STO 6 LECTURE 19

Note Title 4/25/2006

TODAY

NETWORK INFORMATION THEORY

└ Multiple Access Channel

Review of last lecture

- Gaussian Channel with Noise σ^2
  Power P

$$\text{Capacity} = \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right)$$

- Colored Channel $n \times n$ with error covariance matrix K_Z & power nP

$$\text{Capacity} = \max_{K_x} \frac{1}{2} \log \frac{|K_x + K_z|}{|K_z|}$$

s.t.

$$\mathrm{tr}(K_x) \leq nP$$

- Turns out this is capacity without feedback

- What can you do with feedback?

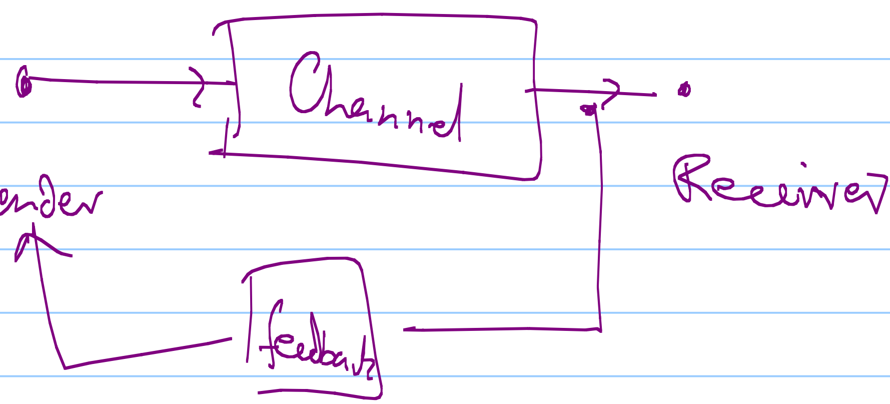
Try to pick X s.t. $|K_{x+z}|$ is maximized .... yields

$$C_{FB,n} = \max_{K_x} \frac{1}{2} \log \frac{|K_{x+z}|}{|K_z|}$$

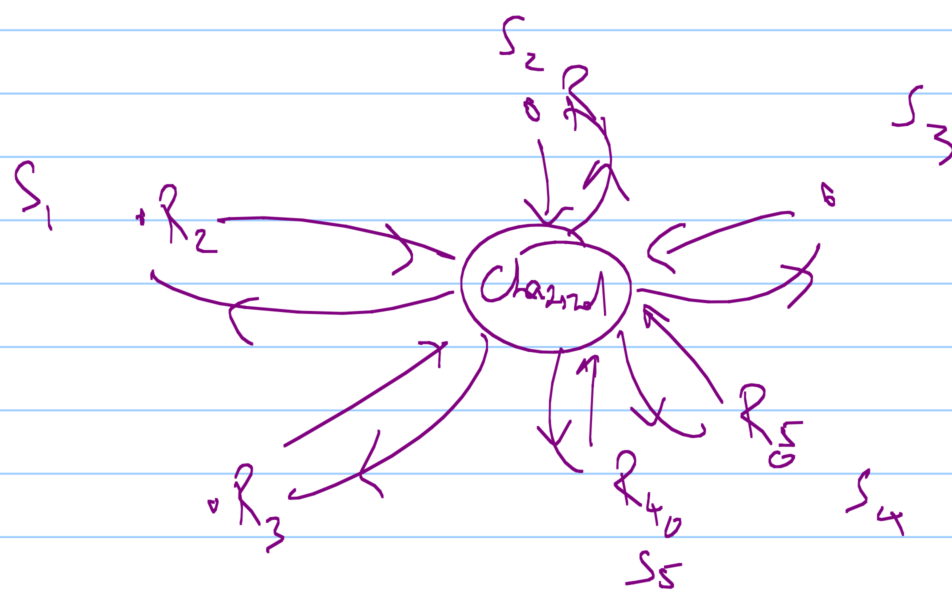
Moving On

Network Information Theory

So far, Information Theory = study of



Real life ... much more complex.



