STO6 LECTURE 21

Note Title 5/2/2006

TODAY

NETWORK INF. THEORY (CONTD.)

- CORRELATED SOURCE CODING
- CODING WITH SIDE INFORMATION
- BROADCAST CHANNEL.

Recall

Network Inf. Theory

Generic model : - Many sources $X_1 \ldots X_k$ (Correlated)

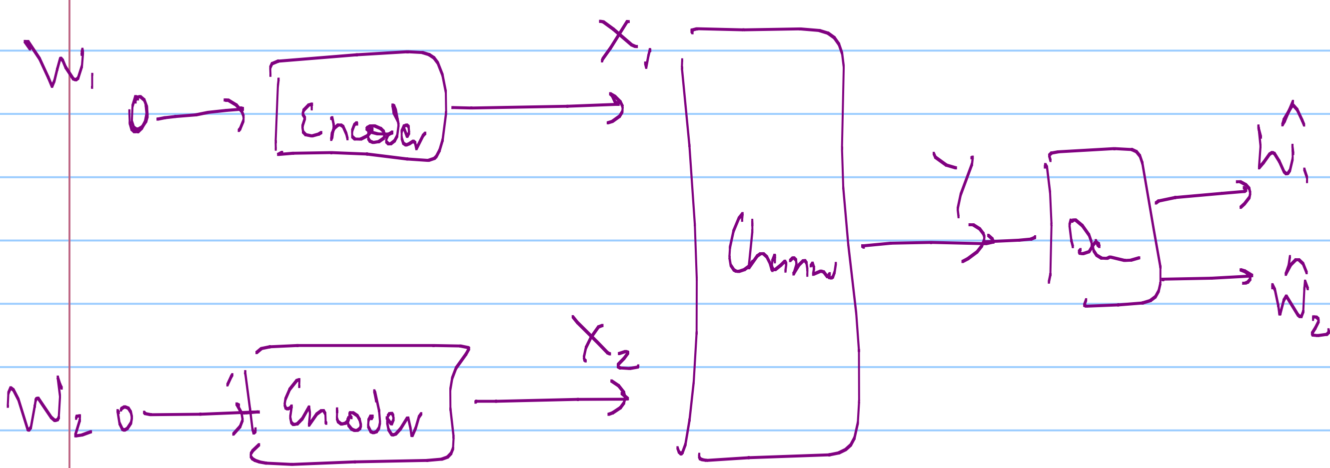
- Many receivers $Y_1 \ldots Y_\ell$

- "Network Channel" : $P_{(X_1 \ldots X_k) | (Y_1 \ldots Y_\ell)}$

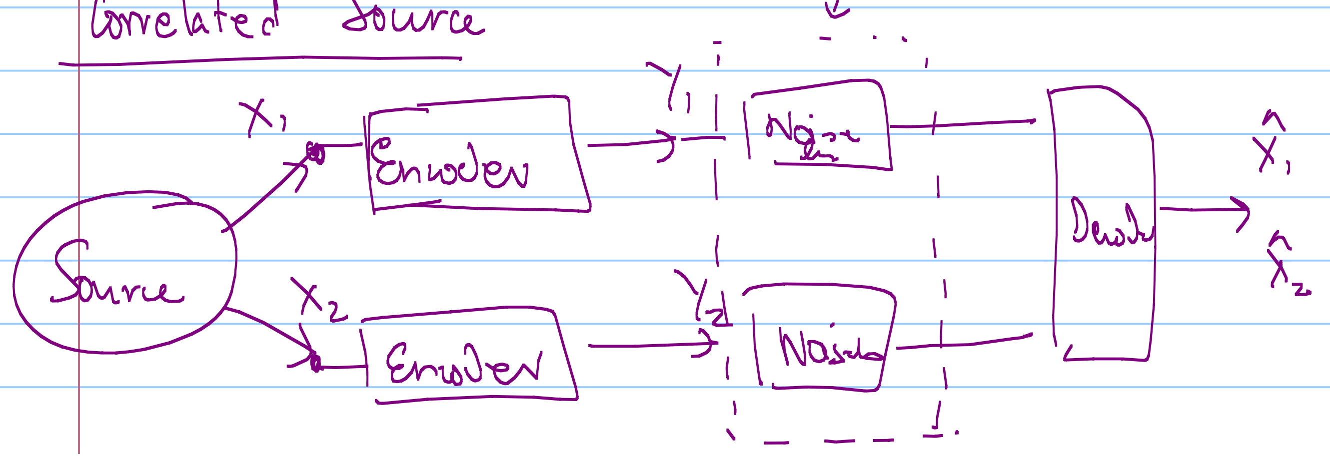
Focus on "individual rates" (e.g $X_i \xrightarrow{R_{ij}} Y_j$)

