Today: Communication Complexity

- Definitions
- Connection to Circuit Depth
- Basic Properties
- Parity Lower Bound

Setting: Alice & Bob separated by some distance; A have two portions of the input to a problem they wish to solve.

Alice \leftarrow x

Bob \leftarrow y

wish to compute \[ z \] s.t. \( (x, y, z) \in \mathbb{R} \)
Protocol \( \Pi \) computes \( R \) if

- \( \Pi \) specifies who will speak next as a function of \( b_1, \ldots, b_k \).
- \( \Pi \) specifies what will be said \( f(x, b_1, \ldots, b_k) \) if Alice ...
- \( \Pi \) specifies when to stop (for \( b_1, \ldots, b_k \)).
- \( \Pi \) specifies \( Z = f(x, b_1, \ldots, b_k) = g(y, b_1, \ldots, b_k) \).

\( (x, y, z) \in R \)
Complexity:

- $CC(R) = \min_{\pi} \max_{n, y} \{ \# \text{ bits exchanged} \}$

   for function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

- $CC(f) = CC((x, y, f(x, y)))$

   for "partial" function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1, ?\}$

- $CC(f) = CC(R)$ where $(x, y, 1) \in R$

   if $f(x, y) \in \{1, 0\}$

   $(x, y, 0) \in R$

   if $f(x, y) \in \{1, ?, \}$
**CC vs. Circuit Complexity**

For function $f : \{0, 1\}^n \rightarrow \{0, 1\}$

Let $R_f = \{(x, y, i) \mid f(x) = f(y)$

or $(f(x) \neq f(y)$

and $x_i \neq y_i)$ \}$

Theorem [KARCHMER - WINDERSON]

For every $f$

$$CC(R_f) = \text{Circuit-Depth}(f) + \Theta(1)$$

\[\uparrow\]

Over $\{\text{NOR, 2-AND, 2-OR}\}$. 

Proof

\[ \text{CC}(R_f) \leq \text{Depth}(f) \]

\[ f(x) = 1 \]

\[ \text{OR} \]

\[ f_0 \]

\[ f_1 \]

\[ f_0(y) = 1 \quad f_1(y) = 0 \]

\[ \text{OR} \quad f_q(y) = 1 \quad f_q(y) = 0 \]

Alice knows which one sends bit to Bob

Now have \( f_0 \) of depth one less
\[ \text{Depth } (f) \leq \text{CC } (R_f) \]

Proof by Induction on partial function \( f \):
Assume Alice has \( x \) s.t. \( f(x) = 1 \)
Bob has \( y \) s.t. \( f(y) = 0 \)
Say Alice goes first and sends
\[ b_i = b_i(x) \]
Let
\[ f_i = f(x) \mid x \mid b_i(x) = 1 \text{ or } f(x) = 0 \]
\[ f_0 = f \mid x \mid b_i(x) = 0 \text{ or } f(x) = 0 \]

By induction
\[ \triangle C_0 \text{ computes } f_0 \]
\[ \triangle C_1 \text{ computes } f_i \]
Claim:

OR

$\text{computes } f.$

Proof by Picture

$C_0 = 1$

$C_0 = 0$

$b_i = 1$

$f = f_0 = f_0 = 0$

$b_i = 0$

$\varepsilon_0, \varepsilon_1^n$

$f = 1$

$C_1 = 1$

$C_0 = 0$
Lower Bounds Based On $CC(R_f)$.

- Mostly for monotone functions with monotone circuits

- Simple proof that depth (parity) $\geq \log(n^2)$
  \[ \Pi \]
  formula size (⊕) $\geq n^2$

- Proof: Pick random $x$ s.t. $⊕(x) = 0$
  Pick random $i \in [n]$
  $x \rightarrow$ Alice

  $y = x \oplus i \rightarrow$ Bob

  By PHP: Bob must receive $\geq \log n$
  bits from Alice

  Alice must also $\ldots$

  total $\geq 2 \log n \cdot = \log(n^2)$
GENERAL RESULTS IN CC

• Not well-developed for relations, partial functions.

• Functions are quite well-understood \([\text{Yao, Kushilevitz-Naor}]\)

LOWER BOUNDS BY TILING

• Fix protocol \(T\) & bits exchanged

\[
\overline{b} = 0110\ldots 0
\]

• What does the set

\[
\mathcal{S}_b = \left\{(x,y) \mid \Pi(x,y) = \overline{b}\right\}
\]

look like?
• Claim: \( \exists S_A \subseteq \{0,1\}^n \text{ and } S_B \subseteq \{0,1\}^n \)

\[ S = S_A \times S_B \]

• Proof: \( S_A \) = projection of \( S_5 \) to 1st coord.

\[ S_B = \ldots 2^{nd} \ldots \]

\[ \forall (x,y) \in S_A \times S_B \]

\[ Tr(x,y) = 1 \quad : \]

