w \in \mathcal{L}(P)$, and let $v \lessdot w$ in the weak Bruhat order, i.e. $v = ws$ for a transposition $s = (i\, i+1)$ and $\ell(w) > \ell(v)$. We need to show that $v \in \mathcal{L}(P)$. Indeed, s permutes two consecutive elements w_i and w_{i+1} in one line presentation of w and $w_i > w_{i+1}$. Hence, $w_i \ngeq_P w_{i+1}$, since the greater one appears to the left than the smaller one. Thus, v is a linear extension.

Conjecture 1.3 would be implied by the following conjecture, which was communicated by Farley [Fa2]:

Conjecture 1.4. *Let $n \geq 2k+1$, $k \geq 1$. Let D_k be the set of permutations of S_n with k descents. Let A_k be the set of permutations with k ascents. There is a bijection $\sigma : D_k \to A_k$ which satisfies $\sigma(x) \geq x$ in weak Bruhat ordering.*

σ^{-1} maps every element of H_{n-1-k} to D_k. Since $\sigma^{-1}(w) \leq w$ in weak order for any w and $\mathcal{L}(P)$ is an ideal, $\sigma^{-1}(w) \in H_k$. Thus, σ^{-1} is an injection from H_{n-1-k} to H_k, $h_{n-1-k} \leq h_k$

As far as we know, there is no known combinatorial proof of Conjecture 1.4. In Section 2, we introduce basic definitions and theorems. In Section 3 we prove Conjecture 1.4 for case $k = 1$. In Section 4 we prove it for the case $k = 2$. In Section 5 we prove generalized Conjecture 1.4 for $k = 1$ and permutations of type $\mathcal{B}_n$.

2 Preliminaries

Let $\mathcal{W}$ be a finite reflection group acting on the euclidean space V with associated *root system* Φ. For every root we consider the reflection across the hyperplane orthogonal to the root. These reflections generate the group $\mathcal{W}$. It is well-known that a root system has *a base* or *a simple system* Δ such that $\Delta \subseteq \Phi$ is a vector space basis for the $\mathbb{R}$-span of Φ and every root is linear combination of simple roots with all coefficients of the same sign (all non-negative or all non-positive.) According to the sign of coefficients we call root w *positive* or *negative* and write $w \succ 0$ or $w \prec 0$, respectively. Let Φ^+ be the set of all positive roots and Φ^- be the set of all negative roots. Let S be the set of simple generators $\{s_\alpha \mid \alpha \in \Delta\}$. Let T be the set of all reflections $T = \{t_\alpha \mid \alpha \in \Phi\}$.

The *length* $\ell(w)$ of w is the smallest k for which w can be written as a product of k simple reflections. This expression is called *reduced.* A *descent* of w is a simple generator s such that $\ell(ws) < \ell(w)$. An *ascent* of w is a simple generator s such that $\ell(ws) > \ell(w)$. The longest element of $\mathcal{W}$ is denoted by w_0.

For the group $\mathcal{W}$, define A_k to be the set of all elements with k ascents. Similarly, we define D_k to be the set of elements with k descents.

The *weak Bruhat order* on $\mathcal{W}$ is defined as follows. The covering relation $v \gtrdot w$ holds iff $\ell(v) = \ell(w) + 1$ and $v = ws$ for a simple reflection. Equivalently, $ws \gtrdot w$ if a simple generator s is an ascent of w; and $w \gtrdot ws$, if s is a descent of w. Throughout the paper, we mean the weak Bruhat order for inequality signs with elements of the Weyl group.

An *inversion* of w is a positive root α which is sent to a negative one by w. The set of all inversions is denoted by $I(w)$. Define the set $Inv(w) = \{\beta = w\alpha \mid \alpha \in I(w)\}$ to be the set of images of inversions. As a fact, $\ell(w) = \#Inv(w) = \#I(w)$.

