6.893 Approximability of Optimization Problems September 20, 1999

Lecture 4

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1 A $\frac{4}{3}$-approximation algorithm for MAX-SAT

In the previous lecture we saw a $\frac{1}{1-\frac{1}{e}}$-approximation algorithm for MAX-SAT based on rounding the (real-valued) solution to a related instance of LINEAR-PROGRAMMING (LP). Today, we will describe a $\frac{4}{3}$-approximation algorithm for MAX-SAT due to Goemans and Williamson. This algorithm is obtained by combining the rounding procedure from the last lecture with another (simpler) random rounding strategy.

Recall that an instance of MAX-SAT consists n boolean variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. The goal is to determine the maximum number of clauses it is possible to satisfy simultaneously.

The LP-relaxation of this instance consists of $n + m$ real variables $Y_1, \ldots, Y_n, Z_1, \ldots, Z_m$ subject to the constraints:

1. $0 \leq Y_i \leq 1$ for all i,
2. $0 \leq Z_j \leq 1$ for all j and,
3. for each clause C_j of the form $x_{i_1} \vee \bar{x_{i_2}} \vee \cdots$, we have a constraint $Z_j \leq Y_{i_1} + (1 - Y_{i_2}) + \cdots$ (the sum of all variables in the clause, where negated variables are subtracted from 1).

The objective function (to be maximized) is $\sum_j Z_j$.

A solution to the LP-relaxation of the MAX-SAT instance is **integral** if all the Y_i's (and hence Z_j's) are either 0 or 1. Non-integral solutions are termed **fractional**. The process of obtaining an integral solution from a fractional one is called **rounding**.

Clearly, maximal assignments of the variables $x_1, \ldots, x_n$ in the original instance correspond to maximal integral solutions to the LP-relaxation instance, where $\sum_j Z_j$ gives the number of satisfied clauses.

Many problems have an LP-relaxation similar to the one for MAX-SAT. In general, the **integrality gap** for a problem Π is the supremum, over all instances of Π, of

$$\frac{\text{optimal fractional solution}}{\text{optimal integral solution}}$$

Since optimal fractional solutions ar easy to find, the integrality gap provides an upper bound on the (non-constructive) approximability of Π. Interestingly, all known results on integrality gaps are algorithmic, and so the corresponding upper bounds are in fact constructive.

We will show a $\frac{4}{3}$ integrality gap for MAX-SAT by combining two different methods.

1.1 LP method

This is the method described and analyzed last time in class. It was shown that

Lemma 1 *If $y_1, \ldots, y_n, z_1, \ldots, z_m$ is a fractional solution then by choosing to assign $X_i = 1$ with probability y_i, the probability that clause C_j is satisfied is at least*

$$\left[1 - \left(1 - \frac{1}{k_j}\right)^{k_j}\right] z_j$$

where k_j is the size of C_j.

By linearity of expectations, the expected number of satisfied clauses is thus at least

$$\left[1-\left(1-\frac{1}{k}\right)^k\right]\sum_j z_j$$

where k is the maximum size of a given clause.

How to find assignments that exceed the expected value? Picking them at random based on an optimal fractional solution is a reasonable strategy: since $\sum_j z_j$ is an upper bound on the number of clauses satisfied in an assignment, the probability of choosing an assignment whose number of satisfied clauses is below $\left[1-(1-\frac{1}{k})^k-\epsilon\right]\sum_j z_j$ is bounded by

$$\frac{\epsilon}{(1-\frac{1}{k})^k+\epsilon}$$

so that choosing the best among $O(\frac{1}{\epsilon})$ independent samples is likely to find a good approximation. We also have the following fact:

Fact 2 *Using calculus, we can show*

$$0<\left[\left(1-\frac{1}{k}\right)^k-\frac{1}{e}\right]\in O(\frac{1}{k})$$

Thus, picking the best among $O(k)$ random assignments (chosen according to the distribution given by the y_i's) is likely to give a solution within $1-\frac{1}{e}$ of the fractional optimum and hence of the integral optimum.

Remark This method can in fact be significantly improved using either the methods of limited independence (k-wise is sufficient) or conditional expectations (proceed one variable at a time, checking which assignment to that variable leaves the best expectation value on the remaining ones).

