Lecture 09

Note Title 3/7/2007

Today

- More Alternation
- Polynomial Hierarchy
- The "Infinite Hierarchy" Assumption
- Karp Lipton

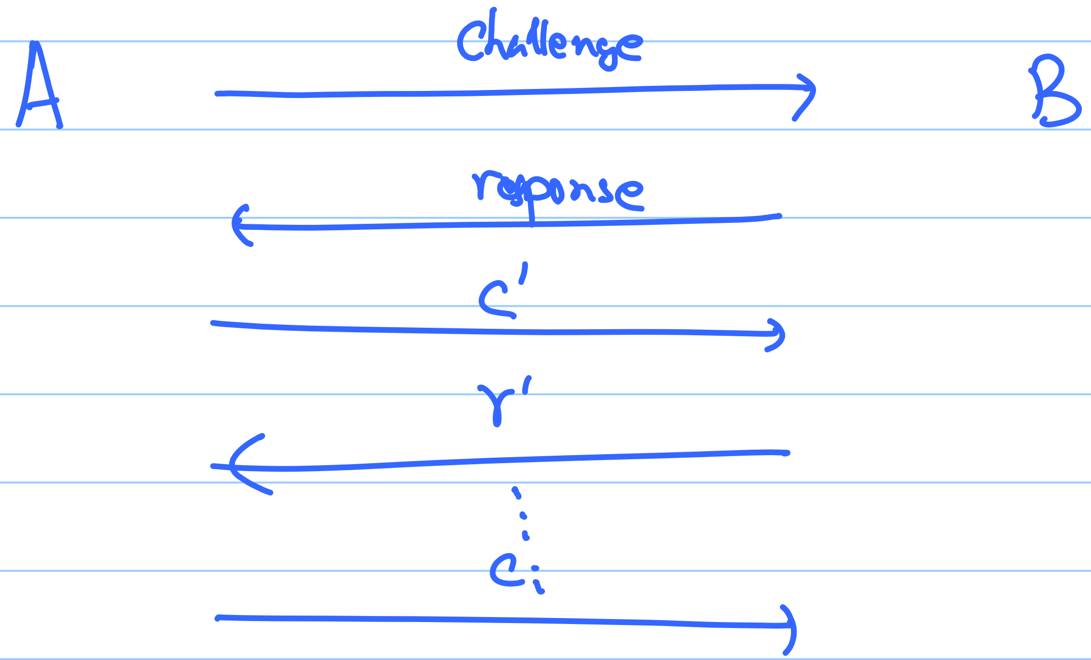
$$\text{I.H.A.} \Rightarrow NP \not\subseteq P/_{poly}.$$

Yesterday: Alternation Gives New Insights On Time + Space.

Today: Alternation Interesting In Itself.

DEBATES & ALTERNATION

Imagine setting up a debate between two players:



A claims X is true

B " " " not true

Question: Why does the listener want to hear this debate?

- Presumably to learn whether X is true or not.

But why does the listener not decide this on his/her own?

- Presumably he lacks computational power to decide; while debaters know more, at least in reference to X.

- But why an exchange? Why not have A, B send their statements to you?
- Presumably, makes a difference

- How to decide # rounds? Should we stop after 2? Who should go first?

  - Believe these make differences too!

How to study formally?

- **Create Model of computation**

  Verifier / Referee : polynomial time alg.

  $$V$$

  $$L = \{X \mid X \text{ is "true"}\}$$

  Completeness : X is true $\Rightarrow \exists a_1 \forall a_2 \ldots Q_k a_k$

  $$V(X, a_1 \ldots a_k) = \text{true}$$

  Soundness : X not true $\Rightarrow \forall a_1 \ldots \overline{Q_k} a_k$

  $$V(x, a_1 \ldots a_k) = \text{false}.$$

Question: How complex an assertion X can be debated like this?

Maybe needs 4 rounds ("Is Lewis Libby Guilty"

0 rounds ("Does raising taxes improve quality of life"

6 rounds ("Who'll make a better president"

— x —

- Does increasing # rounds help?
- Does it matter who speaks first?

— x —

- $\Sigma_i^P = \{ L \mid$ ("$x \in L$") prosecution speaks first & i rounds of debate $\}$
- $\Pi_i^P = \{ L \mid$ ("$x \notin L$") defense speaks first & i rounds of debate $\}$

Equivalently

- $\Sigma_i^P = \{ L \mid$ L decided by ATM in polytime with i alternations starting with $\exists$-quantifier $\}$
- $\Pi_i^P = \{ L \mid$ L decided by ATM in polytime with i alternations starting with $\forall$-quantifier $\}$

(Only subtlety $\supseteq$ work postponed to end).

Belief So Far

- $\forall i \quad \Sigma_i^p \neq \Sigma_{i+1}^p$ ← IHA

Facts

- **Defn:**

$$i\text{-}\exists\text{-TQBF} = \{ 3CNF\ \phi \mid \exists x_1 \in \{0,1\}^n, \forall x_2 \in \{0,1\}^n, \vdots\ Q_i x_i \in \{0,1\}^n\ \phi(x_1 \ldots x_i) = \text{true} \}$$

- $i\text{-}\exists\text{-TQBF}$ is Σ_i^p-Complete
- $\Sigma_i^p = \{ L \mid \bar{L} \in \Pi_i^p \}$
- $\Sigma_i^p = NP^{(i-1)\text{-}\forall\text{-TQBF}}$ [remember relativization?]