Lemma 2.1. *If $\alpha \in \Delta$, then s_α permutes $\Phi^+ \setminus \{\alpha\}$.*

Proof. Let $\beta \in \Phi^+$, $\beta \neq \alpha$. Write $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$, with nonnegative integer coefficients k_γ. Then by definition, $s_\alpha \beta = \beta - \langle \beta, \alpha \rangle \alpha$. Since β is not equal to α, at least one of the coefficients k_γ is positive for some $\gamma \neq \alpha$. Hence, $s_\alpha \beta$ has a positive coefficient k_γ, which means it is a positive root, not equal to α. Since s_α permutes all the roots, we are done. □

One of the conclusions of this lemma is that s_α is a descent of w iff $w\alpha \prec 0$ for $\alpha \in \Delta$. Since every simple generator is a descent of w_0, $w_0 \Delta \subseteq \Phi^-$. Hence, $I(w_0) = \Phi^+$.

Proposition 2.2. *$v \leq w$ is equivalent to $Inv(v) \subseteq Inv(w)$.*

Proof. Consider the case when $v = ws_\alpha$ for a descent s_α of w. Then $w\alpha \prec 0$. By the above lemma, $Inv(w) = Inv(ws_\alpha) \cup \{w\alpha\}$. In general, $v \leq w$ if there is an ascending chain of elements

$$v \lessdot v_1 \lessdot \ldots \lessdot v_k = w$$

and the claim holds.

On the other hand, assume $Inv(v) \subseteq Inv(w)$. We proceed by induction on the difference of lengths. Assume $v \neq w$. If for some ascent s_α of v $Inv(vs_\alpha) = Inv(v) \cup \{-v\alpha\} \subseteq Inv(w)$ then the induction step holds. Otherwise, for any ascent s_α of v, $-v\alpha = w\beta$ for some negative root $\beta \in \Phi^-$. If $u = w^{-1}v$, then $u\alpha \in \Phi^+$. For a descent s_α of v, $v\alpha \in Inv(v)$, hence, $v\alpha \in Inv(w)$, $u\alpha \in \Phi^+$.

Thus, we have $u\Delta \subseteq \Phi^+$. This means that u has no descents, which is only possible if the reduced word of u is $1_{\mathcal{W}}$, and $v = w$, which leads to a contradiction, as desired. □

Corollary 2.3. *$v \leq w$ iff $I(v^{-1}) \subseteq I(w^{-1})$.*

Proof. Note that $\beta \in Inv(v)$ iff $\beta = w\alpha \prec 0$ for some $\alpha \succ 0$, or $w^{-1}(-\beta) = -\alpha \prec 0$, i.e. $-\beta \in I(w^{-1})$. □

We will consider in detail two types of root systems.

Type $\mathcal{A}_n$. Define Φ to be the set of vectors $e_i - e_j$ $(1 \leq i \neq j \leq n+1)$, where vectors e_i are the standard basis of $\mathbb{R}^{n+1}$. For the base Δ take $\alpha_1 = e_1 - e_2, \ldots, \alpha_n = e_n - e_{n+1}$. $\mathcal{W}$ is the symmetric group S_{n+1} which acts by permuting the basis vectors.

Type $\mathcal{B}_n$. Let $V = \mathbb{R}^n$. Φ consists of $2n$ vectors $\pm e_i$ and roots $\pm e_i \pm e_j$ $(i < j)$. For Δ take $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$, and $\alpha_n = e_n$. The group $\mathcal{W}$ is a semidirect product of S_n (permuting e_i) and $(\mathbb{Z}/2\mathbb{Z})^n$ (acting by sign changes of e_i.)

Let A be an alphabet. If $w = \sigma_1\sigma_2\ldots\sigma_k$ is a word ($k \geq 0$, $\sigma_1, \ldots, \sigma_k \in A$), then its *length* is k. A *subword* of w is a word $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_l}$ ($l \leq k$, $1 \leq i_1 < \ldots < i_l \leq n$.) We call w *a shuffle* of words w_1, w_2, $\ldots w_k$ if there is some partition of $[n] = \bigcup_{i=1}^k A_i$ into disjoint sets such that for each $1 \leq i \leq k$, $A_i = \{i_1, i_2, \ldots, i_j\}$, $i_1 < i_2 < \ldots < i_j$ and $w_i = \sigma_{i_1}\ldots\sigma_{i_j}$ (by $[n]$ we mean the set $\{1, 2, \ldots, n\}$.)

