Today

- Non-uniform classes = TMs with advice
- Circuits, Formulae, Branching programs.
- Formulae $\leq$ BPs $\leq$ Branching programs
- Depth and Width; Formula depth to size relationship.
- Quadratic lower bound on formula size (Neciporuk).
- Barrington's result on bounded-width branching programs.

Non-uniform computation

- One way to view non-uniform computation, is in the form of a two input Turing machine.
  - $M(x, y)$ in class say $\mathcal{C}$.
  - M decides L with advice $a = \langle a_1, \ldots, a_n \ldots, \rangle$ if for every $x \in \{0,1\}^n$ $M(x, a_n) = 1 \Leftrightarrow x \in L$.
  - If $|a_n| = \ell(n)$, then L said to be in $\mathcal{C}/_\ell$.
- Most important subcase: $P/_{\text{poly}}$ = polysize circuits = non-uniform polytime.

Recall other non-uniform measures

- Formula: Circuits whose DAG is a tree.
- Branching programs: Diff. model of computation, where nodes represent decisions to be taken, and end point determines answer.
- Circuits: Fully powerful non-uniformity.

Formulae, BPs, Circuits

- f has formula size $s(n)$ implies f has BP size at most $O(s(n))$.
  - By induction: f_1 and f_2, f_1 or f_2, $\overline{f_1}$.
- f has BP size $s(n)$ implies f has circuit size at most $O(s(n))$.
  - Induction on size of BP; note that declaring any other node of BP to be start node gives BP of smaller size. Assume there is a circuit of size $c.(s-1)$ computing all other functions. Now to compute start node, will add c extra nodes. Note BP f can be written as $x_1 \cdot f_1 + \overline{x_1} \cdot f_2$.

Brief History

- Formula size lower bounds: Best known over any basis $\tilde{\Omega}(n^3)$ or so. Will show an $\tilde{\Omega}(n^2)$ lower bound today.
- BP size - slightly behind.
- Circuit size: Only $4.5n$, over AND, OR, NOT basis.
- Easy to show existence of functions that need size $2^{\Omega}n$.

Neciporuk's lower bound for Formula size

- f is the Distinctness function.
  - f takes $n\ell$ bits as input. (Will later set $\ell = 2\log_2 n$.)
  - View input as $X_1, \ldots, X_n$ where $X_i = \langle x_{i,1}, \ldots, x_{i,\ell} \rangle$.
  - $f(X_1, \ldots, X_n) = 1$ iff $\exists i \neq j$ s.t. $X_i = X_j$.
  - Thm: f needs formulae of size at least $\Omega(n^2)$, provided $\ell \geq 2\log_2 n$.

Proof

- Basic ideas: Counting, and restrictions.
- Claim 1: Number of leaves involving variables from X_i is at least $\Omega(n)$.
  - Assume o.w. Let # leaves $= k$.
  - Then # formulae obtained by restricting other variables to $0/1$ is at most $2^{O(k)}$.
  - But # functions obtained by other restrictions is at least $\binom{2^\ell}{n-1}$.

Moving on

- Size lower bounds very weak in unrestricted case. So restrict models and then prove strong lower bounds.
- Example: consider monotone functions (changing input bit from 0 to 1 can not change output from 1 to 0), and prove lower bounds for circuits without negation.
- Example: Restrict "depth" of circuit/BP, "width" of branching program.

Depth, Width

- Depth of DAG = length of longest path in DAG.
- Width of Layered DAG:
  * DAG is layered if vertices are partitioned into layers $L_1, \ldots, L_k$ and all edges run from L_i to L_{i+1}.
  * Width of Layered DAG is max number of vertices in Layer.
- Can define width of unlayered DAG also, but not as clean.

Basic observations about depth

- Depth of circuit = parallel time; Can even allow unbounded fan-in OR/AND, to simulate CRCW models.
- Circuit depth = formula depth.
- Formula depth = Theta(log Formula size).
- Thus Formula size poly(n) = Log n depth.
- In upcoming lecture: Show limitations of poly size circuit of constant depth (with unbounded fan-in OR/AND).
- Depth of BP = time. (Size = space).

- Major recent breakthrough: Ajtai shows explicit functions requiring nearly linear space to be done in nearly linear time. (Won't cover.)

Basic observations about width

- Width of BP = non-uniform space.
- Fair amount of intuition obtained by unravelling DFAs. (DFA unravelling leads to Layered read-once branching program.) Thus O(1)-width BPs can count modulo O(1).
- Early belief: possibly can't do anything else. Can't compute majority?
- Easy to rule out poly size width-2 BPs computing majority. Hard result: width-3 BPs can't compute majority. Hope in the 80s ... will eventually rule out all O(1) width (or even sublinear width).

- Hopes crashed by beautiful result of Barrington: uses insight from group theory, and the existence of non-Abelian groups.

- Our proof: algebraic - due to Ben-Or+Cleve.

Barrington's Theorem

- Every function with poly size formula can be computed with width 5 bp.

- Ben-Or Cleve Proof:
  - wlog: prove for log-depth arithmetic formula. (Why does this suffice? Exercise!)
  - Prove for 3-register machines = width 8 bps. (slightly weaker).

Define Register Machines

Ben-Or + Cleve proof of Barrington