ST06 LECTURE 17

Note Title 4/13/2006

TODAY:

Additive White Gaussian Noise Channel
(AWGN)

AWGN Channel

Input Signal : $X_1 \ldots X_n$ $X_i \in \mathbb{R}$

Noise : $Z_1 \ldots Z_n$ $Z_i \in \mathcal{N}(0, \sigma_z^2)$

Z_i i.i.d.
(ind. of $X_1 \ldots X_n$ also)

Received Signal : $Y_1 \ldots Y_n$

$$Y_i = X_i + Z_i$$

Capacity = ?

- Actually infinite unless there is some constraint on $X_1, \ldots X_n$.

- Standard channels : $X_i \longrightarrow$ voltage.

  $\Rightarrow$ Power incurred $= X_i^2$

  $\Rightarrow$ average Power $= \frac{1}{n} \sum_{i=1}^{n} X_i^2$

- Typical Constraint : $\frac{1}{n} \sum X_i^2 \sim \sigma_x^2 \leq P$

Given $\underset{\text{Signal}}{\overset{\uparrow}{P}}$, $\underset{\text{Noise}}{\overset{\uparrow}{\sigma^2}}$, what is capacity of channel?

Recall

- Differential Entropy $h(W) = \frac{1}{2} \log 2\pi e \sigma^2$

  if $W \sim \mathcal{N}(0, \sigma^2)$

Capacity of Channel $= \max\limits_{\substack{P_x \\ \sigma_x^2 \le P}} \left\{ I(X;Y) \right\}$

$$= \max \left\{ h(Y) - h(Y|X) \right\}$$

$$= h(Y) - h(X+Z|X)$$

$$= h(Y) - h(Z)$$

$$= h(Y) - \frac{1}{2} \log 2\pi e \sigma^2$$

What about Y ?

$$Y = X + Z \quad ; \quad X ; Z \text{ ind.}$$

$$\Rightarrow \text{var}(Y) = \text{var}(X) + \text{var}(Z)$$
$$\leq P + \sigma^2$$

Entropy maximized if Y is Gaussian

& Thm $h(Y) \leq \frac{1}{2} \log 2\pi e (P + \sigma^2)$

[Achieved if $X \sim N(0, P)$]

Get Capacity $\leq \frac{1}{2} \log \frac{2\pi e (P+\sigma^2)}{2\pi e \sigma^2}$

$$= \frac{1}{2} \log \left[1 + \frac{P}{\sigma^2} \right]$$

Asides:

For Discrete r.v. X

& Continuous Y

$$h(Y) - h(Y|X) = H(X) - H(X|Y)$$

Let Y_ϵ = ϵ- discretization of Y

recall ✓ $h(Y) = \lim_{\epsilon \to 0} \{ H(Y_\epsilon) + \log \epsilon \}$.

$\forall \epsilon$

✓ $H(Y_\epsilon) - H(Y_\epsilon|X) = H(X) - H(X|Y_\epsilon)$

✓ $H(X|Y) = \lim_{\epsilon \to 0} H(X|Y_\epsilon)$

cont. r.v.

if $X_1 \ldots X_n \in \mathbb{R}$ have dif entropy

$h(X_1 \ldots X_n)$

then

$$h(Y_1 \ldots Y_n) = h(X_1 \ldots X_n) + \log |A|$$

where $\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$

& A invertible

Proof omitted

But needed to see $h(X+Y \mid X)$

$= h(Y \mid X)$!!

Finally AEP for channel coding

$X_1 \dots X_n \longrightarrow Y_1 \dots Y_n$

where $\{(X_i, Y_i)\}_{i=1}^{n}$ i.i.d.

- Jointly typical set has volume $2^{h(X,Y)n}$
- if $\tilde{X}, Y$ i.i.d with marginal dist. P_X, P_Y

then $P_r[\tilde{X}_n, Y_n \in A^{(n)}] \approx 2^{-I(X;Y)n}$

Coding for AWGN Channel

Message Set $= \{1 \dots M\}$

$$M = 2^{Rn}$$

Random Encoding

$$m \longrightarrow X_1(m) \dots X_n(m)$$

$$X_j(m) \sim N(0, P-\epsilon)$$

Decoding: Given $Y_1 \dots Y_n$

if $\exists\,!$ $m \in \{1 \dots M\}$ s.t.

$$\frac{1}{n} \sum_{i=1}^{n} |X_i(m) - Y_i|^2 \leq (\sigma^2 + \epsilon)$$

then output m else $\boxtimes$

$$\Pr[\text{decoding error}] \le \Pr[E_0] + \Pr[E_1] + \Pr[E_2]$$

E_0 : Power $\frac{1}{n}\sum_i X_i(m)^2 > P$

$$\Pr_{m,\text{Encoding}}[E_0] \longrightarrow 0 \quad (\text{LLN});$$

E_1 : Power $\frac{1}{n}\sum z_i^2 > \sigma^2 + \epsilon$

$$\Pr_z[E_1] \longrightarrow 0 \quad (\text{LLN});$$

E_2 : $\exists\ m' \neq m$ s.t. $E_2(m')$

$E_2(m')$: $\sum |Y_i - X_i(m')|^2 \le \sigma^2 + \epsilon$

$$\Pr\left[E_2(m')\right] \le 2^{-I(X;Y)n} \qquad [\text{AEP}]$$

$$\Pr\left[E_2\right] \le 2^{Rn - I(X;Y)n} \qquad [\text{Union Bound}]$$

Thus if $R < I(X;Y)$

$$\Pr[E_2] \to 0 \quad ;$$

Conclude: Error Prob. $\to 0$ if $R < I(X;Y)$.