3 Case $k = 1$ for type $\mathcal{A}_{n-1}$

In this section, we focus on the root systems of type $\mathcal{A}_{n-1}$, i.e. on permutations of S_n. We prove the conjecture 1.4 for the case $k = 1$.

By definition, for $w = w_1 \ldots w_n \in S_n$, $I(w) = \{e_i - e_j \mid 1 \leq i < j \leq n, w_i > w_j\}$.

Consider the set of words in alphabet $\{a_1, \ldots, a_k\}$ of length n with the following property — for each i in $[k-1]$ it has a subword $a_{i+1}a_i$. Call the set of such words T_k. Define the inversion set for a word $v = v_1v_2\ldots v_m \in T_k$ to be the set $I(v) = \{e_i - e_j \mid 1 \leq i < j \leq m, v_i > v_j$ assuming $a_1 < a_2 < \ldots < a_m\}$. This definition is similar to the definition of inversions given before for reflection groups. We may say that a pair (i, j), $i < j$ is an inversion of a word an element or a permutation, which is synonymous to saying that $e_i - e_j$ is in the inversion set.

Assume we have a permutation $w = w_1 w_2 \ldots w_n$ in S_n with k descents $(w_i = w[i])$:

$$w_1 < w_2 < \ldots < w_{t_1} > w_{t_1+1} < w_{t_1+2} < \ldots < w_{t_k} > w_{t_k+1} < \ldots < w_n$$

The inverse permutation w^{-1} written in one line presentation is a shuffle of words $\sigma_i = (t_{i-1}+1)(t_{i-1}+2) \ldots t_i$ of alphabet $[n]$, where $i \in \{1, 2, \ldots, k+1\}$, and $t_0 = 0$, $t_{k+1} = n$. Consider a word v in alphabet $\{a_1, \ldots, a_{k+1}\}$ obtained from w^{-1} by substituting all letters of σ_i in w by letter a_i. Since $w_{t_i+1} < w_{t_i}$, the letter a_{i+1} corresponding to w_{t_i+1} appears to the left of letter a_i corresponding to w_{t_i} in v for all i in $[k]$. Therefore, every such word is in T_{k+1}.

Proposition 3.1. *Let $1 \leq k \leq n-1$. This map $\Phi_k : D_k \to T_{k+1}$ is a bijection with the property*

$$I(\Phi_k(w)) = I(w^{-1})$$

There exists a bijection $\Psi_k : A_k \to T_{k+1}$ such that

$$I(\Psi_k(w)) = \Phi^+ \setminus I(w^{-1})$$

Proof. The letters in every σ_i are sorted in increasing order. If $1 \leq i < j \leq k+1$ all letters of σ_i are less than any letter of σ_j. Hence, for every inversion pair (i, j) of w^{-1}, i belongs to some permutation σ_k with higher index than that of j, i.e. it is substituted by the letter a_k with higher index. This agrees with the definition of inversion set for T_{k+1}.

It is clear how to reconstruct $w \in D_k$ from a word in T_{k+1}. The condition imposed on T_{k+1} ensures that there are exactly k descents.

Consider the following bijection $f : w \mapsto v$, $v[i] = w[n+1-i]$, $v = ww_0$ ($w_0 = n \ldots 1$ is the longest element) from A_k to D_k. Since w_0 is an involution and w_0 sends every positive root a negative one and vice versa,

$$I(f(w)^{-1}) = I(w_0 w^{-1}) = \Phi^+ \setminus I(w^{-1})$$

Define $\Psi_k = \Phi_k f$. Then

$$I(\Psi_k(w)) = I(\Phi_k f(w)) = I(f(w)^{-1}) = \Phi^+ \setminus I(w^{-1})$$

and we are done. $\square$

According to the criterion, $v \le w$ if and only if $I(v^{-1}) \subseteq I(w^{-1})$. Hence, for $v \in D_k$ and $w \in A_k$, $v \le w$ if and only if $I(\Phi_k(v)) \cap I(\Psi_k(w)) = \emptyset$.