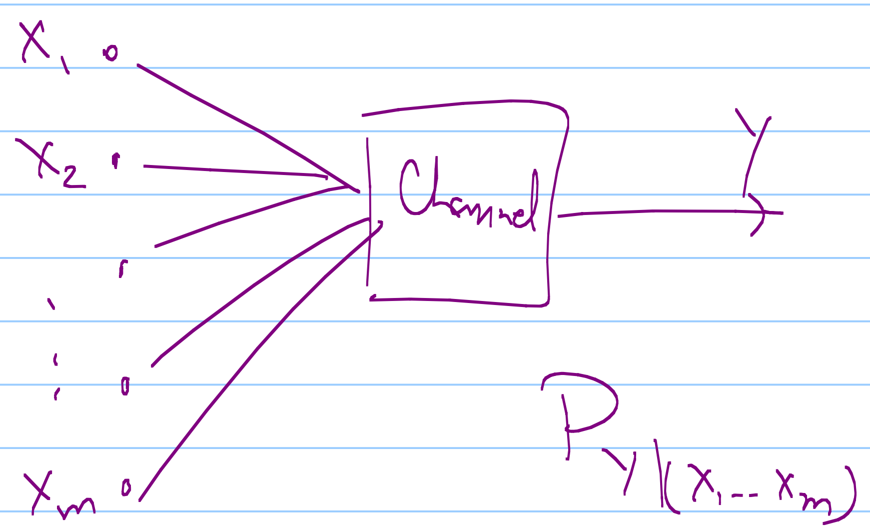
- Many senders +
- Many receivers
- Complex channels : mixes various conversations intentionally/otherwise.
- Network Information Theory studies many problems in this setting.
- Multitude of distinct settings
- many unresolved
- some better understood.
- One such = Multiple access channel.

Multiple-Access Channels

Setting : many m senders

: one receiver

: one channel



Suppose Sender S_i generating information at rate R_i. $i = 1 \dots m$.

Can all m transmit feasibly ?

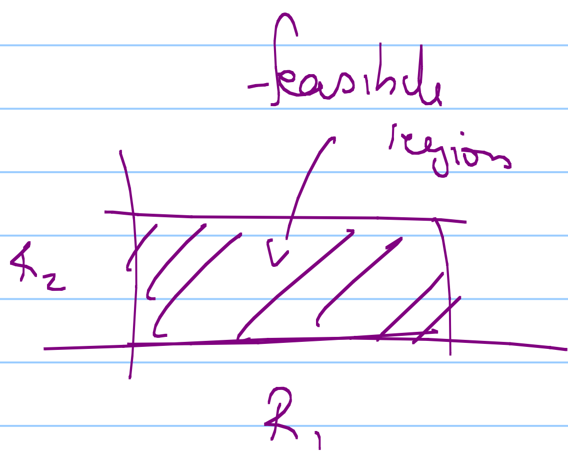
Simple Examples

Parallel channels

$$Y = (X_1 + Z_1 ,\ X_2 + Z_2)$$

Can achieve rates $R_1 \le Cap(X_1 \to X_1 + Z_1)$

$R_2 \le Cap(X_2 \to X_2 + Z_2)$



Channels.

More interesting

$$Y = X_1 + X_2 + Z \pmod 2$$

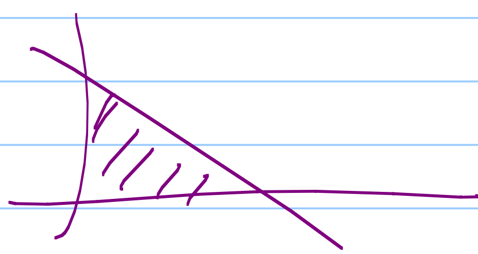
$X_1, X_2 \in \{0,1\}$; $\quad Z = \text{Bern}(p)$

Setting $X_2 = 0$ can achieve

$R_1 = 1 - H(p)$; $R_2 = 0$

Sim. $R_2 = 1 - H(p)$; $R_1 = 0$

time sharing $\Rightarrow$ $R_1 + R_2 = 1 - H(p)$



Can't do better!

