Today:
- (Degraded) Broadcast Channel

Review of "Network" I.T.:
- Multiple Access Channel

\[
\begin{array}{c}
\text{Rate region}
\end{array}
\]

- Correlated Sources

\[
\begin{array}{c}
\text{Rate region}
\end{array}
\]
Capacities needed

\[ H(x_2|x_1) \]

\[ H(x_1|x_2) \]

Together: Often enable us to achieve capacity

But not always best: Chung's Example

Correlated source produces \( W_1 = W_2 \) \( \rho_{w_1w_2} \leq \rho \)

Channel: \( (X_1, X_2) \rightarrow X_1 + X_2 + Z \)
Making $X_1, X_i$ ind. $\Rightarrow$ Signal power $= 2P$

$\Rightarrow$ Capacity $\leq R_1 + R_2 = \frac{1}{2} \log\frac{2P}{\varepsilon^2}$

Setting $X_1 = X_2 \Rightarrow$ Signal power $= 4P$

$\Rightarrow$ Capacity $= \frac{1}{2} \log\frac{4P}{\varepsilon^2}$

Can realize this if $W_1 = W_2$

Also talked about side channels.

Try coding à la Nisan-Wolf as an exercise....
Today

Broadcast Channels: Single Source

Many receivers.

Source sends \( X \)

\( R_1 \) sees \( Y_1 \)

\( R_2 \) sees \( Y_2 \)

Channel characterized by

\[ P_{Y_1|X}, \quad P_{Y_2|X} \]

Goal of Receivers - Is some common info

Some disjoint info
Rate triple \((R_0, R_1, R_2)\) achievable if

\[
\begin{align*}
\mathcal{X} & : S_0 \times S_1 \times S_2 \rightarrow \mathbb{R}^n_x \\
D_1 & : \mathbb{R}^n_x \rightarrow S_0 \times S_1 \\
D_2 & : \mathbb{R}^n_x \rightarrow S_0 \times S_2 \\
(S_i = \xi_1, \ldots, 2^{nR_i} \xi_i)
\end{align*}
\]

\[w_0, w_1, w_2 \rightarrow x(w_0, w_1, w_2) \xrightarrow{\text{Channel}} (y_1, y_2) \]

\[\sqrt{d_1} \quad \sqrt{d_2}
\]

\[\mathbb{P} \left[ (w_0, w_1, w_2) \neq (\hat{w}_0, \hat{w}_1, \hat{w}_2, \hat{w}_2) \right] \rightarrow 0 \]
Example

\[ S_x = S_y = S_{y_2} = \$0,123 \]

\[ P_1 \leq P_2 \]

Consider a special case where

\[ P_1(1-\alpha) + (1-P_1)\alpha = P_2 \]
Claim: \( P_1 \leq P_2 \leq \frac{1}{2} \) and \( \alpha \leq \frac{1}{2} \) exists

\[
\text{dither channel} \equiv \text{any other channel } P_{y_1|x} \quad \text{with same } P_{y_1|x} \quad P_{y_2|x}
\]

So what's the rate region of BSC Broadcast Channel?

How to encode?

Fix \( R_0 = 0 \)

\[\text{(Aside: } (0, R_1, R_2) \text{ achievable} \Rightarrow (R_0, R_1, R_2 - R_0) \text{ achievable})\]
Clearly $R_1 + R_2 \leq 1 - H(p_2)$ achievable.

But can we do better?
"Pack" $2^n R_2$

spheres of
radius $\approx \eta$

into $50,125 h$

Then "pack" $2^n R_i$

spheres of radius

$n P_i$ into each

$R_i$ finds $x$ & so knows its position relative to

$R_2$: View $\omega$ as "computation" & adds $\omega$

error to decode
Informal analysis

\[
\frac{\text{Vol}((x \& \rho_1)n)}{\text{Vol}((\rho_1)n)} = 2^n \left( H(x \& \rho_1) - H(\rho_1) \right)
\]

\[R_1 \leq H(x \& \rho_1) - H(\rho_1)\]

\[1 - H(\gamma \& \rho_2) \geq R_2 \leq 1 - H(\gamma \& \rho_1) + H(\gamma \& \rho_1) - H(\rho_1) \]

\[R_1 + R_2 = \max_g \left\{ 1 - H(\gamma \& \rho_2) + H(\gamma \& \rho_1) - H(\rho_1) \right\} \geq 1 - H(0 \& \rho_2) + H(0 \& \rho_1) - H(\rho_1) = 1 - H(\rho_2) \]
Optimizing $\gamma$ yields

\[ 1 - H(\gamma) \]

More generally

\[ X \rightarrow Y_1 \rightarrow Y_2 \quad [\text{De graded B.c.}] \]

\[(R_1, R_2) \quad \text{whence} \quad \exists \ U \]

\[ U \rightarrow X \rightarrow Y_1 \rightarrow Y_2 \quad \text{s.t.} \]

\[ R_2 \leq I(U; Y_2) \]

\[ R_1 \leq I(X; Y_1 | U) \]
**Encoding**: \( P_u \)

Given \((W_1, W_2)\)

Let \( U(W_2) \) \( \sim \) \( P_u \) i.i.d. over \( i, w_2 \)

Let \( X(W_1, W_2) \) \( \sim \) \( P_{X|U}(U(W_2)) \) i.i.d. over \( i, w_1, w_2 \)

Transmit \( X(W_1, W_2) \)

**Decoding**: Typical Set

**Analysis**: Two separate issues

\( R_1 \) \( \text{con} \) \( \to 0 \) if \( I(X; Y_i | U) \) small

\( R_2 \) \( \text{con} \) \( \to 0 \) if \( I(X; Y_2 | U) \) small