Converse Proof Similar

But What is really going on?

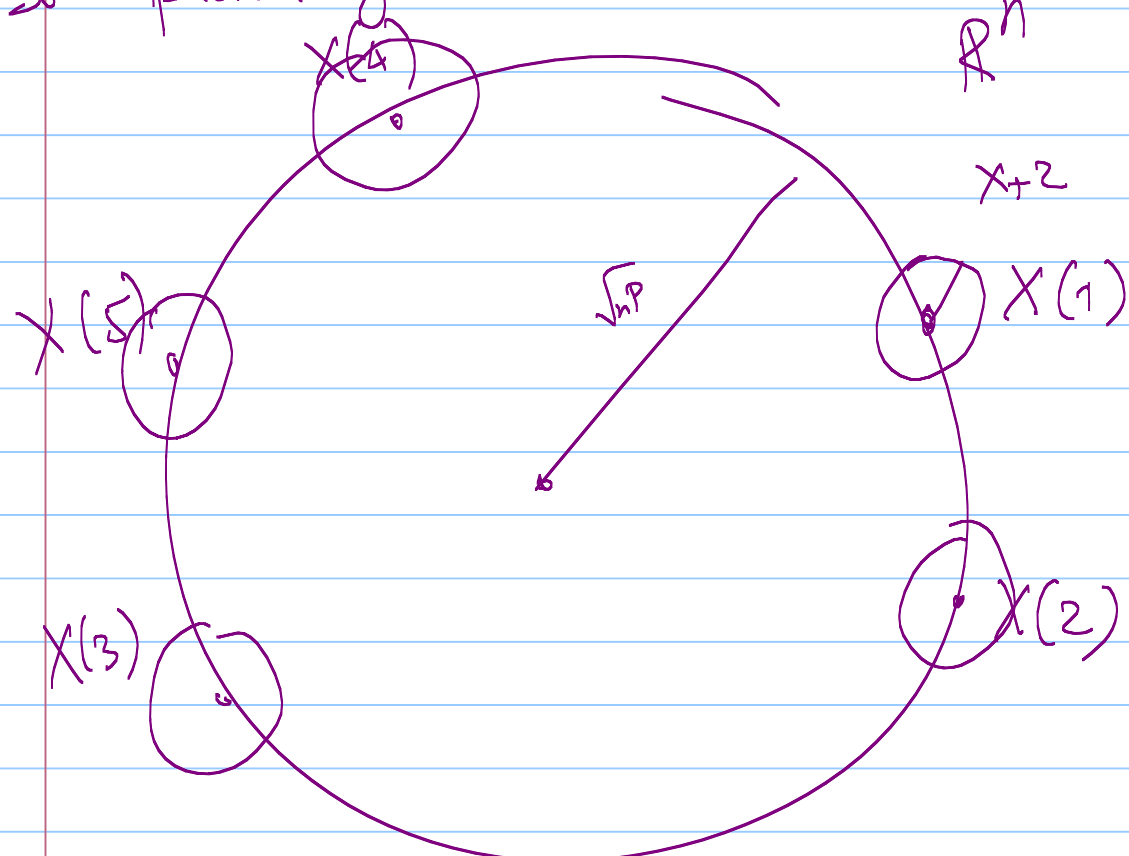
$$X_1(m) \ldots X_n(m) \sim N(0, P)$$

$$\Rightarrow \quad \sum X_i^2(m) \longrightarrow n \cdot P \qquad \text{w.h.p.}$$

$\Rightarrow$ w.h.p. $\underline{X}(m)$ lies in on sphere of radius $\sqrt{nP}$ in n-dim space.

Furthermore, by nice-ness of $N(0,P)$, $\underline{X}$ lies in a "random direction".

So pictorially



- Want many points on Ball of radius $\sqrt{n(P+\sigma^2)}$ s.t. balls of radius $\sqrt{n\sigma^2}$ around them are <u>almost</u> disjoint.

- Volume bound

Vol. of $^{\text{Big}}$ $\sim (\sqrt{nP})^n$

Vol. of small $\sim (\sqrt{n\sigma^2})^n$

Can't hope to pack more that

$$\left(\frac{P}{\sigma^2}\right)^{n/2} \text{ of these}$$

$$\sim 2^{\left(\frac{n}{2} \log \frac{P+\sigma^2}{\sigma^2}\right)} \quad \ldots$$