Proposition 3.2. *Let $n \ge 2k+1$ and $k \ge 1$. If there exists a bijection $\tau : T_{k+1} \to T_{k+1}$ such that $I(x)$ and $I(\tau(x))$ are disjoint for any $x \in T_{k+1}$, then there is a bijection $\sigma : D_k \to A_k$ such that $\sigma(w) \ge w$.*

Proof. The combined bijection $\sigma = \Psi_k^{-1}\tau\Phi_k$, $D_k \stackrel{\Phi_k}{\to} T_{k+1} \stackrel{\tau}{\to} T_{k+1} \stackrel{\Psi_k^{-1}}{\to} A_k$, works. □

Consider the case of permutations with one descent. The set T_2 is a set of words in alphabet $\{a, b\}$ with $a < b$ containing subword ba. Pick the following subset T^* in T_2:

$$\{bab\dots b, bbab\dots b, \cdots, b\dots bab, b\dots ba\}$$

These words have pairwise disjoint inversion sets.

Let us group words in $T_2 \setminus T^*$ into pairs in the following way. Pick any word u in $T_2 \setminus T^*$ and find the last occurrence of a starting from the left end of the word:

$$u = \dots abb \dots b = vabb \dots b \text{ where } v \text{ is some word.}$$

Construct a word u^* by reverting all b's to a's and all a's to b's in v. We obtain a word which belongs to $T_2 \setminus T^*$ as well (since v must contain at least one a and at least one b by construction.) Notice that if we apply the same operation to u^* we will get the initial word u, so this operation is an involution on $T_2 \setminus T^*$. A pair (i, j) is an inversion iff ith letter is b and jth letter is a. Hence, u and u^* have no common inversions.

Thus, we can define a bijection $\tau_2 : T_2 \to T_2$ satisfying $I(v) \cap I(\tau_2(v)) = \emptyset$ for any $v \in T_2$:

- enumerate elements of T^* and set τ_2 sending kth element to the $(k+1)$th and the last to the first (we can do this for $n \ge 3$ since $\#T^* = n - 1 \ge 2$);
- put $\tau_2(u) = u^*$, $\tau_2(u^*) = u$ for every pair as above.

For example, when $n = 3$, we have:

$$baa \mapsto aba, bba \mapsto bab, bab \mapsto bba, aba \mapsto baa.$$

Proposition 3.3. *For $n \ge 3$ there is a bijection $\sigma : D_1 \to A_1$ with the property $\sigma(w) \ge w$.*

Proof. Follows from Proposition 3.2. □

4 Case $k = 2$ for type $\mathcal{A}_{n-1}$

The method we used above can be successfully applied to the case $k = 2$. It turns out that we can describe the bijection on T_3 for most of its words using the bijection τ_2 above.

Proposition 4.1. *Let $n \geq 5$. Then there is a bijection $\sigma : D_2 \to A_2$ satisfying the property $\sigma(w) \geq w$.*

Proof. When $n = 5$ then $D_2 = A_2$ and $\sigma = id$ works. Assume from now on that $n \geq 6$.

It suffices to find a bijection $\tau_3 : T_3 \to T_3$ such that $I(x)$ and $I(\tau_3(x))$ are disjoint. The rest will follow from Proposition 3.2. Assume the letters of the alphabet of T_3 are a, b,c, and they are ordered as $a < b < c$.

We introduce the notation $T(k, L, W)$ to be the set of words in alphabet L of length k containing every word from W as a subword. We denote by $TR(k, L, W)$ all such words which contain every letter of the alphabet L. For example, $T_3 = T(n, \{a, b, c\}, \{ba, cb\})$. For clarity, we sometimes leave out braces and commas when listing sets.

Let the sum of two sets of words $X + Y$ be the set of words obtained by *concatenation* of words $x \in X$ and $y \in Y$ in this order (we may take a single word as X or Y.)

The words of T_3 fall into the following classes according to the last letter:

- $TR(n-1; a, b, c; ba) + b$;
- $T(n-1; a, b, c; cb)+a = (TR(n-1; a, b, c; cb)+a)\cup(T(n-1; b, c; cb)+a)$;
- $T(n-1; a, b, c; ba, cb) + c$.

Our strategy is to subdivide words into sets (C_i for $i = 1, 2, \ldots, 7$) and establish bijections within them.

