LECTURE 08

Note Title 3/5/2007

TODAY:
- ALTERNATION vs. TIME vs. SPACE
- FORTNOW'S THEOREM:

$$\text{SAT} \notin \bigcup_c \text{TIME}(n \log^c n)$$

$$\text{OR} \quad \text{SAT} \notin L$$

FIRST: A CLARIFICATION

- What is TIME$(f(n))$?
- ANSWER: Problems solvable by Deterministic TM with finite # tapes/heads in time $f(n)$.

$$\text{finite} \neq 1$$

- So PALINDROME $\in$ TIME$(O(n))$!

(while problem 6 in pset 2 expects you to show that 1-tape TM can't solve problem in time n^2)

- On problem 1 of pset 1: Everyone assumed 1 tape & showed/asserted

$$\mathrm{TIME}(o(f(n))) \subsetneq \mathrm{TIME}(f(n) \log f(n))$$

- Claim 1: for 1 tape m/c don't need $\log f(n)$ extra time

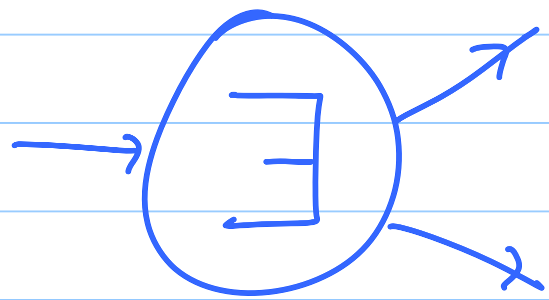
EXTRA CREDIT EXERCISE

- Claim 2: Above assertion holds for multitape m/c. Check this

READING EXERCISE

ALTERNATION :

ATM : Consists of a Turing Machine with 2 special states



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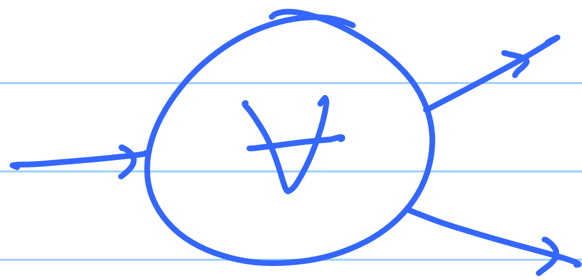
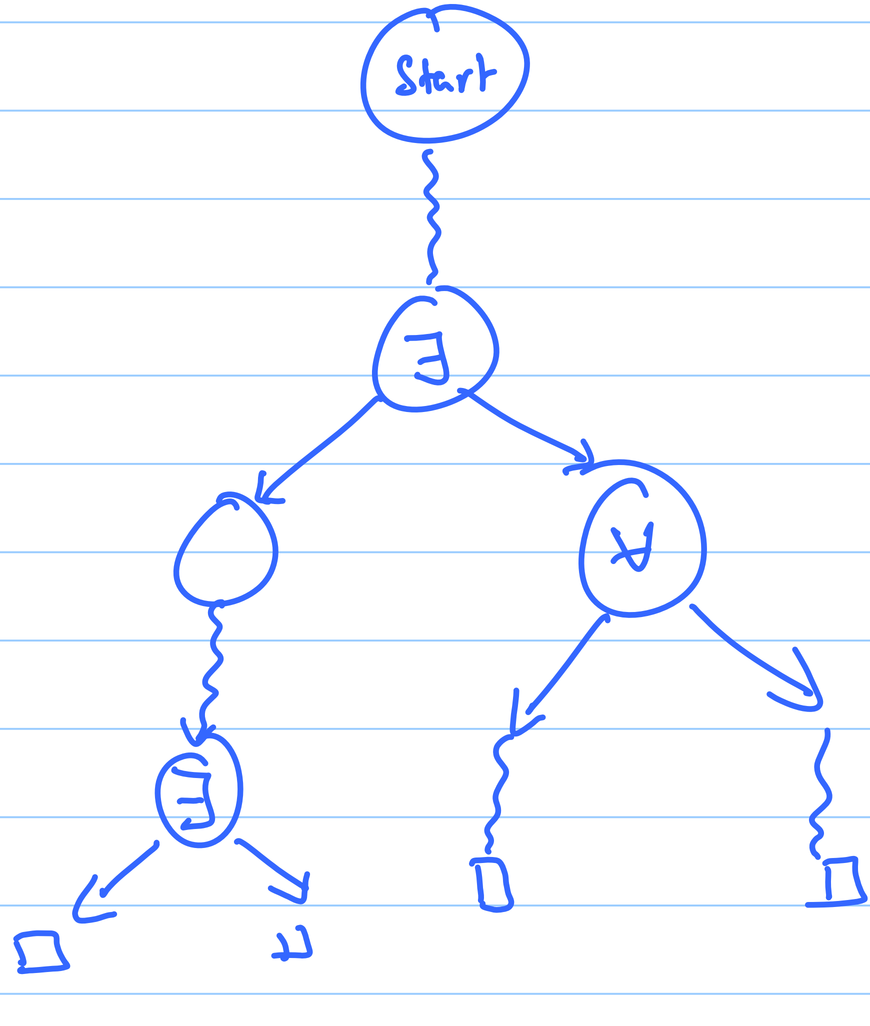