Within this setting considered

M.A. Channel



Correlated Source



Main differences

$$\begin{cases} W_1, W_2 & \text{independent in M.A.} \\ X_1, X_2 & \text{correlated in Slep.-Wolf} \end{cases}$$

$$\begin{cases} Y \neq (X_1, X_2) & \text{in M.A.} \\ Y = (Y_1, Y_2) & \text{in Slep. Wolf.} \end{cases}$$

<u>Defn.</u> (R_1, R_2) acheivable in Slep. Wolf

if $\exists$

$$f_1 : \Omega_{X_1}^n \to \{1 \dots 2^{nR_1}\}$$

$$f_2 : \Omega_{X_2}^n \to \{1 \dots 2^{nR_2}\}$$

and $g : \{1 \dots 2^{nR_1}\} \times \{1 \dots 2^{nR_2}\} \to \Omega_{X_1}^n \times \Omega_{X_2}^n$

s.t. $P_{err}(n) \xrightarrow[n \to \infty]{} 0$

where $P_{err} = \Pr\left[(\hat{X}_1, \hat{X}_2) \neq (X_1, X_2) \right]$

where

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] & Y_1 & \\
X_2 \arrow[r, "f_2"] & Y_2 \arrow[r, "g"] & (\hat{X}_1, \hat{X}_2)
\end{tikzcd}$$

Thm: (R_1, R_2) achievable iff

$$R_1 \geq H(X_1 \mid X_2)$$

$$R_2 \geq H(X_2 \mid X_1)$$

$$R_1 + R_2 \geq H(X_1, X_2)$$

Proof: Converse: Standard argument; more tedious

Coding?

Pick $f_1 : \Omega_{x_1}^n \to \{1 \dots 2^{nR_1}\}$ at random

$f_2 : \Omega_{x_2}^n \to \{1 \dots 2^{nR_2}\}$ "

Decoding: Given (Y_1, Y_2) if

$\exists$ unique $(\hat{X}_1, \hat{X}_2)$ s.t.

1. $(\hat{X}_1, \hat{X}_2)$ jointly typical

2. $Y_1 = f_1(\hat{X}_1)$, $Y_2 = f_2(\hat{X}_2)$

then output $(\hat{X}_1, \hat{X}_2)$ else ERROR

Analysis

Part 1: if (X_1, X_2) = message then

Prob. that (X_1, X_2) not typical $\to 0$
(LLN)

Part 2 :

(a) $\left.\begin{array}{l}\exists\ \hat{X}_2 \neq X_2 \quad \text{s.t.} \quad (X_1, \hat{X}_2) \text{ jointly typical} \\ \quad \& \quad f_2(\hat{X}_2) = f_2(X_2)\end{array}\right\} E_{1a}$

fix $\hat{x}_2, x_2$

$$\Pr_{f_2}\left[f_2(\hat{x}_2) = f_2(x_2)\right] \leq 2^{-nR_2}$$

$$\#\ \hat{X}_2 \quad \text{s.t.} \quad (\underline{X}_1, \underline{\hat{X}}_2) \text{ jointly typical} = ?$$

$$\underline{X}_1 \text{ typical} \Rightarrow \Pr[\underline{X}_1] \leq 2^{-(H(X_1)-\epsilon)n}$$

