6.441 Transmission of Information April 25, 2006

Lecture 18

Lecturer: Madhu Sudan *Scribe: Xiaomeng Shi*

Review of Last Lecture

Gaussian Channel

- Noise $\sim \mathcal{N}(0, \sigma^2)$
- Input power constraint P
- Capacity achieving input is Gaussian with variance P
- Capacity: $\frac{1}{2}\log(1+\frac{P}{\sigma^2})$.

Colored Gaussian Channels

- Blocks of n elements transmitted each time
- The additive noise $Z \in \mathbb{R}^n$ is multivariate gaussian with covariance matrix K_Z (noise with memory)
- Input signal $X \in \mathbb{R}^n$ has covariance matrix K_X
- Input power constraint is nP, ie. $\text{trace}(K_X) \leq nP$.
- Capacity (without feedback):

$$C_n = \frac{1}{2}\log\frac{|K_X + K_Z|}{|K_Z|}$$

- Unlike memoryless channels, in channels with memory, feedback may increase capacity by as much as $\frac{1}{2}$ bit. A colored Gaussian channel with feedback has capacity

$$C_{FB,n} = \max_{K_X : \text{tr}(K_X) \leq nP} \frac{1}{2}\log\frac{|K_{X+Z}|}{|K_Z|} \leq C_n + \frac{1}{2}$$

Network Information Theory

So far, we have only considered single transmitter single receiver communication systems as shown on the left side of Figure 1. More generally, practical communication systems are more complex and may contain multiple senders and/or multiple receivers in various configurations. The second plot in Figure 1 illustrates such a system. The channel is not dedicated to one communication link, but shared between multiple users. Network information theory studies problems in such settings. There are many unsolved problems in network information theory, but some special networks are better understood than others. One example is the multiple access (MA) channel.

The Multiple Access Channel

The multiple access has many (m) senders and one receiver:
An example of MA channels is the ethernet. A MA channel can be charaterized by its input alphabet $\Omega_{X_1}, \Omega_{X_2}, \ldots, \Omega_{X_m}$, output alphabet Ω_Y, and probability transition function $P_{Y|X_1,\ldots,X_m}$. One question we would like to ask is, suppose the sources generate information at rate $R_i, i \in \{1, \ldots, m\}$. Is it feasible to transmit all messages correctly? Next we look at some simple examples of multiple access channels to study what rates are feasible.

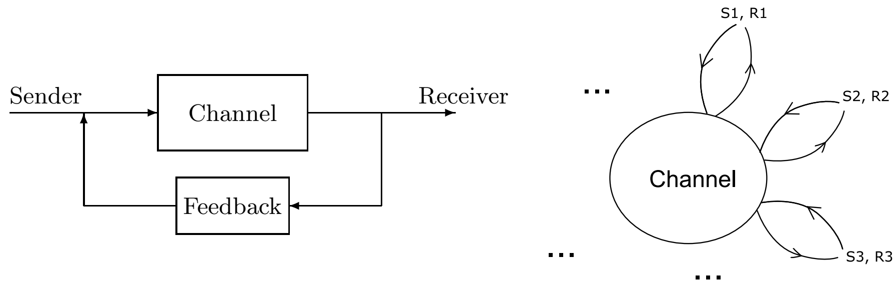


Figure 1: Single Channel vs. Network

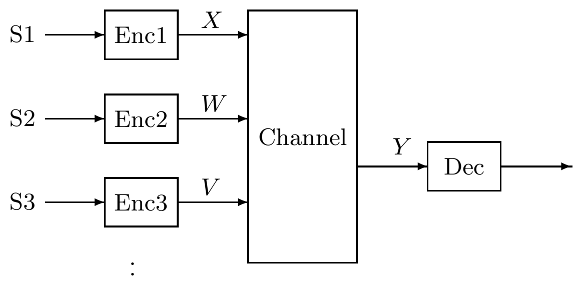


Figure 2: Multiple Access Channel

Examples of Multiple Access Channels

Parallel Channel: $Y = (X_1 + Z_1, X_2 + Z_2)$

Achievable Rates: $R_1 \leq C_1(X_1 \to X_1 + Z_1)$, $R_2 \leq C_2(X_2 \to X_2 + Z_2)$

