NETWORK INFORMATION THEORY

- Multiple Access Channel

Review of last lecture

- Gaussian Channel with Noise $\sigma^2$
  
  
  \[ \text{Capacity} = \frac{1}{2} \log \left(1 + \frac{P}{\sigma^2}\right) \]

- Colored Channel $n \times n$ with error covariance matrix $K_z$ & power $nP$
\[ \text{Capacity} = \max_x \frac{1}{2} \log \frac{|K_x + K_2|}{|K_2|} \]

\[ \text{or} \]

\[ \text{tr}(K_x) \leq n \rho \]

- Turns out this is capacity without feedback

- What can you do with feedback?

Try to pick \( X \) s.t. \( |K_{x+2}| \) is minimized .... yields

\[ C_{FB,n} = \max_x \frac{1}{2} \log \frac{|K_{x+2}|}{|K_x|} \]

\[ \underline{\text{moving in}} \]
Network Information Theory

So far, Information Theory = Study of

Real life: much more complex.

Diagram:

Sender -> Channel -> Receiver

Feedback

\[ S_1, R_1, R_2, R_3, R_4, R_5, S_2, S_3, S_4, S_5 \]
- Many senders
- Many receivers
- Complex channels: mixes various conversations intentionally
  otherwise.
- Network Information Theory studies many problems in this setting.
- Multitude of distinct settings
- Many unresolved
- Some better understood.
- One such = multiple access channel.
Multiple Access Channels

Setting: many m senders
  one receiver
  one channel

\[ X_1, X_2, \ldots, X_m \]

Channel

\[ P_Y(Y|X_1, \ldots, X_m) \]

Suppose sender \( S_i \) generating information at rate \( R_i \), \( i = 1 \ldots m \).

Can all \( m \) transmit feasibly?
Simple Examples

Parallel channels

\[ Y = (X_1 + Z_1, X_2 + Z_2) \]

Can achieve rates

\[ R_1 \leq \text{Gap}(X_1 \rightarrow X_1 + Z_1) \]

\[ R_2 \leq \text{Gap}(X_2 \rightarrow X_2 + Z_2) \]

Region

- Feasible channels.
Interesting

\[ Y = X_1 + X_2 + Z \pmod{2} \]

\[ X_1, X_2 \in \{0,1\}, \quad Z = \text{Bern}(p) \]

Setting \( X_2 = 0 \) can achieve

\[ R_1 = 1 - H(p) \quad \text{and} \quad R_2 = 0 \]

Thus,

\[ R_2 = 1 - H(p) \quad \text{and} \quad R_1 = 0 \]

Time sharing \( \Rightarrow R_1 + R_2 = 1 - H(p) \)

Can’t do better!

Why?
Often cute examples

\[ Y = X_1 \cdot X_2 \quad R_1 + R_2 \leq 1 \]

achievable

How?

Why is this best?

\[ Y = X_1 + X_2 \quad (\text{not mod 2!}) \]

\[ \frac{1}{2} \]

Why?

Etc.
Gaussian M-A Channel

\[ Z \in \mathcal{N}(0, \sigma^2) \]

\[ \text{Var} \left[ X_1 \right] \leq P \ ; \ \text{Var} \left[ X_2 \right] \leq P \]

What do we mean by rule \((R_1, R_2)\)?

\( X_1 \) is chosen from \(2^{R_{in}}\) possibilities uniformly

\( X_2 \in \ldots (\text{i.i.d. of } X_1) \)
Decoder should recover both source

Wp.

(Note: Don’t consider \((x_i, x_2) \in \Sigma^{(R_1 + R_2)}_n\

That would be a different channel.

Thm.: \((R_1, R_2)\) achievable if

\((R_1, R_2)\) is in the convex hull

of all \((R_1, R_2)\) set.

\[
R_1 \leq I(x_i; Y | x_2) \bigg|_{\text{ind.}}
\]

\[
R_2 \leq I(x_2; Y | x_1)
\]

\[
R_1 + R_2 \leq I((x_1, x_2); Y)
\]
- Won't prove in general
- Why convex hull? Time sharing

Special Case
Gaussian M.A. Channel

What does bound say: \((R_1, R_2)\) achievable

\[
\begin{align*}
R_1 & \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{\sigma^2} \right) \\
R_2 & \leq \frac{1}{2} \log \left( 1 + \frac{P_2}{\sigma^2} \right) \\
R_1 + R_2 & \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2}{\sigma^2} \right)
\end{align*}
\]
Proof of upper bound

\[ \text{Proof: } w = p_1 + p_2 \]

\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{p_1 + p_2}{\sigma^2} \right) \]

\[ R \leq I(y; x_1) \leq I(\hat{y}; x_1) = \frac{1}{2} \log \left( 1 + \frac{p_1 + p_2}{\sigma^2} \right) \]
Encoding:
- Pick \( x_1(w_i) \in \mathbb{R}^n \sim N(0, p_1) \)
  \[ x_2(w_2) \in \mathbb{R}^n \sim N(0, p_2) \]
- \( w_1 \in \{1, \ldots, 2^{r_1} \} \)
- \( w_2 \in \{1, \ldots, 2^{r_2} \} \)

Decoding:
- Given \( y \) if \( \exists! \) \( (w_1, w_2) \)
- s.t. \( (x_1(w_1), x_2(w_2), y) \) is jointly typical then output \( (w_1, w_2) \)
Analysis: Usual analysis

\[ \Rightarrow \text{w.h.p. for actual msg.} \]

\[ (X_1(w_1), X_2(w_2), y) \text{ is typical.} \]

\[ \text{Prob. } (X_1(w'_1), X_2(w'_2), y) \text{ typical for } (w'_1, w'_2) \neq (w_1, w_2) \]

Case 1: \[ w'_1 = w_1, \quad w'_2 = w_2 \]

\[ \Pr \left[ \right] \leq 2^{-I(X_2; Y | X_1) \cdot n} \]

Case 2: \[ w'_1 \neq w_1, \quad w'_2 \neq w_2 \]

\[ \Pr \left[ \right] \leq 2^{-I(X, X_2; Y) \cdot n} \]

\[ 2^{(R_1 + R_2) \cdot n} \text{ a.s. } w'_1, w'_2 \]