Why? Suppose \( (x,y') \in S_5 \)

\( (x, y) \in S_5 \)

then for all Alice knows, Bob might have \( y' \ldots \) so should be consistent with \( 1 \) ...
Conclusion: Let

$$N_0 = \min \# \text{ "rectangle" needed to tile } f^{-1}(0)$$

$$N_i = \text{ " } f^{-1}(r)$$

Thus $$CC(f) \geq \log N_0, \log N_i.$$ 

Example: $$EQ? (x, y) = 1 \text{ if } x = y$$

$$= 0 \quad \text{ otherwise}.$$ 

$$f\text{-Matrix} \quad -y \rightarrow$$

$$\begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{array}$$

$$N_i = 2^n \Rightarrow CC(f) \geq n$$
**Rank Lower Bound**

**Theorem:** Let $M_f = -y \rightarrow f(x, y)$. Then $\text{CC}(f) \geq \log \text{rank}(M_f)$

**Proof:** Follows from $\text{rank}(M_f) \leq N_f$

$N_f$: Express $M_f = \leq \text{rank one matrices}$

$\text{rank} (A+B) \leq \text{rank}(A) + \text{rank}(B)$. (Which field? Any one ---- or Rationals ----.)
Example

Inner Product Function:

\[ \langle x, y \rangle = \sum x_i y_i \pmod{2} \]

\[ \text{Rank} \left( M_f \right) = ? \]

Let's use fundamental trick of Complexity

\[ M_f \rightarrow N_f \]

0 \rightarrow 1

1 \rightarrow -1

\[ M_f \rightarrow J - 2M_f \uparrow \]

all 1's matrix; rank (1)
Claim: \( \text{Rank}(N_f) = 2^n \)

Proof: \( N_f \cdot N_f^T = 2^n \cdot I \)

\[ \text{rank} = 2^n \]

\[ \text{rank}(AB) \leq \text{rank}(A) \cdot \text{rank}(B) \]

\[ \Rightarrow \text{rank}(N_f) \geq 2^n \]

Claim: \( \text{Rank}(M_f) \geq 2^n - 1 \)

Proof: \( N_f = J - 2M_f \)

\[ \Rightarrow \text{rank}(N_f) \leq \text{rank}(J) + \text{rank}(M_f) \]

\[ \Rightarrow \text{rank}(M_f) \geq 2^n - 1 \]

Note: Proof doesn't work over \( GF(2) \)?
Other Interesting Topics in Communication Complexity

- Multiparty Communication:
  
  \[ \begin{array}{l}
  \text{[Chandra, Furst, Lipton; } \\
  \text{Babai, Nisan, Szegedy]} \\
  \end{array} \]

  \( k \)-players; \( k \) inputs \( x_1, \ldots, x_k \in \{0,1\}^n \)

  Catch: \( P_i \) has all inputs except \( x_i \).

  Compute: \( f(x_1, \ldots, x_k) \)

  Example Result (without proof):

  \[ \text{GIP} (x_1, \ldots, x_k) = \sum_{j=1}^{n} \prod_{i=1}^{k} (x_{i,j}) \pmod{2} \]

  \[ \text{CC (GIP)} \geq \frac{n}{f(k)} \ldots \]
Probabilistic Communication Complexity

\[ (x, y) \]

\[ A \quad \rightarrow \quad B \]

Independently \[ \mathcal{R} \]

Share a common ind. random string

\[ P_{x,y} = \text{Prob} \left[ \bigwedge_{r \in \mathcal{R}} (x, y) \text{ outputs } 1 \right] \]

Want: \[ \forall x, y \]

\[ |f(x, y) - P_{x,y}| \leq \frac{1}{2} - \epsilon \]

\[ CC_\epsilon(f) \]
Example:

Set disjointness $\text{Diss}(x, y) = 1$

$(\Rightarrow)$ if $\not\exists i : x_i \land y_i = 0$

$\forall x \in \{0, 1\}^n \quad \text{CC}_e(\text{Diss}) \geq \Omega(n)$.

Example

$\text{CC}_e(\text{EQ}?) = O(1)$.

Proof: Pick error amending code

$E : \{0, 1\}^n \to \{0, 1\}^{n^2}$

with $\Pr_i [E(x)_i = E(y)_i] \leq 0.9$

$\text{TR}_R : A$ sends $E(x)_R$ to $B$. 
Open Questions

- Obvious ones due to $\text{CC}(R_f)$.

- log-rank conjecture

$\forall f \quad \text{CC}(t) \leq \text{poly} \left( \log(\text{rank}(M_t)) \right)$

- Direct Sum Questions:

How does $\text{CC}(f_1 + f_2)$ depend on $f_1$, $f_2$?

$\left[ f_1 + f_2 \rightarrow \text{compute} \quad f_1(x, y_1) \right.$

$\left. \land f_2(x, y_2) \right]$