Bijection on the set C_1

Let L_k, $k \geq 4$ be the set of words in $TR(k; a, b, c; ba)$ containing at least *three* letters different from c. Similarly, let M_k be the set of words $TR(n-1; a, b, c; cb)$ which contain at least *three* letters different from a. Let

$$C_1 = \bigcup_{k=4}^{n-1} (L_k + b\,\underbrace{c \ldots c}_{n-1-k}) \cup (M_k + a\,\underbrace{c \ldots c}_{n-1-k})$$

We define bijections $\rho_k : L_k + b\,\underbrace{c \ldots c}_{n-1-k} \to M_k + a\,\underbrace{c \ldots c}_{n-1-k}$ for $4 \leq k \leq n-1$ as follows. Fix k and consider a word w in the first set. The subword w' of

w consisting of all letters a and b except the last b has length at least three. Apply bijection τ_2 used in the proof of Proposition 3.3 to w'. Substitute all letters of w' in w with $\tau_2(w')$ to obtain v. Finally change the first k letters in v as $b \to c$, $a \to b$, and $c \to a$ and $k+1$th letter to a. For example,

$$c\mathbf{b}cc\mathbf{aa}ccc\ bcc \mapsto c\mathbf{a}cc\mathbf{ba}ccc\ bcc \mapsto a\mathbf{b}aa\mathbf{cb}aaa\ acc$$

By construction, the resulting word is in the second set. Since all steps in construction can be reversed, this map is a bijection. One can check that $I(x) \cap I(\rho_k(x)) = \emptyset$. Hence, ρ_k gives a bijection on the set $(L_k + b\underbrace{c\dots c}_{n-1-k}) \cup (M_k + a\underbrace{c\dots c}_{n-1-k})$ such that every word has no common inversions with its image (define it on the second set by sending x to $\rho_k^{-1}(x)$.) The set of these bijections gives the required bijection on C_1.

Denote $(x) = x\dots x$, where the number of letters x can attain any nonnegative value. From now on we use notation like $\{(b)c(c)b(b)a\}$ to denote the set of words of length n of the form inside the braces, which is $(b)c(c)b(b)a$.

The words of $T_3 \setminus C_1$ are the following:

1. $T(k; b, c; cb) + a\underbrace{c\dots c}_{n-1-k}$ $(3 \le k \le n-1)$;

2. $\{(c)b(c)a(c)b(c)\}$ (except $babc\dots$ since it is not in T_3);

3. $\{(a)c(a)ba(a)(c)\}$.

Bijection on the set C_2

$$C_2 = \bigcup_{k=3}^{n-2} (T(k; b, c; cb) + c\underbrace{b\dots b}_{n-2-k}a) \cup (T(k; b, c; cb) + a\underbrace{c\dots c}_{n-1-k})$$

C_2 is from the first group of words above.

Define bijection π_1 on this set by sending a word $w + c(b)a$ where $w \in T(k; b, c; bc)$ to the word $\tau_2(w) + a(c)$ and vice versa. Then π_1 is well-defined since τ_2 is a bijection on $T(k; b, c; bc)$. Notice that for any $x \in C_2$, $I(\pi_1(x)) \cap I(x) = \emptyset$.

The rest of words of the first set are $cbac(c)$, $cbc(b)a$, and words of the form $(b)c(c)b(b)a$. Assume now that $n \ge 7$. The case $n = 6$ will be considered separately.

Bijection on the set C_3

$$C_3 = \{(c)b(c)a(c)b(c)\} \cup \bigcup_{k=0,3,\ldots,n-5} \{(a)c(a)ba(a)\underbrace{c\ldots c}_{k}\}$$

Every word in the second set has exactly three letters different from c. So two words have no common inversions as long as they do not have any of these letters in the same position. Similarly, two words from the third set do not have common inversions if they have different positions of the first two letters different from a.

The following lemma will give bijections on the set $\{(c)b(c)a(c)b(c)\}$ and each of the sets $\{(a)c(a)ba(a)(c)\}$ with the same number of c's at the end (which is not equal to 1 or 2 and less than $n-4$.) Thus, we define bijection π_2 on C_3 by combining all these bijections.