$$(X_1, \hat{X}_2) \text{ typical} \Rightarrow \Pr[(X_1, \hat{X}_2)] \geq 2^{-(H(X_1,X_2)+\epsilon)n}$$

$$\Rightarrow \#\ X_2 \leq 2^{[H(X_1,X_2) - H(X_1) + 2\epsilon]n}$$

$$\leq 2^{(H(X_2|X_1) + 2\epsilon)n}$$

$$\Rightarrow \Pr\left[E_{1a}\right] \le 2^{(-R_2 + H_2(X_2|X_1) + 2\epsilon)n}$$

$$\rightarrow 0 \quad \text{if} \quad R_2 > H_2(X_2|X_1)$$

(b) $\exists\ \hat{X}_1, \ldots \ (\hat{X}_1, X_2)$ jointly typical.

(c) $\exists\ \hat{X}_1 \neq X_1,\ \hat{X}_2 \neq X_2 \ \ldots$

Conclude: (R_1, R_2) achievable if

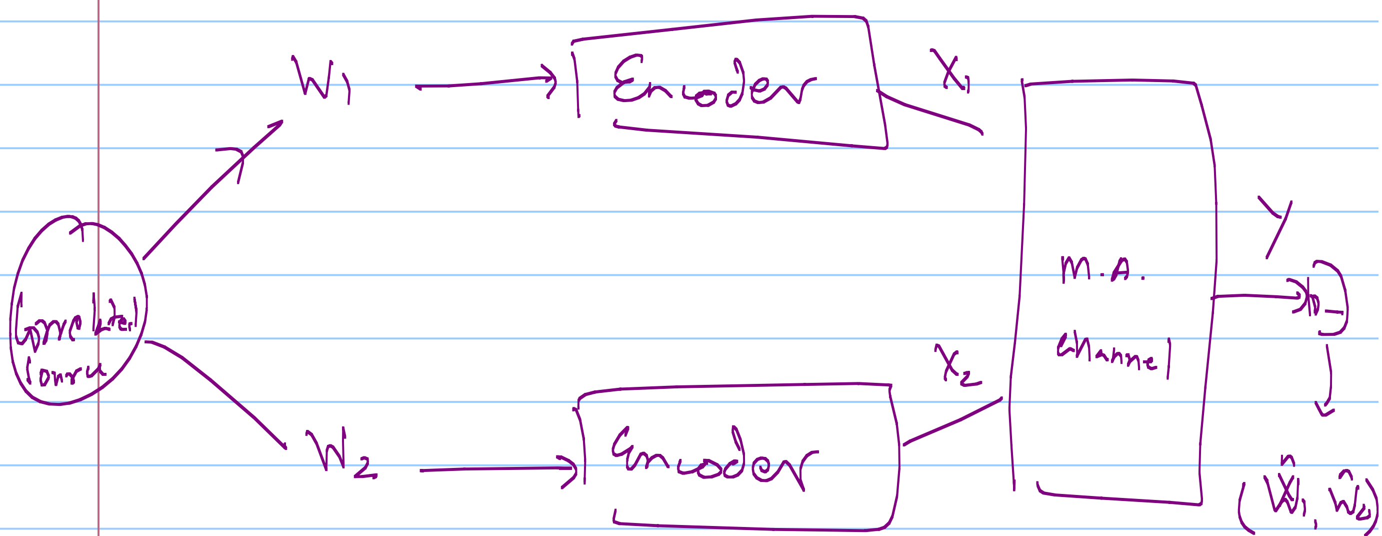
$$R_1 \geq H(X_1|X_2) \qquad (2b)$$

$$R_2 \geq H(X_2|X_1) \qquad (2a)$$

$$R_1 + R_2 \geq H(X_1, X_2) \qquad (2c)$$

Back to M.A; Recall W_1, W_2 ind.

Now suppose not.



