1 Overview

In this lecture, we will continue with the theme of network information theory.

- Correlated-Sources Coding
- Side Information (an aside)
- Broadcast Channel (We ran out of time and this will be covered in next lecture)

Channel is characterized by transition probabilities.

\[ R_{ij} = \text{requested rate from } X_i \rightarrow Y_j \]

2 Correlated Source Coding

Where the arrows indicate how “hard” either case is (increasing in arrow direction).

Back to the Correlated Source Coding problem:

Definition: \((R_1, R_2)\) achievable if \(\exists f_1, f_2\)

\[
\begin{align*}
  f_1 : \Omega^n_{X_1} &\rightarrow \{1 \ldots 2^{nR_1}\} \\
  f_2 : \Omega^n_{X_2} &\rightarrow \{1 \ldots 2^{nR_2}\}
\end{align*}
\]
\[ g : \{1 \ldots 2^{nR_1}\} \times \{1 \ldots 2^{nR_2}\} \rightarrow \Omega^n_{X_1} \times \Omega^n_{X_2} \]

\[ (X_1, X_2) \rightarrow (f_1(x), f_2(x)) \xrightarrow{g} (\hat{X}_1, \hat{X}_2) \]

such that

\[ P_{err}^n = \Pr[(X_1, X_2) \neq (\hat{X}_1, \hat{X}_2)] \xrightarrow{n \to \infty} 0 \]

\[ X_1 \text{ is a source of entropy } H(X_1) = H_1. \]

\[ X_2 \text{ is a source of entropy } H(X_2) = H_2. \]

\[ I(X_1; X_2) = I \]

### 3 Slepian-Wolf Theorem

\((R_1, R_2)\) achievable iff

\[ R_1 \geq H(X_1 | X_2) = H_1 - I \]

\[ R_2 \geq H(X_2 | X_1) = H_2 - I \]

\[ R_1 + R_2 \geq H(X_1, X_2) = H_1 + H_2 - I \]

**ENCODING**

Pick \(f_1\) at random

Pick \(f_2\) at random

Decoding \(Y_1, Y_2\): if \(\exists\) unique \((\hat{X}_1, \hat{X}_2)\) such that:

\(1\) \(Y_1 = f_1(\hat{X}_1), Y_2 = f_2(\hat{X}_2)\).

AND

\(2\) If \((\hat{X}_1, \hat{X}_2)\) are jointly typical then output \((\hat{X}_1, \hat{X}_2)\) else ERROR.
ANALYSIS:
Error Type 1 \((X_1, X_2)\) is not jointly typical: \(Pr \rightarrow 0\) (LLN).

Error Type 2:

\(\exists \tilde{X}_1 \neq X_1, \tilde{X}_2 \neq X_2\) such that \((\tilde{X}_1, \tilde{X}_2)\) satisfy 1 and 2.

To bound \(Pr[\text{⑤}]\):
Fix \(\tilde{X}_1, \tilde{X}_2, X_1, X_2\) with \(\tilde{X}_1 \neq X_1\) and \(\tilde{X}_2 \neq X_2\)

\[Pr_{f_1, f_2}[f_1(\tilde{X}_1) = f_1(X_1) \text{ and } f_2(\tilde{X}_2) = f_2(X_2)] = 2^{-n(R_1 + R_2)}\]

Union bound over \((\tilde{X}_1, \tilde{X}_2)\) jointly typical. Number of jointly typical \((\tilde{X}_1, \tilde{X}_2)\) ≤ \(2^{-n(H(X_1, X_2) + \epsilon)}\). If \(R_1 + R_2 > H(X_1, X_2)\) then \(Pr[\text{⑤}] \rightarrow 0\).

\(\exists \tilde{X}_1 \neq X_1\) such that \((\tilde{X}_1, X_2)\) satisfy 1 and 2.

Fix \(\tilde{X}_1, X_1\) with \(\tilde{X}_1 \neq X_1\)

\[Pr_{f_1, f_2}[f_1(\tilde{X}_1) = f_1(X_1)] = 2^{-n(R_1)}\]

Union bound over \((\tilde{X}_1, X_2)\) jointly typical. Number of \(\tilde{X}_1\) s.t. \((\tilde{X}_1, X_2)\) jointly typical = \(2^{-n(H(X_1|X_2) - \epsilon)}\), \(Pr[X_1, X_2] \leq 2^{-n(H(X_1|X_2) - \epsilon)}\). If \(R_1 > H(X_1|X_2)\) then \(Pr[\text{⑤}] \rightarrow 0\).

\(\exists \tilde{X}_2 \neq X_2\) such that \((X_1, \tilde{X}_2)\) satisfy 1 and 2. Similarly to ⑤: If \(R_2 > H(X_2|X_1)\) then \(Pr[\text{⑥}] \rightarrow 0\).

if \(\exists R_1, R_2\)

\[R_1 \geq H(W_1|W_2)\]
\[R_2 \geq H(W_2|W_1)\]
\[R_1 + R_2 \geq H(W_1, W_2)\]

and \((R_1, R_2)\) are achievable for MA channel \((R_1, R_2) \in \text{convex hull } (\tilde{R}_1, \tilde{R}_2)\) s.t. \(P_{Y|(X_1, X_2)}\)

\[\tilde{R}_1 \leq I(X_1; Y|X_2)\]
\[\tilde{R}_2 \leq I(X_2; Y|X_1)\]
\[\tilde{R}_1 + \tilde{R}_2 \leq I(Y; (X_1, X_2))\]

the transmission is feasible.
4 Side Information

X₂ —→ Decoder
Suffices if:

\[ R_1 \geq H(X_1 | X_2) \]
\[ R_2 \geq H(X_2 | X_1) \]
\[ R_1 + R_2 \geq H(X_1, X_2) \]

For example: \( X_1 = Z_1 Z_2 \) and \( X_2 = Z_2 Z_3 \).
Then:

\[ R_1 \geq H(X_1 | X_2) = H(Z_1) \]
\[ R_2 \geq H(X_2 | X_1) = H(Z_3) \]
\[ R_1 + R_2 \geq H(X_1, X_2) = H(Z_1) + H(Z_2) + H(Z_3) \]

Instead, since we don’t care about \( X_2 \):

\[ R_1 \geq H(Z_1) \]
\[ R_2 \geq 0 \]
\[ R_1 + R_2 \geq H(Z_1) + H(Z_2) = H(X_1) \]

\((R_1, R_2)\) suffices if \( \exists \hat{X}_2 \) s.t. \( X_1 \rightarrow X_2 \rightarrow \hat{X}_2 \)

s.t.

\[ R_1 \geq H(X_1 | \hat{X}_2) \]
\[ R_2 \geq H(\hat{X}_2 | X_1) \]
\[ R_1 + R_2 \geq H(X_1, \hat{X}_2) \]

What we want, or what would be nice to have is:

\[ R_2 \geq H(\hat{X}_2 | X_1) = 0 \]
\[ R_1 + R_2 \geq H(X_1, \hat{X}_2) = H(X_1) \]

OR

\[ R_1 \geq H(X_1 | \hat{X}_2) \]
\[ R_2 \geq I(\hat{X}_2; X_1) \]
THEOREM:
Side information problem is realizable for $(X_1, X_2)$ if $(R_1, R_2)$
if $\exists \hat{X}_2$ s.t. $X_1 \rightarrow X_2 \rightarrow \hat{X}_2$

\[
R_1 \geq H(X_1|\hat{X}_2) \\
R_2 \geq I(X_1; \hat{X}_2)
\]