Today

Conclude: Broadcast Channel

Recall

\[ \text{Channel} = P(\mathbf{Y}_1, \mathbf{Y}_2 | \mathbf{X}) \]
Fact: Channels with $P_{Y_1,Y_2|x} \triangleq P'_{Y_1,Y_2|x}$

"Equivalent" if

$P_{Y_1|x} = P'_{Y_1|x}$

$P_{Y_2|x} = P'_{Y_2|x}$

"Equivalence": $(R_1, R_2)$ achievable on $P$ if $(R_1, R_2)$ achievable on $P'$. 

Defn: $(R_1, R_2)$ achievable on Channel with $P$ if

$\exists \mathcal{E}_i, D_i, D_2$ s.t. if

$(w_1, w_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ \quad $\mathcal{S}_i = \{0, \ldots, 2^{nR_i}\}$

$(w_1, w_2) \xrightarrow{X^n} Y_1^n, Y_2^n \xrightarrow{D_i, D_2} \hat{W}_1, \hat{W}_2$

$\Pr[err] = \Pr[(w_1, w_2) \neq (\hat{W}_1, \hat{W}_2)] \rightarrow 0$
Degraded Channel

\[ X \rightarrow Y_1 \rightarrow Y_2 \]

Stochastically degraded channel: \( P \)

\[ P \text{ s.d. it } \exists \text{ equivalent } P' \text{ s.t. } P' \]

\[ \text{is Degraded} \]

\[ \therefore P'(Y_2|Y_1, X) = P'(Y_2|Y_1) \]

\[ \therefore P(Y_2|X) = P'(Y_2|X) \]

\[ P(Y_1|X) = P'(Y_1|X) \]

\[ \Rightarrow \exists P' \text{ s.t.} \]

\[ P(Y_2|X) = P'(Y_2|X) \]

\[ \leq \sum_{Y_1} P'(Y_1, Y_2|X) \]

\[ = \sum_{Y_1} P'(Y_1|X) ?(Y_2|Y_1, X) \]
\[ \sum_{y_1} p(y_1 | x) p'(y_2 | y_1) \]

Theorem: For (stochastically) degraded channel \( \mathcal{P} \)

(Coding): \( (R_1, R_2) \) achievable \( \iff \exists U \)

\[ U \rightarrow x \rightarrow Y_1 \rightarrow Y_2 \quad \text{and} \]

\[ R_1 \leq I(X; Y_1 | U) \]

\[ R_2 \leq I(Y_2; U) \]

(Converse): \( (R_1, R_2) \) achievable \( \Rightarrow \exists U \)

\[ \text{with} \quad |U| \leq \min \left\{ |\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2| \right\} \]

\[ U \rightarrow x \rightarrow Y_1 \rightarrow Y_2 \]

\[ R_1 \leq I(X; Y_1 | U) \]

\[ R_2 \leq I(Y_2; U) \]
(won't prove converse)

Coding Theorem Proof

Encoding $E$

Step 1:
Given $W_2$ generate $U^n = (U_1, \ldots, U_n)$
$U_i \sim P_u$ i.i.d.

Step 2:
Given $W_1$, $U^n$ generate $X^n = (X_1, \ldots, X_n)$
$X_i \sim P_{x|u}(U_i)$ independently
Decoding

Receiver 1: if $\exists \hat{\omega}_1, \hat{\omega}_2$ s.t.
\[
(U(\hat{\omega}_2), x(\hat{\omega}_1, \hat{\omega}_2), Y_1^n) \text{ jointly typical, output } \hat{\omega}_1, \text{ else error}
\]

Receiver 2: if $\exists \hat{\omega}_1$ s.t.
\[
(U(\hat{\omega}_1), Y_2^n) \text{ jointly typical output } \hat{\omega}_1, \text{ else error.}
\]

Analysis:

Receiver 2: Same as $U \rightarrow Y_2$

Channel:

OK if $R_2 \leq \ldots$
Receiver 1:

1. @ \((U(w_1), x(\hat{w}_1, w_2), y)\) typical

\[
\Pr[ ] \leq 2^{-I(x; y_1 U) N}
\]

2. \((U(\hat{w}_2), x(\hat{w}_1, \hat{w}_2), y)\) typical

\[
\Pr[ ] \leq 2^{-I(x, U ; y)} N
\]

\[
R_1 \leq I(x; y_1 U)
\]

\[
R_1 + R_2 \leq I(x, U ; y)
\]

\[
= I(x, y_1 U) + I(U; y)
\]

redundant since \(R_1 \leq I(x, y_1 U)\)

\[
R_2 \leq I(U; y_2) \leq I(U; y)
\]
General Setup
Many nodes acting as receivers and transmitters.

Intermediate nodes can:
1. Store & forward
2. Re-compute & forward
Example

\[ S \]

\[ R=2a+ib \]

\[ a, b \]

What can be achieved?

\[ H(Y_1, Y_2, Y_3 \mid X_1, X_2, X_3, X_4) = 0. \]
\[ R_1 + R_2 + R_3 \leq C_1 + C_2 + C_3 + C_4 \]

Not always achievable.

Don't even know what kind of entropies achievable.

\[ \exists \text{ given } H_{\mathcal{E}_1}, H_{\mathcal{E}_2}, \ldots, H_{\mathcal{E}_s}, \ldots, H_{\mathcal{E}_m} \]

\[ S \subseteq \mathcal{E}_1, \ldots, \mathcal{E}_m \]
Does there exist a distribution

\[ P(x_1, \ldots, x_m) \rightarrow \forall s \]

\[ H_s = H\{\{x_i\}_i \in s\} \]

\(H_s\)'s satisfy some constraints (Chain Rule)

\[ H_{S \cup T} \leq H_s + H_T \]

But other constraints apply! [Hi, Yeng...]


\( \text{even better} \)

\[ H_{S \cup T} + H_{S \cap T} \leq H_s + H_T \]

(See [Zeger's talk].)