Putting two together we get X-mission achievable if $\exists$ R_1, R_2 s.t.

$$R_1 \geq H(W_1 | W_2)$$

$$R_2 \geq H(W_2 | W_1)$$

$$R_1 + R_2 \geq H(W_1, W_2)$$

(R_1, R_2) in convex hull of $\tilde{R}_1, \tilde{R}_2$ s.t.

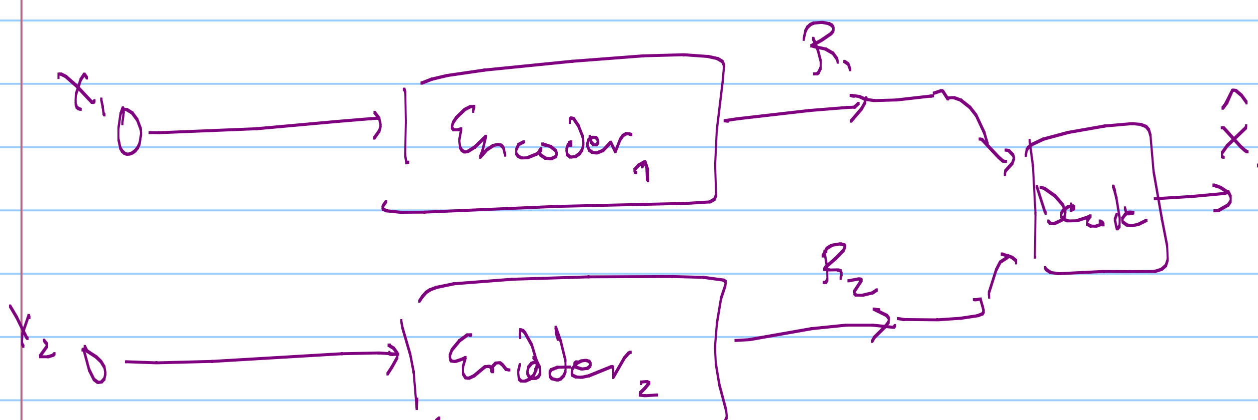
$$\tilde{R}_1 \leq I(X_1 ; Y | X_2)$$

$$\tilde{R}_2 \leq I(X_2 ; Y | X_1)$$

$$\tilde{R}_1 + \tilde{R}_2 \leq I((X_1, X_2) ; Y)$$

Implication = ? Is this tight ?

Side Information : Similar to Slepian - Wolf.



Main Difference : Not interested in X_2.

How should we encode ?

Example

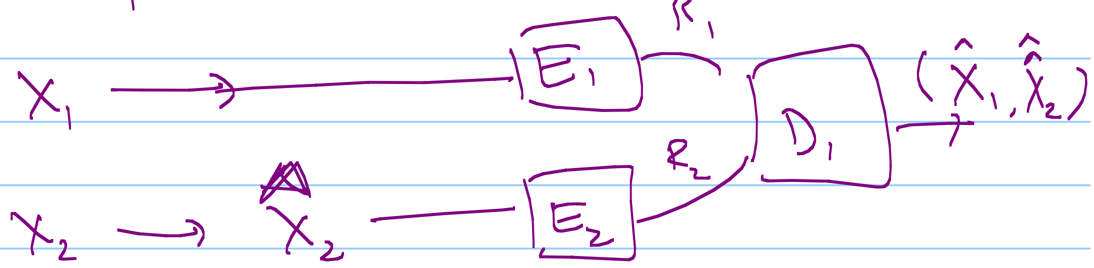
$X_1 = Z_1 Z_2 \qquad X_2 = Z_2 Z_3$

Z_1, Z_2, Z_3 ind. with entropy H_1, H_2, H_3.

When is it feasible to recover X_1 ?