Note that the limiting factor in the calculation of the bound on the integrality gap by this method is the presence of large clauses. For $k=1$, the factor is in fact 1. For $k=2$, it's $\frac{3}{4}$, for $k=3$ it's $\frac{19}{27}$ and so on down to $1-\frac{1}{e}$.

1.2 An upper bound

But is the integrality ratio really that low? How much better than $1-\frac{1}{e}$ can we get? Consider the following CNF formula:

$$(x_1\vee x_2)\wedge(\overline{x_1}\vee x_2)\wedge(x_1\vee\overline{x_2})\wedge(\overline{x_1}\vee\overline{x_2})$$

Clearly the integral maximum (and minimum!) is 3. On the other hand, the fractional maximum is 4 (attained by setting all $Y_i=\frac{1}{2}$). Thus, the integrality gap for MAX-SAT can be bounded above by $\frac{3}{4}$. It turns out that this bound is always attainable. This is somewhat surprising,given the results of the previous section.

1.3 Random rounding

This method independently assigns variables a value of either 0 or 1 with probability $\frac{1}{2}$. For a clause of length k, the probability that it is satisfied is $\left(1-\frac{1}{2^k}\right)$. Thus, the expected number of satisfied clauses is $\sum_j\left(1-\frac{1}{2^{k_j}}\right)$.

We notice that although this performs poorly for small clauses (with respect to the number of clauses, not $\sum_j z_j$), it does extremely well for large ones, tending to 1 as $k\to\infty$.

1.4 Joint method

Thus it seems that the best approach would be to try both methods, taking the better of the two integral solutions obtained. How can we analyse this?

For any two random variables A and B, $E\max(A,B) \geq \frac{E(A)+E(B)}{2}$. We can obtain a lower bound on the performance of this algorithm by supposing that rather than choosing the better of the two solutions we pick one at random. The probability that a clause C_j of size k is satisfied is then

$$\begin{aligned}
P_{joint}[C_j \text{ satisfied}] &= \frac{1}{2}P_{random}[C_j \text{ satisfied}] + \frac{1}{2}P_{\mathrm{LP}-relaxation}[C_j \text{ satisfied}] \\
&= \frac{1}{2}\left[1-\frac{1}{2^k}\right] + \frac{z_j}{2}\left[1-\left(1-\frac{1}{k}\right)^k\right] \\
&\geq \left[1-\frac{1}{2^{k+1}}-\frac{1}{2}\left(1-\frac{1}{k}\right)^k\right] z_j
\end{aligned}$$

It turns out that for $k=1,2$ the factor of z_j in the equation above is exactly $\frac{3}{4}$ and is strictly superior to that for all $k>2$. This yields the desired result:

Theorem 3 *The integrality gap for* MAX-SAT *is exactly $\frac{3}{4}$. Moreover the method above yields an explicit construction of an integral solution within $\frac{3}{4}$ of the optimal fractional one.*

2 Deterministic rounding for VERTEX-COVER

We have already encountered the unweighted VERTEX-COVER problem in the first lecture and we gave a 2-approximation greedy algorithm at that time. In this section we give a 2-approximation algorithm for the weighted version. We also prove some structure results for the unweighted case.

Definition 4 WEIGHTED-VERTEX-COVER (WVC):

- Input: A graph $G=(V,E)$ and weights $\{w_u \mid u \in V\} \in \mathbf{N}^+$.
- Feasible set: Covers of V: $C \subseteq V$ such that $\forall e \in E$, $e \cap C \neq \emptyset$ (**Note:** edges are taken to be subsets V of size 2).
- Objective function: $w(C) = \sum_{u\in C} w_u$

There is a natural reduction to INTEGER-PROGRAMMING for WVC:

Variables	$(\forall u \in V)$	$x_v \in \{0,1\}$
Constraints	$(\forall \{u,v\} \in E)$	$x_u + x_v \geq 1$
Objective	$\min \sum_{u\in V} w_u x_u$	

There is a natural LP-relaxation of this problem, in which $x_u \in [0,1]$. Given a solution $(x_u)_{u\in V}$ to the relaxation, we can construct an integral solution (x'_u) deterministically (**deterministic rounding**) by setting $x'_u = 1$ if $x_u \geq \frac{1}{2}$ and $x'_u = 0$ otherwise. The solution thus obtained remains feasible since for every edge $e=\{u,v\}$, the constraint $x_u + x_v \geq 1$ implies that either x'_u or x'_v is 1. Moreover, the value of the objective function can be at most doubled since each x_u is no more than doubled. Thus, the integrality gap for WVC is at least $\frac{1}{2}$ and we get a 2-approximation algorithm for WVC using a LP algorithm as a subroutine.