Why?

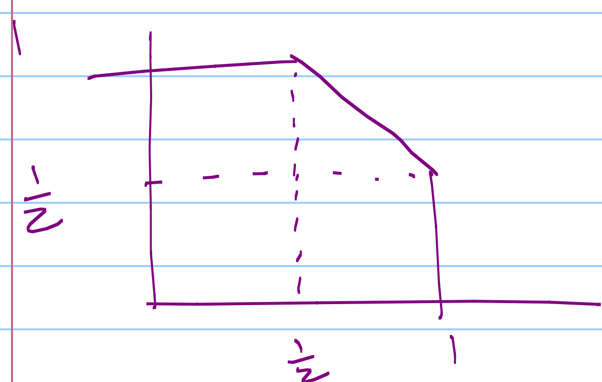
Other cute examples

- $Y = X_1 \cdot X_2$ $\qquad$ $R_1 + R_2 \leq 1$ achievable

  How?

  Why is this best?

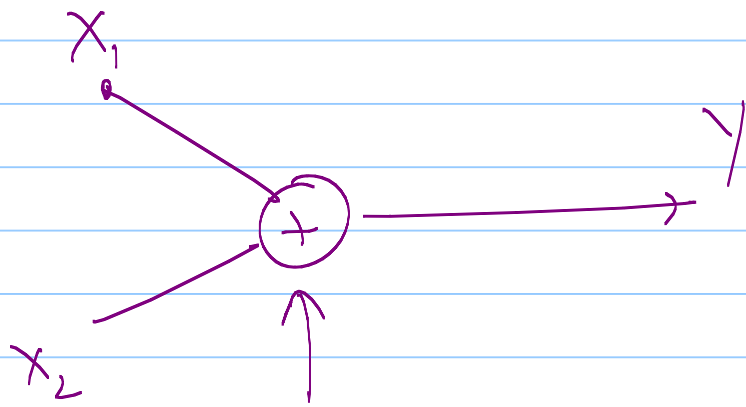
- $Y = X_1 + X_2$ $\qquad$ (Not mod 2!)



Why?

Etc.

Gaussian M-A-Channel



$Z \in N(0, \sigma^2)$

$\mathrm{Var}[X_1] \leq P$; $\mathrm{Var}[X_2] \leq P$.

What do we mean by rate $(R_1 ; R_2)$?

$X_1 \in$ chosen from $2^{R_1 n}$ possibilities uniformly

$X_2 \in$ – – (ind. of X_1)

* Decoder should recover both stream whp.

(Note : Don't consider $(X_1, X_2) \in 2^{(R_1+R_2)n}$ sized set.)

That would be a diff. channel.

Thm: (R_1, R_2) achievable iff

(R_1, R_2) is in the convex hull of all (R_1, R_2) sat.

$$\left.\begin{aligned} R_1 &\le I(X_1 ; Y \mid X_2) \\ R_2 &\le I(X_2 ; Y \mid X_1) \\ R_1 + R_2 &\le I((X_1, X_2) ; Y) \end{aligned}\right] \begin{matrix} X_1\, X_2 \\ \text{ind.} \end{matrix}$$

- Won't prove in general
- Why convex hull? Time sharing.

Special Case

Gaussian M.A. Channel

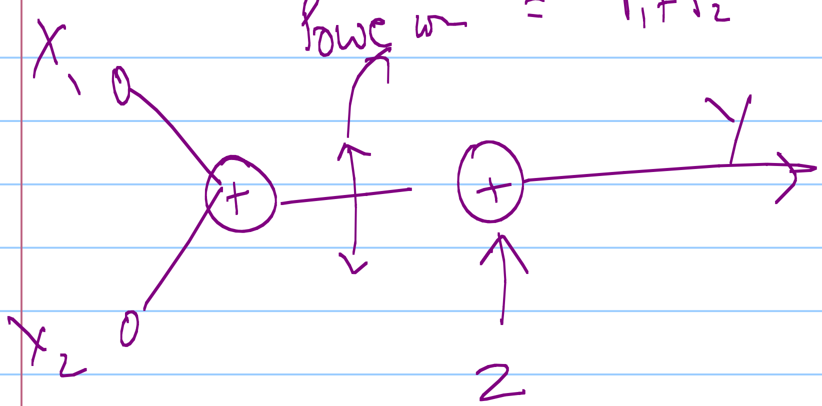
What does bound say: (R_1, R_2) achievable iff