Answer iff feasible to encode (X_1, Z_2) in Slep.-Wolf setting

Generally: feasible if $\exists\ \hat{X}_2$



if $\exists \hat{X}_2$ $\quad X_1 \to X_2 \to \hat{X}_2$

s.t. $\quad R_1 \geq H(X_1 | \hat{X}_2)$

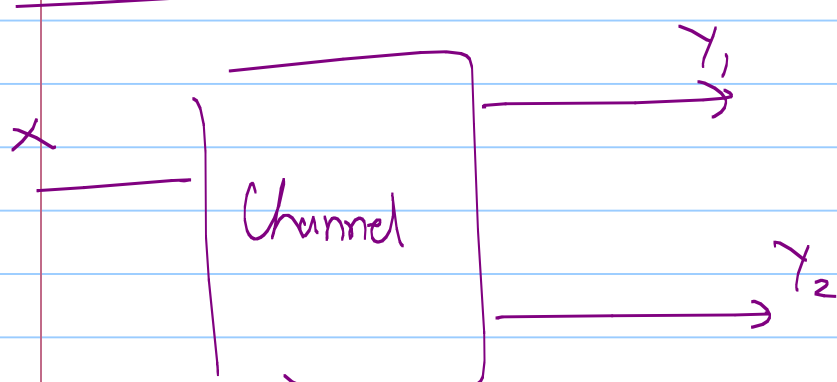
$$R_2 \geq H(\hat{X}_2 | X_1)$$

$$R_1 + R_2 \geq H(X_1, \hat{X}_2)$$

$$\Rightarrow \quad R_2 \geq H(X_1, \hat{X}_2) - H(X_1 | \hat{X}_2)$$

$$\geq I(X_1 ; \hat{X}_2)$$

BROADCAST CHANNELS



Defn: (R_0, R_1, R_2) acheivable if

$\exists \ \underline{X} : \{1 \ldots 2^{nR_0}\} \times \{1 \ldots 2^{nR_1}\} \times \{1 \ldots 2^{nR_2}\} \to \Omega_X^n$

& Decoders $D_1 : \Omega_{Y_1}^n \to \{\ ^{R_0}\} \times \{\ ^{R_1}\}$

$D_2 : \ ^n_{Y_2} \to \quad ^{R_0} \quad\quad ^{R_2}$

s.t.

$Pr\left[\text{Decoding Error}\right] \to 0$

Channels $X \to (Y_1, Y_2)$ & $X' \to Y_1', Y_2'$ equiv.

if marginals equivalent.