- Machine entering $\exists$ states accepts if one of the outgoing paths accepts.
- Machine entering $\forall$ state accepts if both outgoing paths accept.

(Natural extension of non-determinism)

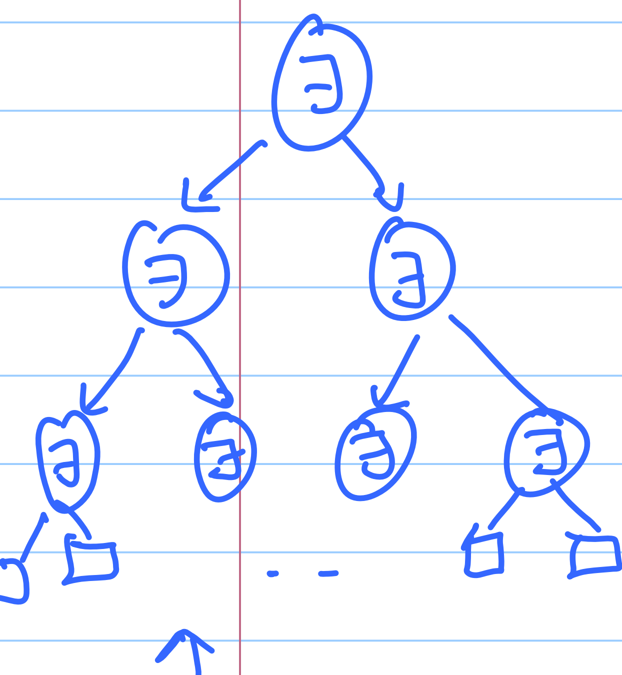
Computation of ATM

Represented by huge tree, with binary branching for $\exists$ & $\forall$ nodes.

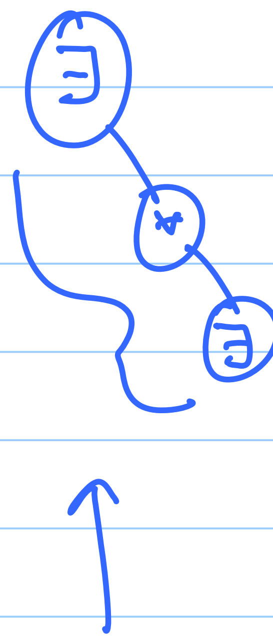


Resources

- Time ≡ Depth of tree
- Space ≡ as usual
- Alternations ≡ # switches from ∃ state to ∀ state in tree (max over paths) +1.



1 alternation



3 alternations.

Why study alternations ?

