Today

- Bounded depth circuits: $AC^0$
- Parity & $AC^0$ (mod "Switching Lemma")
- Proof of Switching Lemma.

**Warning:** These notes contain only the proof of the lemma;

No statement;

No context!
Furst-Saxe-Sipser Stitching Lemma

- Let $f$ be a DNF formula with $s$ wires, on $n$ inputs.

Let $p$ be a random restriction of $x_1, \ldots, x_n$

\[ x_i \leftarrow 0 \quad \text{w.p.} \quad \frac{1-p}{2} \]
\[ x_i \leftarrow 1 \quad \text{w.p.} \quad \frac{1-p}{2} \]
\[ x_i \leftarrow x_i \quad \text{w.p.} \quad p \]

- Then if $p < -\frac{f(s,n)}{\log s}$

$f_{\leq p}$ is a function of $C$ inputs.
Proof: Do the restriction in 2 stages

First stage: $X_i \leftarrow X_i$ w.p. $\sqrt{p}$.

Second stage: $X_i \leftarrow X_i$ w.p. $\sqrt{p}$.

Claim:

- First Stage: All terms have $\leq C$ variables

- Second Stage: $f$ depends on $\leq C$ variables
Proof of first stage Claim: (relatively easy)

\text{Case 1:} \quad \text{fanout large} \geq 2 \log s

\Pr[\text{output }= 0] \leq \left(\frac{1}{2} + \sqrt{p}\right)^2 \log s

\leq \frac{1}{5^2}

\Pr[\exists \text{ and gate with output } = 0] \leq \frac{1}{5}

\text{Case 2:} \quad \text{fanout} \leq 2 \log s

\Pr[\exists \text{ unrestricted gate}] \leq \left(2 \log s\right) \cdot \left(\frac{1}{\sqrt{p}}\right)
So if \((\sqrt{p}) \ll \frac{1}{s}\) then ---

\[c \sim \text{depends on } S \text{ vs. } \frac{1}{p}\]

if \(s = p^{-a}\) then \(c \approx 2a\).
Proof of Stage 2:

- More complicated: induction on c
  (from first stage)

- Why? Terms from first level overlap
  (share variables)

\[ T_1 \lor T_2 \ldots \lor T_m \]
Case 1: Many disjoint $T_i$'s. \( \Omega(2^{\log s}) \)

\[
\Pr \left[ T_i = 1 \right] \geq \left( \frac{1}{2} - \frac{1}{\sqrt{p}} \right)^c
\]

\[
\geq \frac{1}{3}
\]

\[
\Pr \left[ \forall i: T_i = 1 \right] \leq \left( 1 - \frac{1}{3^c} \right)^{\Omega(3^c \log s)}
\]

\[
\leq \frac{1}{5}
\]

Case 2: (The hard, inductive, case)

Less than $3^c \log s$ disjoint $T_i$'s

dot $T_1, \ldots, T_k$ be maximal disjoint set.

Let \( H = \bigcup_{i=1}^k T_i \)
$[n] = H \cup X$

$[n]_p = H' \cup X'$

Want to show: $f|_p (H' \cup X')$ depends on $b_c$ variables

**Step 1:** $\mid H' \mid$ is small (as in first stage)

**Step 2:** A assignment $\rho'$ to $H'$

$f|_{\rho' \cup \rho'} (x')$ depends on few variables

Induction: every term depends on only $c - 1$ variables

$\Rightarrow$ depends on $\leq b_{c-1}$ variables
Step 3: \( f(p_{uv}, x') \) depends on variables.

\[ m_{bc} = C' + 2b_{c-1} \]