$$\left\{\begin{aligned} R_1 &\le \frac{1}{2}\log\left(1+\frac{P_1}{\sigma^2}\right) \\ R_2 &\le \frac{1}{2}\log\left(1+\frac{P_2}{\sigma^2}\right) \\ R_1+R_2 &\le \frac{1}{2}\log\left(1+\frac{P_1+P_2}{\sigma^2}\right) \end{aligned}\right\}$$

Proof of upper bound

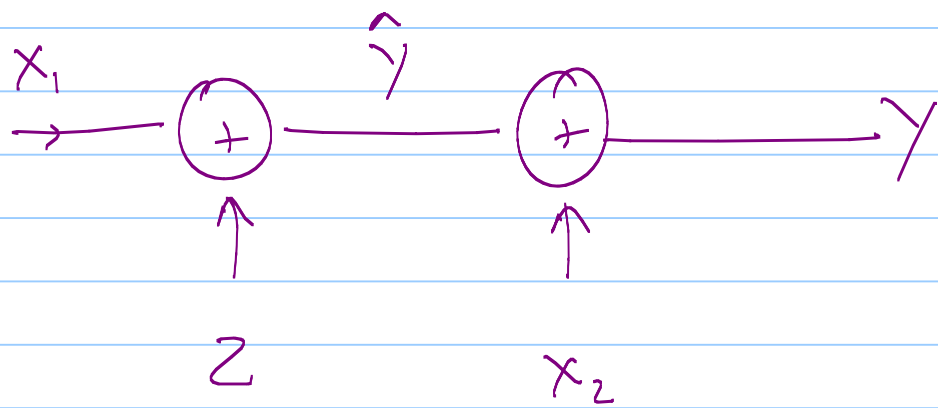
Power = P_1+P_2

1



$$R_1+R_2 \le \frac{1}{2}\log\left(1+\frac{P_1+P_2}{\sigma^2}\right)$$

2



$$R_1 \le I(Y;X_1) \le I(\hat{Y};X_1) = \frac{1}{2}\log\left(1+\frac{P_1}{\sigma^2}\right)$$

Encoding:

- Pick $X_1(W_1)_i \in \mathbb{R}^n \sim N(0, P_1)$

  $X_2(W_2)_i \in \mathbb{R}^n \sim N(0, P_2)$

- $W_1 \in \{1 \ldots 2^{R_1 n}\}$
- $W_2 \in \{1 \ldots 2^{R_2 n}\}$

Decoding

Given Y if $\exists!$ (W_1, W_2)

s.t. $(X_1(W_1), X_2(W_2), Y)$ is jointly

typical then output (W_1, W_2)

Analysis: Usual analysis

$\Rightarrow$ w.h.p. for actual msg.

$(X_1(w_1), X_2(w_2), Y)$ is typical.

Prob. $(X_1(w_1'), X_2(w_2'), Y)$ typical

for $(W_1', W_2') \neq (W_1, W_2)$ = ?

Case 1: $W_1' = W_1$ $\quad W_2' \neq W_2$

$$P_r[\quad] \leq 2^{-I(X_2;Y|X_1)\cdot n}$$

$2^{R_2 n}$ such W_2'

Case 2: $W_1' \neq W_1$, $W_2' \neq W_2$

$$P_r[\quad] \leq 2^{-I(X_1 X_2;Y)\cdot n}$$

$2^{(R_1+R_2)n}$ s.t. W_1', W_2'