6.441 Transmission of Information	May 2, 2006

Lecture 20

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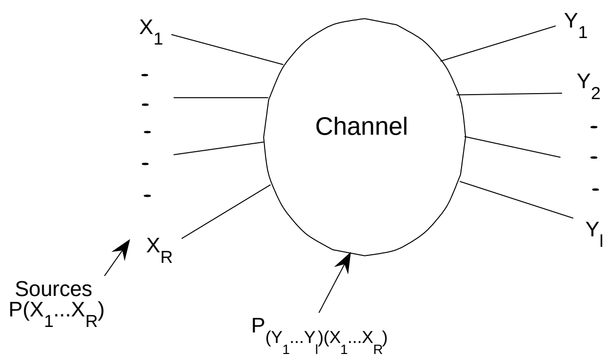
1 Overview

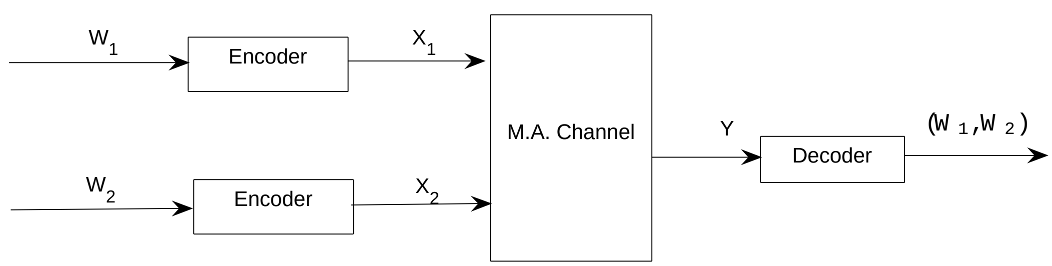
In this lecture, we will continue with the theme of network information theory.

- Correlated-Sources Coding
- Side Information (an aside)
- Broadcast Channel (We ran out of time and this will be covered in next lecture)



Channel is characterized by transition probabilities.
R_{ij} = requested rate from $X_i \to Y_j$

MULTIPLE ACCESS



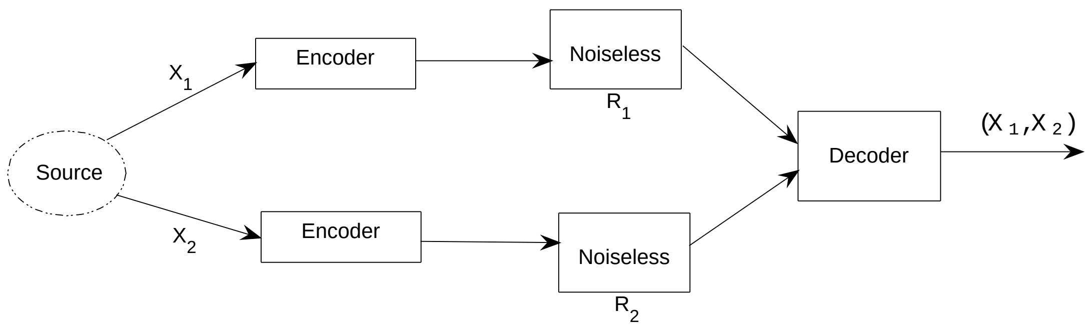
2 Correlated Source Coding

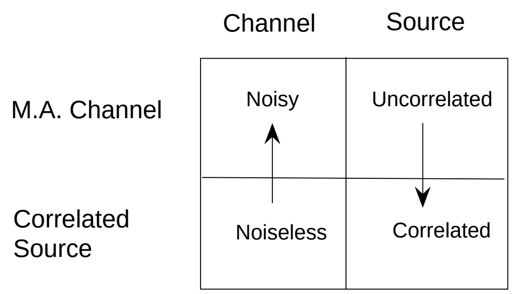
Where the arrows indicate how "hard" either case is (increasing in arrow direction).

Back to the Correlated Source Coding problem:

Definition: (R_1, R_2) achievable if $\exists\ f_1, f_2$

$$\left[\begin{array}{l} f_1 : \Omega_{X_1}^n \to \{1 \dots 2^{nR_1}\} \\ f_2 : \Omega_{X_2}^n \to \{1 \dots 2^{nR_2}\} \end{array}\right.$$





$$g : \{1 \dots 2^{nR_1}\} \times \{1 \dots 2^{nR_2}\} \to \Omega^n_{X_1} \times \Omega^n_{X_2}$$

$(X_1, X_2) \longrightarrow (f_1(x), f_2(x)) \xrightarrow{\text{g}} (\hat{X}_1, \hat{X}_2)$

such that

$P^n_{err} = Pr[(X_1, X_2) \neq (\hat{X}_1, \hat{X}_2)] \underset{\text{n}}{\longrightarrow} 0$