Exercise 1 *Show that the integrality gap for* WVC *is exactly $\frac{1}{2}$.*

Solution: Consider the complete graph K_n with equal weights on all vertices. Clearly any optimal integral solution will have size $n-1$—only one vertex can *not* be in the cover. On the other hand, the optimal fractional solution sets $x_u = \frac{1}{2}$ for all u. The value of the objective functio is tehn $\frac{n}{2}$. Thus the integrality gap is at most $\frac{n}{2(n-1)}$. Since the argument works for all n and the integrality gap is the sup over all instances, it must be exactly $\frac{1}{2}$.

2.1 The Nemhauser-Trotter theorem

The algorithm above, although it is very simple, hides just what is "hard" about WVC. We give some results on how the optimal solutions to WVC instances are structured and a 2-approximation algorithm due to Hochbaum. The result implies that the hardest cases of WVC are those for which the fractional solution $x_u = \frac{1}{2}$ is actually optimal for the corresponding LP instance.

The first key result, due to Nemhauser and Trotter, describes an optimal fractional solution, for both the weighted and unweighted cases.

Theorem 5 (Nemhauser-Trotter) *If L is a* LP *instance constructed from an instance of* WVC, *one of L's optimal solutions is in $\{0, \frac{1}{2}, 1\}^V$. Such a solution is called **semi-integral**.*

Proof Pick the optimal solution $(x_u)_{u\in V}$ which minimizes

$$\epsilon = \min_{u\in V \text{ s.t. } x_u \neq \frac{1}{2}} \left| x_u - \frac{1}{2} \right|$$

This solution is in some sense the "closest" to the vector $(\frac{1}{2})^V$ (even though this distance is not metric). To prove the theorem, we must show that this solution is in fact in $\{0, \frac{1}{2}, 1\}^V$. Suppose for contradiction that it is not.

Let $V_x = \{u \in V \mid x_u = x\}$ and partition the vertices into five sets:

$$S^< = \{u \in V \mid x_u < \frac{1}{2} - \epsilon\},\ S^- = V_{1/2-\epsilon},\ S^0 = V_{1/2},\ S^+ = V_{1/2+\epsilon},\ S^> = \{u \in V \mid x_u > \frac{1}{2} + \epsilon\}$$

Notice that there can be no edges between $S^<$ and the sets $S^<$, S^-, S^0 or S^+, or between S^- and the sets S^- or S^0. Moreover, note that the only edges for which the corresponding constraints are actually tight are between $S^<$ and $S^>$, S^- and S^+ and within S_0.

There are now two cases to handle:

1. Suppose $w(S^-) \leq w(S^+)$. Then we can in fact set all variables in S^+ and S^- to $\frac{1}{2}$. All constraints will be maintained or increased since there can be no edges between S^+ and $S^<$. Moreover, the objective function will decrease by $\epsilon(w(S^+) - w(S^-)) \geq 0$. However, this constradicts the minimality of the "closeness" of the original (x_u) to $(\frac{1}{2})$. Thus this case cannot actually arise.

2. Suppose $w(S^-) > w(S^+)$. Consider the effect of adding some small quantity δ to all variables in S^+ and subtracting it from all variables in S^-. This will clearly reduce the objective function by $\delta(w(S^-) - w(S^+)) > 0$. Moreover, the only constraints which will be decreased are those connecting variables in S^- and $S^>$ or in S^+ and $S^<$. However, these constraints are not initally tight so there is some $\delta > 0$ such that none of them will be violated. The existence of such a δ contradicts the optimality of the initial solution (x_u) and thus proves the theorem.