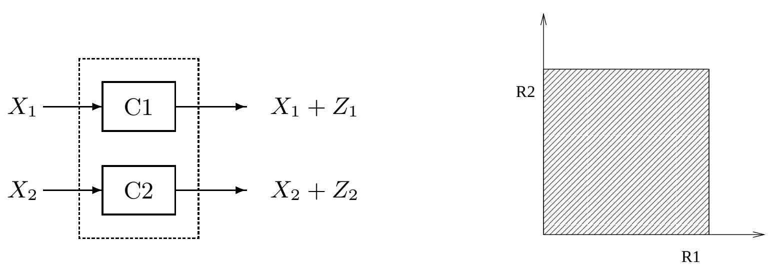


Figure 3: Parallel MA Channel

Binary Symmetric: $Y = X_1 + X_2 + Z(mod2), X_1, X_2 \in \{0, 1\}, Z \sim Bern(p)$

- Setting $X_2 = 0$ achieves $R_1 = 1 - H(p)$
- Setting $X_1 = 0$ achieves $R_2 = 1 - H(P)$
- Time sharing between these two points gives a straight line $R_1 + R_2 = 1 - H(p)$.

Binary Erasure MA Channel: $Y = X_1 + X_2$

The binary erasure MA channel (first plot in Figure 5) adds its two inputs.
First, note:

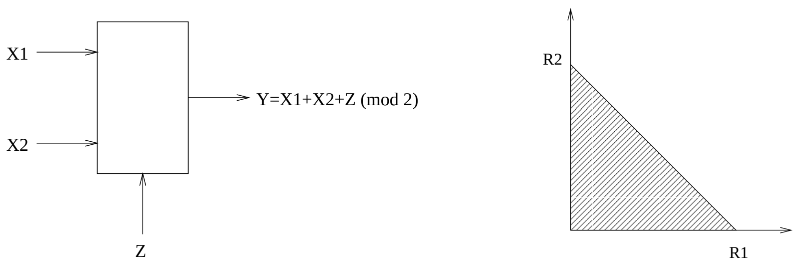


Figure 4: Binary Symmetric MA Channel

- Set $X_2 = 0 \Rightarrow$ noiseless channel with rate $R_1 \leq 1$.
- Set $X_1 = 0 \Rightarrow R_2 \leq 1$
- Time sharing gives a triangular shaped capacity region as in the symmetric channel $Y = X_1 + X_2 + Z(\text{mod } 2)$ case.

Can we do better?

The answer is yes:

- Assume $R_1 = 1$, ie., X_1 is always transmitted reliably.
- Decode X_2, regarding X_1 as noise (second plot in Figure 5), $X_1 \sim$Bern$(\frac{1}{2})$.
- The MA channel looks like a BEC for X_2 (last plot in Figure 5), $R_2 = \frac{1}{2}$.

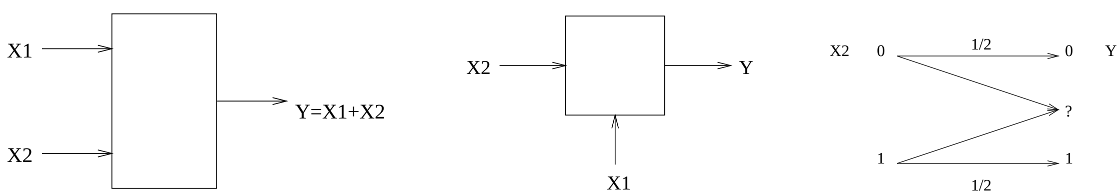


Figure 5: Binary Erasure MA Channel

$\therefore (1, 0), (1, \frac{1}{2}), (\frac{1}{2}, 1), (0, 1)$ are achievable rate pairs.