- $\Sigma_i^P = \Sigma_{i+1}^P \overset{①}{\iff} \Sigma_i^P = \Pi_i^P$

② $\Rightarrow \Sigma_j^P = \Pi_j^P = \Sigma_i^P \quad \forall\, j \geq i$

Proof:

① $\Longrightarrow \quad \Pi_i^P \subseteq \Sigma_{i+1}^P = \Sigma_i^P$ $\boxed{\checkmark}$

① $\Longleftarrow$ Consider $L \in \Sigma_{i+1}^P$ given by referee V.

Consider question $x \in L$?

i.e. $\exists y_1 \forall y_2 \ldots Q_i y_i \quad V(x, y_1 \ldots y_i)$?

Consider $L' = \{ (x, y_1) \mid \forall y_2 \ldots Q_i y_i$

$V(x, y_1, \ldots y_i)\ ? \}$

$L' \in \Pi_i^P = \Sigma_i^P \Rightarrow \exists V'$ s.t.

$L' = \{ (x, y_1) \mid \exists z_2, \ldots \bar{Q}_i z_i \quad V'(x, y_1, z_2, \ldots z_i)? \}$

$$\Rightarrow L = \{ x \mid \exists (y_1, z_2), \forall z_3 \dots \bar{Q}_i z_i \; V'(x, y, z_2 \dots z_i) \}$$

$L \in \Sigma_i^p$ $\boxed{\times}$

2 $\Longrightarrow$. By induction on $(j-i)$.

. Base case already proven.

. $\Sigma_i^p = \Sigma_{i+1}^p \Rightarrow$

$$\Sigma_{i+2}^p = NP^{\Sigma_{i+1}^p} = NP^{\Sigma_i^p} = \Sigma_{i+1}^p = \Sigma_i^p$$

$\vdots$

$$\Sigma_j^p = \Sigma_i^p$$

$\vdots$

$\boxed{\times}$

" First collapse $\Rightarrow$ total collapse .... "

- Polynomial Hierarchy $= PH = \bigcup_{i \geq 0} \Sigma_i^P$

$= \bigcup_{i \geq 0} \Pi_i^P$

- $\underline{IHA} \Rightarrow P \neq NP$

Infinite Hierarchy Assumption

$\Sigma_0^P \uparrow \quad \Sigma_1^P \uparrow$

But not $\Leftarrow$.

IHA: • Strongest of many assumptions we make;

- But not refuted (so far)
- Might as well try to refute this first....

[Karp-Lipton] Theorem :

$$\text{IHA} \Rightarrow NP \not\subseteq P/_{poly}$$

Motivation: • We tried proving $NP \not\subseteq P/_{poly}$, as a route to proving $NP \neq P$.

• But maybe $NP \neq P$ but $NP \subseteq P/_{poly}$;

• after all $P/_{poly}$ has undecidable problems

• [KL] true, but non-uniformity is not so powerful on its own; unless we really know how to deal with quantifi

Proof :: [Of KL Theorem]:

Recall: Wish to show

$$NP \subseteq P/_{poly} \Rightarrow \Sigma_3^p = \Pi_3^p$$

Idea: • Will try to guess small circuit that solves NP;

• Can verify if C computes SAT in the hierarchy!

• How? $C \equiv SAT$

$$\Leftrightarrow \forall \phi,$$

$$C(\phi)=1 \Rightarrow \exists y \text{ s.t. } \phi(y)=1$$

$$C(\phi)=0 \Rightarrow \forall z \ \phi(z)=0$$

- More crisply:

$C \equiv SAT \iff$

$\forall \phi, z \ \exists y$ s.t.

$$((C(\phi)=1) \text{ and } (\phi(y)=1))$$

$$\text{OR } ((C(\phi)=0) \text{ and } (\phi(z)=0))$$

- $NP \subseteq P/_{poly}$ yields following alg. for satisfiability

$\psi \in SAT \iff \exists C \ \forall \phi, z \ \exists y$ s.t.

$$C(\psi)=1 \ \underline{\text{and}} \left(\begin{array}{l} ((C(\phi)=1) \ \& \ (\phi(y)=1)) \\ \text{OR } ((C(\phi)=0) \ \& \ (\phi(z)=0)) \end{array} \right)$$

- Did we just prove $SAT \in \Sigma_3^P$?

- Additional ideas : • Can use this idea for bottom level of any Σ_i^P / Π_i^P computation;

• Can guess C & verify its correctness in parallel to real computation.

x

Proof: Fix $L \in \Pi_3^P$

$$= \{ \psi \mid \forall x_1 \exists x_2 \forall x_3 \; \psi(x_1, x_2, x_3) = \text{true} \}$$

Let $L' = \{ (\psi, x_1, x_2) \mid \forall x_3 \; \psi(x_1 x_2 x_3) \}$

$L' \in$ co-NP $\subseteq P/_{poly}$

So there exist circuit C deciding L'

parallel

$L = \exists C$

$\forall \phi, z, \ldots, x_1$

$\exists y, \ldots x_2$

$C(\psi, x_1, x_2) = 1$

and $\left[(C(\phi)=1) \text{ and } (\phi(y)=1) \right.$

$\left. \text{or } (C(\phi)=0) \text{ and } (\phi(z)=0) \right]$

Clearly $L \in \Sigma_3^p \Rightarrow \Sigma_3^p = \Pi_3^p$

$\Rightarrow \neg(\text{IHA})$