1. Models interesting problems

$$\text{MINDNF} = \{ (\phi, k) \mid \exists \psi \quad |\psi| \le k \text{ and } \forall x \ (\phi(x) = \psi(x)) \}$$

$$\text{TQBF} = \{ \phi \mid \exists x_1 \forall x_2 \ldots Q_n x_n \ \phi(x_1 \ldots x_n) = \text{true} \}.$$

2. Sheds light about TIME & SPACE

[FORTNOW]

Classical relationships

Theorem: $SPACE(S) \subseteq ATIME(S^2) \subseteq SPACE(S^2)$

Is this familiar? Replace ATIME with NSPACE .... !

Proof:

- $ATIME(S^2) \subseteq SPACE(S^2)$: Immediate since $ATIME(S^2)$ given by depth S^2 computation tree & can explore entire tree with space S^2.
- $SPACE(S) \subseteq ATIME(S^2)$

$REACH(C_1, C_2, 2^S)$

$$= \exists \underline{\underline{C_3}} \text{ s.t. } REACH(C_1, C_3, 2^{S-1}) \text{ } \underline{\text{AND}} \text{ } REACH(C_3, C_2, 2^{S-1})$$

takes S space to write ....

Theorem: $TIME(2^{s}) \subseteq ASPACE(O(s)) \subseteq TIME(2^{O(s)})$

Proof: Second containment: # configurations $= 2^{O(s)}$.

- If graph on configurations = DAG then can determine whether start node accepts or not.
- if graph $\neq$ DAG, divide into strongly connected components;

  $\forall$ state in cycle is rejecting;

  $\exists$ state accepts if it leads to accepting state

  ⋮

  etc
- Can determine accept/reject nature of each configuration.

- First containment:

To determine if content of DTM T tableau in cell $(i,j) = \sigma$ can write ATM as follows:

$\text{CELL}(i, j, \sigma)$:

if $\exists\ \sigma_{-1}, \sigma_0, \sigma_1$ s.t.

$$\left\{ \forall b \in \{-1, 0, 1\} \quad \text{CELL}(i-b, j-1, \sigma_i) \right\}$$

AND $\begin{array}{|c|}\hline \sigma_1 \sigma_2 \sigma_3 \\ \hline \sigma_2 \\ \hline \end{array}$ valid for T

FORTNOW'S THEOREM

Idea:

- Assume $SAT \in LIN \triangleq \bigcup_c n \log^c n$
  & $SAT \in L$
- Then alternations are not useful
  (since $\underbrace{\exists \cdot LIN}_{SAT} \subseteq LIN$)
- But for small space computation, alternation is powerful; few alternations can reduce time significantly.
- $SAT \in L \Rightarrow$ all computation is small space.
- So all computation can be speeded up!

DETAILS : Assume $SAT \in SPACE(c \cdot \log n)$

& $SAT \in TIME(n^{1+\epsilon})$

• STEP 1 :

① : $TIME(T(n)) \subseteq SPACE(c \cdot \log T(n))$

(needs a STRONG Cook's Theorem

SAT complete for NTIME (t(n)))

under LIN-reductions)

• STEP 2

② : $SPACE(S) \subseteq ATIME[a, a \cdot 2^{S/a} \cdot S]$

\# alternations   time

① + ② ⇒ ③ :

$$TIME[T(n)] \subseteq ATIME[a, a(T(n))^{c/a} \cdot \log T(n)]$$

STEP 3

④: $\mathrm{ATIME}[a, t] \subseteq \mathrm{TIME}\left[t^{(1+\epsilon)^a}\right]$

(Induction on a; Cook's Theorem)

③ + ④ : ⑤

$$\mathrm{TIME}(T(n)) \subseteq \mathrm{TIME}\left(T(n)^{\frac{c}{a}\cdot(1+\epsilon)^a}\right)$$

contradiction if

$$\frac{c}{a}(1+\epsilon)^a < 1$$

$\Rightarrow$ $\forall\, c < \infty\ \exists\, \epsilon > 0$ s.t.

$\mathrm{SAT} \in \mathrm{SPACE}[c \cdot \log n]$.

$\Rightarrow$ $\mathrm{SAT} \notin \mathrm{TIME}(n^{1+\epsilon})$.