Time sharing then gives the following achievable rate region:

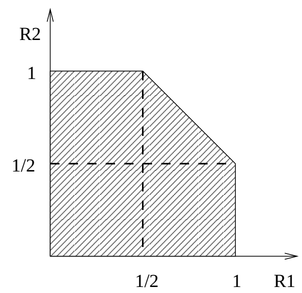


Figure 6: Achievable rate region of Binary Erasure MA Channel

Multiple Access Gaussian Channel

- $var(X_1) \leq P_1, var(X_2) \leq P_2, Z \sim \mathcal{N}(0, \sigma^2)$

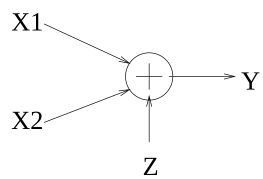


Figure 7: Multiple Access Gaussian Channel

- Set $X_2 = 0 \Rightarrow 0 \leq R_1 \leq \frac{1}{2}\ln(1 + \frac{P_1}{\sigma^2})$
- Set $X_1 = 0 \Rightarrow 0 \leq R_2 \leq \frac{1}{2}\ln(1 + \frac{P_2}{\sigma^2})$
- Decode one input regarding the other as noise $\Rightarrow R_1 + R_2 \leq \frac{1}{2}\ln(1 + \frac{P_1+P_2}{\sigma^2})$

The achievable region of a multiple access gaussian channel has the general shape same as Figure 6, except the vertices on the R_1, R_2 axis are located at $\left(0, \frac{1}{2}\ln(1 + \frac{P_2}{\sigma^2})\right), \left(\frac{1}{2}\ln(1 + \frac{P_1}{\sigma^2}), 0\right)$, and the slanted boundary line is $R_1 + R_2 \leq \frac{1}{2}\ln(1 + \frac{P_1+P_2}{\sigma^2})$.It can also be shown that instead of time-sharing, frequency division multiplexing can achieve the following capacity region:

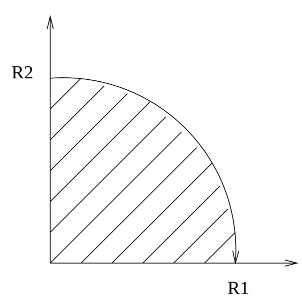


Figure 8: MA Gaussian Channel: Rate pairs achieved by FDM

Achievable Rate Pairs

For a multiple access channel, what does it mean exactly to have a achievable rate pair (R_1, R_2)?

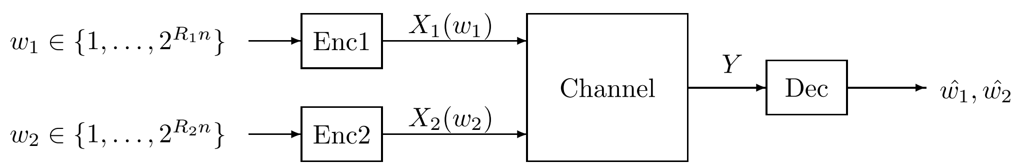
- (R_1, R_2) is achievable if there exist

$$\begin{aligned} \text{Encoding function:} \quad & X_1 : \{1, \dots, 2^{R_1 n}\} \longrightarrow (\Omega_{X_1})^n \\ & X_2 : \{1, \dots, 2^{R_2 n}\} \longrightarrow (\Omega_{X_2})^n \\ \text{Decoding function:} \quad & Y \ \ : (\Omega_Y)^n \longrightarrow \{1, \dots, 2^{R_1 n}\} \times \{1, \dots, 2^{R_2 n}\} \end{aligned}$$

  such that decoding error probability approaches 0 when transmitting the messages w_1, w_2 independently generated (uniformly) on codebooks of size $2^{R_1 n}$ and $2^{R_2 n}$:

$$w_1 \in \text{uniformaly on } \{1, \dots, 2^{R_1 n}\} \qquad w_2 \in \text{uniformaly on } \{1, \dots, 2^{R_2 n}\}$$

- As an illustration:



  If $(\hat{w}_1, \hat{w}_2) = (w_1, w_2)$ with probability $\to 1$, the rate pair (R_1, R_2) is achievable.

What rate pairs are achievable?

Theorem the rate pair $(\tilde{R}_1, \tilde{R}_2)$ is achievable *iff* it is in the convex hull of points (R_1, R_2) such that there exist independent distributions P_{X_1}, P_{X_2} such that

$$\begin{aligned} 0 \le R_1 \le & I_1 = I(X; Y|W) \\ 0 \le R_2 \le & I_2 = I(W; Y|X) \\ R_1 + R_2 \le & I_3 = I(X, W; Y) \end{aligned}$$