X_1 is a source of entropy $H(X_1) = H_1$.
X_2 is a source of entropy $H(X_2) = H_2$
$I(X_1; X_2) = I$

3 Slepian-Wolf Theorem

(R_1, R_2) achievable iff

$$R_1 \geq H(X_1|X_2) = H_1 - I$$
$$R_2 \geq H(X_2|X_1) = H_2 - I$$
$$R_1 + R_2 \geq H(X_1, X_2) = H_1 + H_2 - I$$

ENCODING

Pick f_1 at random
Pick f_2 at random

Decoding$_{(f_1,f_2)}[Y_1, Y_2]$: if $\exists$ unique $(\hat{X}_1, \hat{X}_2)$ such that :

① $Y_1 = f_1(\hat{X}_1)$, $Y_2 = f_2(\hat{X}_2)$.
AND

② If $(\hat{X}_1, \hat{X}_2)$ are jointly typical then output $(\hat{X}_1, \hat{X}_2)$ else ERROR.

ANALYSIS:
Error Type 1 (X_1, X_2) is not jointly typical: $Pr \longrightarrow 0$ (LLN).

Error Type 2:

ⓐ $\exists \hat{X}_1 \neq X_1$, $\hat{X}_2 \neq X_2$ such that $(\hat{X}_1, \hat{X}_2)$ satisfy ① and ②.
To bound Pr[ⓐ]:
Fix $\hat{X}_1$, $\hat{X}_2$, X_1, X_2 with $\hat{X}_1 \neq X_1$ and $\hat{X}_2 \neq X_2$

$$\Pr_{f_1,f_2}[f_1(\hat{X}_1) = f_1(X_1) \text{and} f_2(\hat{X}_2) = f_2(X_2)] = 2^{-n(R_1+R_2)}$$

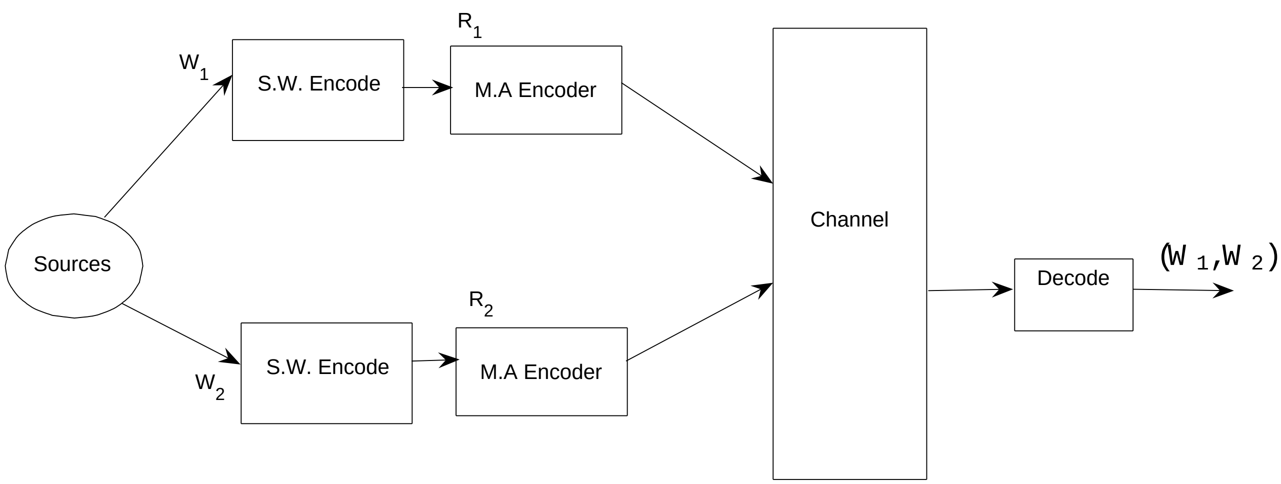
Union bound over $(\hat{X}_1, \hat{X}_2)$ jointly typical. Number of jointly typical $(\hat{X}_1, \hat{X}_2) \leq 2^{n(H(X_1,X_2)+\epsilon)}$.
If $R_1 + R_2 > H(X_1, X_2)$ then Pr[ⓐ]$\longrightarrow 0$.