Lemma 4.2. *For $n \geq 4$ there is a bijection ψ on $\{\{i,j\} \mid 1 \leq i < j \leq n\}$ such that $\psi(S) \cap S = \emptyset$. For $n \geq 7$ there is a bijection ϕ on $\{\{i,j,k\} \mid 1 \leq i < j < k \leq n\} \setminus \{1,2,3\}$ with the property that $\phi(S) \cap S = \emptyset$.*

Proof. Place numbers $[n]$ on a circle in the clockwise order. We represent subsets of $[n]$ by marking the respective numbers on the circle. Two sets are disjoint, if no number is marked for both of them.

To describe bijection ψ we consider two cases:

- Marked numbers are not neighbors on the circle. Define the bijection on the corresponding set by shifting both marked numbers by one position clockwise. The inverse map is shifting one position counter-clockwise, so this is a bijection.

- Marked numbers are consecutive. Shift marked numbers by two positions clockwise. For $n \geq 4$, the sets corresponding to these markings of the numbers in the circle are disjoint, and we define ψ by analogy.

To describe bijection ϕ we consider four cases:

- There is at least one unmarked number between any two marked numbers (from both sides). Then ϕ acts on these sets by shifting one position clockwise. The inverse map is obvious.

- Two marked numbers are consecutive while there are at least two unmarked numbers between any of them and the third one. ϕ shifts it by two position clockwise.

- Two marked numbers are consecutive while the third number is one unmarked number apart. For example, 124. ϕ sends $\{x, x+1, x+3\}$ to $\{x+2, x+4, x+5\}$ and vice versa (numbers are modulo n). Then $\phi(124) = 356$.

- All three numbers are consecutive. For example, 123 and $n12$. We start shifting numbers by three positions clockwise beginning from 123. When n is divisible by 3 then by shifting we will get three groups of sets, in which any two are non-intersecting. Define the bijection on each group in any way, sending cyclically, for instance. If n is not divisible by three, then we will go through every such set by repeated shifting. If $n > 9$, we have

  $$\ldots \mapsto (n-2)(n-1)n \mapsto 123 \mapsto 456 \mapsto \ldots$$

  We define ϕ by sending sets cyclically as above except $(n-2)(n-1)n$ goes to 456. The cases $n = 7$ and $n = 8$ can be described explicitly:
  $n = 7:\ 234 \leftrightarrow 567, 345 \leftrightarrow 671, 456 \leftrightarrow 712$
  $n = 8:\ 456 \mapsto 781 \mapsto 234 \mapsto 567 \mapsto 812 \mapsto 456, 345 \leftrightarrow 678.$

$\square$

Bijection on the set C_4

$$C_4 = \{(b)bc(c)bb(b)a\} \cup \{(a)ca(a)ba(a)c\}$$

The two forms inside the braces describe the 'pattern' of the words. Define bijection π_3 on the set C_4 by substituting letters according to this 'pattern' — A word belonging to one of the subset of C_4 is sent to the word in which each letter is substituted by the respective letter of the form of the other subset. One can check that there are no common inversions between a word and its image. For example, π_3 sends $b\mathbf{bc}ccc\mathbf{bb}a$ to $a\mathbf{ca}aaa\mathbf{ba}c$ and vice versa.

Finally, we list the words, which do not fall into any of the described bijections:

1. $cbc(b)a$, $\{c(c)(b)ba\}$, $\{b(b)(c)cba\}$ which are left from the first set above;
2. $\{(a)cb(a)ac\}$, $\{(a)cb(a)acc\}$, $cba(c)$, $cbaa(c)$, $caba(c)$, $acba(c)$;
3. $\{(a)ca(a)b(a)acc\}$.

Bijection on the set C_5

$$C_5 = \{(a)ca(a)b(a)acc\} \setminus c(a)bacc$$

The bijection π_4 on this set was already described as one of the cases in the proof of the Lemma 4.2.

Bijection on the set C_6

$$C_6 = (\{c(c)bb(b)bba\} \cup \{a(a)cb(a)aac\}) \cup (\{(b)bc(c)cba\} \cup \{(a)cb(a)acc\})$$

We define the bijection π_5 on C_6 in the same say as we did with π_3 — For each word we identify its pattern and then substitute the letters according to the pattern. We grouped two pairs of sets above so that in each group any element has no common inversions with its corresponding one in the other set.