■

Exercise 2 *Given an optimal fractional solution, how can you find a semi-integral one (efficiently)?*

2.2 Hochbaum's algorithm

We now describe an application of the N.-T. theorem due to Hochbaum. Consider the case where all vertices have weight equal to 1. We want to in some sense distill all the difficulty of the vertex matching problem onto some subgrpah of our graph by reducing VC for G to VC for the subgraph.

Using theorem 5, we can partition the graph into three sets V_0, V_1, $V_{\frac{1}{2}}$ according to a given optimal semi-integral solution. There will be no edges between V_0 and $V_{\frac{1}{2}}$ or within V_0. Hence, it is clear that any set C of vertices which includes V_1 and whose intersection with $V_{\frac{1}{2}}$ forms a cover of the induced graph on $V_{\frac{1}{2}}$ will in fact be a cover of G. It turns out that an optimal cover will be of that form:

Lemma 6 *There exists an optimal vertex cover C for G such that $V_1 \subseteq G$, $C \cap V_0 = \emptyset$ and $C \cap V_{\frac{1}{2}}$ is an optimal vertex cover of the induced graph on $V_{\frac{1}{2}}$.*

To prove this lemma, it suffices to show that there exists some optimal vertex cover which includes V_1. The rest will follow since clearly we can do no better than choosing vertices in $V_{\frac{1}{2}}$ which form an optimal cover: vertices in V_0 are of no help since no edges connect V_0 to $V_{\frac{1}{2}}$.

The following lemma is sufficient for the cause:

Lemma 7 *There exists a matching in G which connects every vertex in V_1 to a vertex in V_0.*

Proof We will consider only the bipartite subgraph consisting of V_0, V_1 and the edges connecting them. Given a set of vertices S, denote its neighborhood $\Gamma(S) = \{v \mid \exists s \in S \text{ s.t. } \{u, s\} \in E\}$. Using Hall's theorem on matchings[1], it is sufficient to show that for every set $S \subseteq V_1$, $|\Gamma(S)| \geq |S|$.

To show this, note that we can set all variables in $S \cup \Gamma(S)$ to $\frac{1}{2}$ without violating feasibility. Since all vertices have equal weight, and since there are no edges leaving from V_0, the objective function from the LP instance will decrease by at least $\frac{1}{2}(|S| - |\Gamma(S)|)$. If $\Gamma(S)$ were indeed smaller than S, this would contradict the minimality of the solution. ■

This concludes the proof of lemma 6.

Exercise 3 *Reduce both solving* VC *exactly and approximating it to graphs with perfect matchings.*

Solution: Reverse proof of lemma 7 to show a contradiction when some set has a small neighborhood.

Exercise 4 *Prove lemma 6 for the weighted case.*

Solution: Instead of going through matchings, use an argument similar to the one in the proof of lemma 7 to show directly that whatever part of a given cover lies in V_0 can be "transferred" to V_1 without increasing the objective function. The only necessary observation is that $\Gamma(V_1 \setminus C) \subseteq C \cap V_0$ (in the bipartite graph from lemma 7).

3 Set Cover

The SET-COVER problem is a generalization of VERTEX-COVER.

Definition 8 SET-COVER:

- Input: A finite universe $\mathcal{U} = [n]$ and sets $S_1, S_2, \ldots, S_m \subseteq \mathcal{U}$ such that $\bigcup_{i=1}^{m} S_i = \mathcal{U}$.
- Feasible set: $\{I \subseteq [m] \mid \bigcup_{i \in I} S_i = \mathcal{U}\}$
- Objective: $\min |I|$

[1] Hall's theorem: Let $B = (L, R, E)$ be a bipartite graph. There is a matching in B connecting every vertex in L to a vertex in R if and only if for every $S \subseteq L$, the neighbourhood of L is at least as large as L.

Fact 9 *The following approximability results are known:*

1. *If* $d = \max_{j \in \{1,\ldots,n\}} |\{i \mid j \in S_i\}|$ *then* SET-COVER *is* d*-approximable.*
2. *If* $f = \max_{i \in \{1,\ldots,m\}} |S_i|$, *then* SET-COVER *is* H_f*-approximable, where*

$$H_f = \sum_{k=1}^{f} \frac{1}{k} \in [\ln(f+1)\ ,\ \ln(f)+1]$$

is the f*-th harmonic number.*