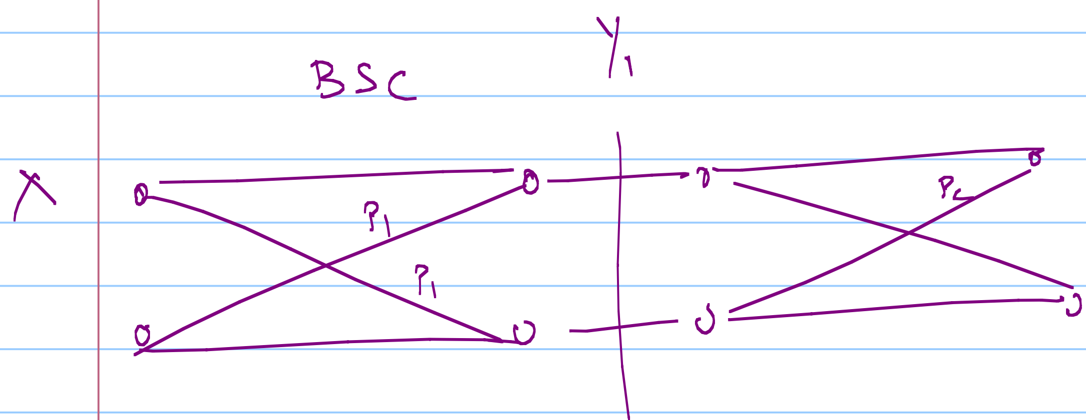
Channel Degraded if

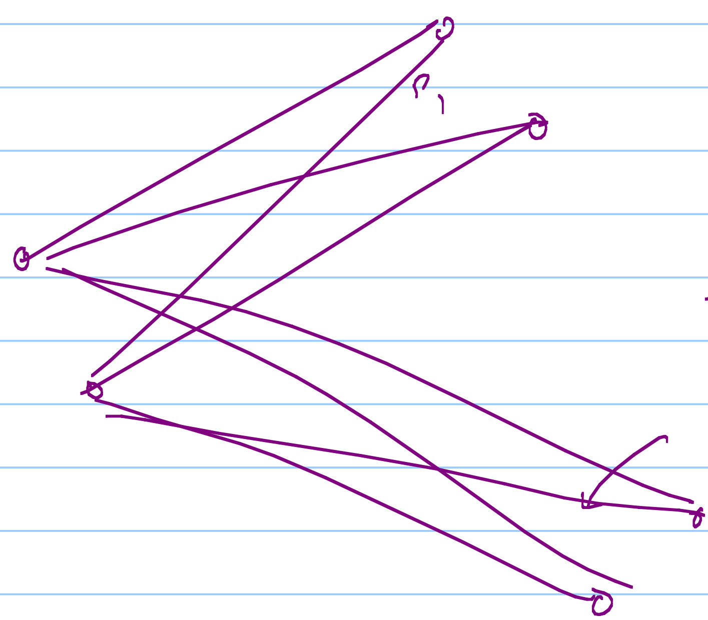
$$X \longrightarrow Y_1 \longrightarrow Y_2$$

Stoch. Degraded if $\exists (X\, Y_1\, Y_2)$ chan

$X \to Y_1 \to Y_2$

Example:



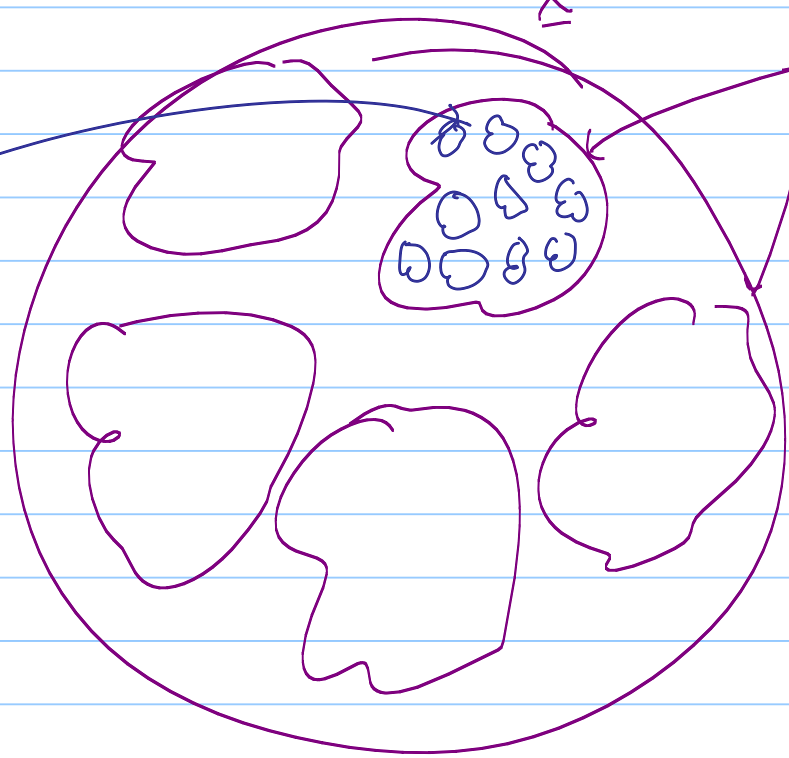


$$p_2' = p_1(1-p_2) + p_2(1-p_1)$$

How to encode for such channels?

Simplify : $\boxed{\text{Degraded} \; ; \quad R_0 = 0 \; ;}$

Ideas:



Need 2^{nR_2} Y_2 regions

2^{nR_1} Y_1 regions / Y_2 region

$\Rightarrow \quad R_2 \leq I(X; Y_2)$

$R_1 \leq I(X; Y_1 | Y_2)$

Even worse

$R_2 \leq I(U; Y_2)$

$R_1 \leq I(X; Y_1 | U)$