Today

- Communication complexity
- Communication complexity and circuit depth (+ monotone version).
- Monotone depth of matching

Communication Complexity

- Communication complexity: Two players Alice and Bob want to compute some function of their joint information. Minimize computation needed.
- Contrast with classical communication (Shannon): Alice and Bob wish to exchange information, with no specific end function in mind. Here goal is fixed.
- Model: A has $x = z_A \in \{0,1\}^n$, B has $y = z_B \in \{0,1\}^n$. A and B wish to compute $f(x,y)$.
- Protocol: Sequence of functions $T_1, \ldots, T_\ell$, $T_i : \{0,1\}^{n+(i-1)} \to \{0,1\}$, and player choices $P_1, \ldots, P_\ell \in \{A, B\}$.

- Protocol computes f if, for every z_A, z_B the sequence $b_1, \ldots, b_\ell$ given by $b_i = T_i(z_{P_i}, b_1, \ldots, b_{i-1})$ ends with $b_\ell = f(z_A, z_B)$.
- Communication complexity of protocol $= \ell$. $CC(f) = \min_{\text{protocols}}\{\ell\}$.

History

- Introduced by Yao.
- Basic versions + variants linked to many intriguing themes in computational complexity.
- Yao also introduced some basic lower bound techniques.
- Log-rank of f-matrix.

Examples

- "Crossing sequence" technique. (Identity function).
- Rank technique. (Inner Product function).
- Log-rank conjecture.

Karchmer-Wigderson Games

- Communication complexity for relations: Given $R \subseteq X_A \times X_B \times D$, and Alice with $x \in X_A$ and Bob with $y \in X_B$, they have to "agree" on $i \in D$ s.t. $R(x, y, i)$ holds.
- Formalization? Exercise!

Canonical relation for function

- Given $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, let $X_A = f^{-1}(0)$, $X_B = f^{-1}(1)$, and let $D = [n]$. $R_f(x, y, i) = 1$ if $x_i \neq y_i$.
- "Total relation", in that for every $x \in X_A$ and $y \in X_B$, $\exists i$ such that $x_i \neq y_i$.
- Canonical game: Determine communication complexity of R_f.

K-W theorem

- Circuit depth$(f) = CC(R_f)$. (Circuit with $\{2-AND, 2-OR, NOT\}$ gates.)
- Given circuit to compute f, suppose $f(x) = f_0(x) \wedge f_1(x)$. Then A plays first and says $b = 0$ if $f_0(x) = 0$, else says $b = 1$. Recurse on f_b.
- Other direction:
  - Make strong inductive hypothesis for partial functions: For every $f : \{0,1\}^n \to \{0,1,*\}$ if $CC(C_f) \leq d$, then there exists $g : \{0,1\}^n \to \{0,1\}$ extending f with circuit depth g being at most d.
  - Suppose A speaks first and announces

$P(x)$. Then there exists a depth $d-1$ circuit computing $f|_{P=0}$ and $f|_{P=1}$.