Week 7 (28 March, 2005): Modus Ponens

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Examinator: dr. Hanno Wupper
Docenten: drs. Ling Cheung, drs. Martijn Hendriks, dr. Stijn Hoppenbrouwers

If you liked last week's reading, here is more from the *Platonic Realms Interactive Mathematics Encyclopedia*. This one is available at `http://www.mathacademy.com/pr/prime/articles/carroll/index.asp`. Carroll's original article will be handed out separately in class.

On Carroll's Paradox

In 1895, in the journal Mind, the Oxford logician and author of *Alice in Wonderland*, Lewis Carroll, published a playful dialogue between Achilles and the Tortoise which brought to light a central problem in logic as it was understood at the time. Specifically, he showed that merely having *axioms* - even the best and most perfect axioms - is not sufficient for determining truth in a system of logic; for one also must be very careful about one's choice of *rules of inference*. In other words, one's assumptions must be explicitly augmented by the exact mechanisms by which one is to deduce consequences from those assumptions.

In his dialogue (which is fully reproduced below), Carroll tackles the single most important rule of first-order logic, *modus ponens*, which says that if a statement P is assumed, and if the *conditional statement* "P implies Q" is also assumed (or previously proved), then the statement Q itself is a logical consequence and may therefore be considered proved. What Achilles learns, to his lasting regret, is that modus ponens must be first granted as a rule of inference, for otherwise no conclusion can ever be reached.

Readers may wish to note that Carroll is drawing on what has become a tradition, starting with Zeno of Elea and continuing with the modern author Douglas Hofstadter, of using a dialogue between the Homeric hero Achilles and the comical figure of the Tortoise to make an important philosophical point. Compare with Zeno's Paradox of the Tortoise and Achilles.

Carroll's humor in this piece disguises a point that is essential to understanding modern logic. Unlike in the classical, Aristotelian conception, modern mathematics relies ultimately on pure formalism in its use of logic. This avoids the infinite regress in which the Tortoise traps Achilles.

This trap is impossible to avoid if logic is not formalized, because, as Douglas Hofstadter points out in *Gdel, Escher, Bach* (p. 170), in order to know how to use a rule (such as a rule of inference) you need a rule telling you how to apply the rule. And then a rule telling you how to apply that rule, and so on. By contrast, in formal logic, rules of inference are reduced to rules of symbol manipulation. Since the symbols themselves are uninterpreted (which is what we really mean by "formal"), we have a system as austere and elegant as chess, where it is understood that the game arises from - and entirely consists in - the rules for moving the pieces on the board.

The formalist solution, while effective, has its own philosophical drawbacks. Not the least of these is that, by reducing logic to uninterpreted symbols, all semantic content is removed from the conclusions of formal logic. In other words, what we would ordinarily consider meaning is lost. How to restore meaning to systems of inference while still avoiding difficulties such as Carroll's Paradox remains a thorny question for philosophers of mathematics.