Bijection on the set C_7 We define the bijection π_6 on the remaining words C_7 of T_3 explicitly:

$$cbc(b)a \leftrightarrow (a)cbac, (c)bbba \leftrightarrow cba(c), (c)bba \leftrightarrow caba(c)$$

$$(c)ba \leftrightarrow cbaa(c), acba(c) \leftrightarrow c(a)bacc, (b)cba \leftrightarrow cba(a)c$$

To sum up, we define the required bijection τ_3 on the union of all sets C_i by acting as one of the respective π_{i-1} or ρ_j:

1. $C_1 = \bigcup_{k=4}^{n-1}(L_k + b\underbrace{c\ldots c}_{n-1-k}) \cup (M_k + a\underbrace{c\ldots c}_{n-1-k})$ with bijections ρ_k;

2. $C_2 = \bigcup_{k=3}^{n-2}(T(k;b,c;cb) + c\underbrace{b\ldots b}_{n-2-k}a) \cup (T(k;b,c;cb) + a\underbrace{c\ldots c}_{n-1-k})$ with bijection π_1;

3. $C_3 = \{(c)b(c)a(c)b(c)\} \cup \bigcup_{k=0,3,\ldots,n-5}\{(a)c(a)ba(a)\underbrace{c\ldots c}_{k}\}$ with bijection π_2;

4. $C_4 = \{(b)bc(c)bb(b)a\} \cup \{(a)ca(a)ba(a)c\}$ with bijection π_3;

5. $C_5 = \{(a)ca(a)b(a)acc\} \setminus c(a)bacc$ with bijection π_4;

6. $C_6 = (\{c(c)bb(b)bba\} \cup \{a(a)cb(a)aac\}) \cup (\{(b)bc(c)cba\} \cup \{(a)cb(a)acc\})$ with bijection π_5;

7. $C_7 = \{cbc(b)a$, $(a)cbac$, $(c)bbba$, $cba(c)$, $(c)bba, caba(c)$, $(c)ba$, $cbaa(c)$, $acba(c)$, $c(a)bacc$, $(b)cba$, $cba(a)c\}$ with bijection π_6.

We go back to the case $n = 6$. Among the bijections above, set of bijections ρ_k, π_1, π_3, and π_5 remain valid. We are left with elements of C_3 and C_7 (C_5 is empty.) If we group elements of $\{(c)b(c)a(c)b(c)\}$ into pairs such that the sets of indexes of b's and a of a pair complement each other, we obtain a simple bijection which satisfies our requirement. Applying Lemma 4.2 to the rest of C_3 we are left with *cccbab* and the following elements of C_7:

$$\{cbcbba, aacbac, cbaaac, bbbcba, ccbbba, cccbba, ccccba, cbaccc, cabacc\}$$

The following list of pairs describes the bijection explicitly: *aacbac* $\leftrightarrow$ *ccbbba*, *cbaccc* $\leftrightarrow$ *cccbab*, *cabacc* $\leftrightarrow$ *ccccba*, *cbcbba* $\leftrightarrow$ *cccbab*, *bbbcba* $\leftrightarrow$ *cbaaac*. □

5 Case $k = 1$ for type $\mathcal{B}_n$

From now on we are working with the reflection group $\mathcal{W}$ of type $\mathcal{B}_n$. When we deal with signed permutations, it is important to choose the one line presentation. We use *barring* of letters of a permutation in S_n. We write $w = w_1w_2\dots w_n$ ($w_1\dots w_n$ is a permutation in S_n with some elements barred, for instance $14\overline{3}\overline{2}$) for $w \in W$ which sends e_i to e_{w_i} if w_i is not barred, or to $-e_{w_i}$ if w_i is barred ($1 \le i \le n$.) If we drop bars from presentation of w we obtain a permutation in S_n, and we call this *underlying* permutation.

Proposition 5.1. *For permutations of type $\mathcal{B}_n$, $n \ge 3$, there exists a bijection $\rho : D_1 \to A_1$ such that $w \le \rho(w)$ in weak Bruhat ordering.*

For $\alpha = e_i - e_{i+1}$ s_α is a descent of w in the following cases:

- $w_i = \overline{x}, w_{i+1} = \overline{y}$ for some $x < y$ in $[n]$;
- $w_i = x > y = w_{i+1}$ for some $x, y \in [n]$;
- w_i is barred and w_{i+1} is not.

s_{e_n} is a descent iff w_n is barred.