ⓑ $\exists \hat{X}_1 \neq X_1$ such that $(\hat{X}_1, X_2)$ satisfy ① and ②.
Fix $\hat{X}_1$, X_1 with $\hat{X}_1 \neq X_1$

$$\Pr_{f_1,f_2}[f_1(\hat{X}_1) = f_1(X_1)] = 2^{-n(R_1)}$$

Union bound over $(\hat{X}_1, X_2)$ jointly typical. Number of $\hat{X}_1$ s.t. $(\hat{X}_1, X_2)$ jointly typical $= 2^{nH(X_1|X_2)}$.
Pr$[X_2] \leq 2^{-n(H(X_2)-\epsilon)}$, Pr$[\hat{X}_1, X_2] \geq 2^{-n(H(X_1,X_2)+\epsilon)}$. If $R_1 > H(X_1|X_2)$ then Pr[ⓑ]$\longrightarrow 0$.

ⓒ $\exists \hat{X}_2 \neq X_2$ such that $(X_1, \hat{X}_2)$ satisfy ① and ②. Similarly to ⓑ: If $R_2 > H(X_2|X_1)$ then Pr[ⓒ]$\longrightarrow 0$.



if $\exists\ R_1, R_2$

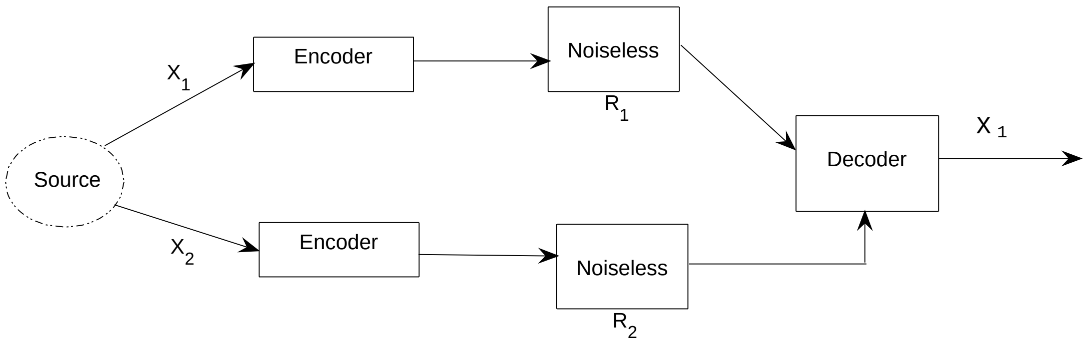
$$R_1 \geq H(W_1|W_2)$$
$$R_2 \geq H(W_2|W_1)$$
$$R_1 + R_2 \geq H(W_1, W_2)$$

and (R_1, R_2) are achievable for MA channel $(R_1, R_2) \in$ convex hull $(\widetilde{R}_1, \widetilde{R}_2)$ s.t. $P_{Y|(X_1,X_2)}$

$$\widetilde{R}_1 \leq I(X_1; Y|X_2)$$
$$\widetilde{R}_2 \leq I(X_2; Y|X_1)$$
$$\widetilde{R}_1 + \widetilde{R}_2 \leq I(Y; (X_1, X_2))$$

the transmission is feasible.

4 Side Information



X_2 ⟶Decoder
Suffices if:

$$R_1 \geq H(X_1|X_2)$$
$$R_2 \geq H(X_2|X_1)$$
$$R_1 + R_2 \geq H(X_1, X_2)$$

For example: $X_1 = Z_1 Z_2$ and $X_2 = Z_2 Z_3$.
Then:

$$R_1 \geq H(X_1|X_2) = H(Z_1)$$
$$R_2 \geq H(X_2|X_1) = H(Z_3)$$
$$R_1 + R_2 \geq H(X_1, X_2) = H(Z_1) + H(Z_2) + H(Z_3)$$

Instead, since we don't care about X_2:

$$R_1 \geq H(Z_1)$$
$$R_2 \geq 0$$
$$R_1 + R_2 \geq H(Z_1) + H(Z_2) = H(X_1)$$

(R_1, R_2) suffices if $\exists\ \hat{X}_2$ s.t. $X_1 \longrightarrow X_2 \longrightarrow \hat{X}_2$
s.t.

$$R_1 \geq H(X_1|\hat{X}_2)$$
$$R_2 \geq H(\hat{X}_2|X_1)$$
$$R_1 + R_2 \geq H(X_1, \hat{X}_2)$$

What we want, or what would be nice to have is:

$$R_2 \geq H(\hat{X}_2|X_1) = 0$$

$$R_1 + R_2 \geq H(X_1, \hat{X}_2) = H(X_1)$$

OR

$$R_1 \geq H(X_1|\hat{X}_2)$$
$$R_2 \geq I(\hat{X}_2; X_1)$$

THEOREM:

Side information problem is realizable for (X_1, X_2) if (R_1, R_2)

if $\exists\ \hat{X}_2$ s.t. $X_1 \longrightarrow X_2 \longrightarrow \hat{X}_2$

$$R_1 \geq H(X_1|\hat{X}_2)$$

$$R_2 \geq I(X_1;\hat{X}_2)$$