The set of permutations of type $\mathcal{B}_n$ having one descent consists of the following permutations:

1. $\mathbf{B_1}$: permutations with no bars: $w_i > 0$ $(1 \le i \le n)$ and for some k

$$w_1 < w_2 < \ldots < w_k > w_{k+1} < w_{k+2} < \ldots < w_n$$

2. $\mathbf{B_2}$: permutations with some barred elements. Then the descent is at the rightmost barred element ($0 \le k < l \le n$):

$$w_1 < w_2 < \ldots < w_k$$

$$w_{k+1} > \ldots > w_l \text{ and they are barred}$$

$$w_{l+1} < \ldots w_n$$

Note that the longest element $w_0 = \overline{1}\overline{2}\ldots\overline{n}$ is an involution which sends every positive root to a negative one, and turns every descent into an ascent and vice versa. It commutes with every element $w \in \mathcal{W}$ $w_0 w = w w_0$. This means $I(w^{-1})$ and $I((w_0 w)^{-1})$ are disjoint and complement each other to Φ^+.

Hence, in order to find a bijection $\rho : D_1 \to A_1$ such that $w \le \rho(w)$, it suffices to find a bijection $\tau : D_1 \to D_1$ such that $I(w^{-1}) \cap I((\rho(w))^{-1}) = \emptyset$.

For any permutation w from the group B_1 above $I(w^{-1}) \subseteq \{e_i - e_j \mid 1 \le i < j \le n\}$. Hence, we can set τ to act as the bijection we already have for the permutations of type $\mathcal{A}_{n-1}$ (see Proposition 3.3.)

For any permutation w from the group B_2 above, presentation of w^{-1} is a shuffle of $x = 12\ldots k$, $y = \bar{l}(\overline{l-1})\ldots\overline{k+1}$, and $z = (l{+}1)\ldots n$. Construct a word of the alphabet $\{a, b, c\}$ from w^{-1} by substituting every letter of x by a, of y by b, and of z by c.

Lemma 5.2. *This gives a bijection $\pi : B_2 \to T(n; a, b, c; b)$.*

Proof. It is clear how to reconstruct a permutation from such a word (subwords corresponding to the same letter are ordered.) We require that there should be at least one letter b in the image word as in the definition of B_2. $\square$

In terms of the word $v = \pi(w)$, $I(w^{-1})$ is a set consisting of:

- e_i if ith letter of v is b;
- $e_i + e_j$ if ith and jth letter of v are both b or one of them is b and another one is c;
- $e_i - e_j$ ($i < j$) if ith letter is b and jth letter is not b, or if ith letter is c and jth letter is a.

$T(n; a, b, c; b) = T_1 \cup T_2 \cup T_3 \cup T_4$ for $T_1 = TR(n; a, b, c)$, $T_2 = TR(n; a, b)$, $T_3 = TR(n; b, c)$, and $T_4 = \{b \ldots b\}$.

We construct bijection τ on the permutations corresponding to $T_1 \cup T_2 \cup T_3$ using π as follows:

- define a bijection on T_1 which changes letters as $b \to a$, $a \to b$, $c\ toc$;
- define a bijection $T_2 \to T_3$ which changes letters as $a \to b$, $b \to c$;
- define a bijection $T_3 \to T_2$ which changes letters as $b \to a$, $c \to b$.

Check that τ satisfies the property that $I(\tau(x))$ and $I(x)$ are disjoint.

The word w of T_4 is $\overline{n}\overline{n-1}\ldots\overline{1}$ For any $v \in B_1$, $w \leq w_0 v$, since

$$I(w^{-1}) = \{e_i \mid 1 \leq i \leq n\} \cup \{e_i + e_j \mid 1 \leq i < j \leq n\}$$

and $I((w_0 v)^{-1})$ contains all these roots. Similarly, $w_0 w \geq v$ for any $v \in B_1$. Since we already have a bijection ρ defined on B_1, pick any $x \in B_1$ which is sent to $y \in w_0 B_1$ and set ρ to send the word x to $w_0 w$ and $w \to y$. Summing up, we proved Proposition 5.1.

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