LECTURE 20

Note Title 4/24/2007

TODAY:

THE PCP THEOREM

(BY GAP AMPLIFICATION [DINUR '05])

Recall: PCP THEOREM (VIEW 2)

Generalized r-Graph k-coloring

Given: $G = (V, E, \text{valid}: E \times [k]^r \to \{0,1\})$

$$E \subseteq \underbrace{V \times V \times \dots V}_{r\text{-times}}$$

Goal: - accept G if G "k-colorable".

- reject G if "$\text{unsat}(G) \geq \epsilon$"

Definitions

- Coloring : $\chi : V \to \{1 \ldots k\}$
- $e = (v_1 \ldots v_r)$ Satisfied by χ if

  $\text{valid}\left(e, \chi(v_1), \ldots, \chi(v_r)\right) = 1.$
- G is k-colorable if $\exists \chi$ s.t.

  $\forall e \in E$, e is satisfied by χ.
- $$\text{Unsat}_\chi(G) = \frac{|\{e \in E \mid e \text{ not satisfied by } \chi\}|}{|E|}$$
- $$\text{Unsat}(G) = \min_\chi \{\text{unsat}_\chi(G)\}$$

Reductions: $(k, r, \epsilon) \to (k', r', \epsilon')$

means $\exists$ linear time reduction T

r-graph $G \longrightarrow r'$-Graph G'

G k colorable $\Longrightarrow$ G' k'-colorable

$\text{unsat}(G) \geq \epsilon \Longrightarrow \text{unsat}(G') \geq \epsilon'$.

Easy Reductions (Complexity TQE):

(i) $(k, r, \epsilon) \longrightarrow (2, r\log k, \epsilon)$

(ii) $(k, r, \epsilon) \longrightarrow \left(k^r, 2, \frac{\epsilon}{r}\right)$ [FRS]

(iii) $(k, 2, \epsilon) \longrightarrow \left(3, 2, \frac{\epsilon}{f(k)}\right)$ [PY] [Petrank]

(iv) $(2, r, \epsilon) \to \left(2, 3, \frac{\epsilon}{r}\right)$ [Cook]

Problem with easy reductions

Gap always reduces.

One neo-classical reduction

(based on expander walks)

(V) $\left(k, r, \epsilon\right) \longrightarrow \left(k, c\cdot r, \sim \underset{1-(1-\epsilon)^c}{\underset{\shortparallel}{\epsilon\cdot c}}\right)$

Still:

$$\frac{\log k \times r}{\epsilon}$$ usually gets worse.

[Dinur]'s Key Insight

I. $(k, 2, \epsilon) \longrightarrow (K, 2, c\cdot\epsilon)$

Quantifiers are important! So ....

$\forall\, c, k,\ \exists\, K$ s.t. $\forall\, \epsilon(\cdot)$

$$(k, 2, \epsilon(\cdot)) \rightarrow (K, 2, c\cdot\epsilon(\cdot))$$

What? Why is this interesting?

"Observation"

II (Monday's lecture $\Longrightarrow$)

$\exists\, \delta\ \ \forall\, K$

$$(K, 2, \epsilon) \longrightarrow (2, 4, \epsilon\cdot\delta)$$

$\mathrm{I} + \mathrm{II} + \text{(ii)} =$

III. $(16, 2, \epsilon) \longrightarrow (16, 2, 2\cdot\epsilon)$

<u>Proof</u>: Let δ be from $\underline{\mathrm{II}}$.

Set $c = \dfrac{8}{\delta}$,

Let K be as in I. $(R=16)$

Then

$$(16, 2, \epsilon) \xrightarrow{\mathrm{I}} (K, 2, \epsilon\cdot c)$$

$$\xrightarrow{\mathrm{II}} (2, 4, \epsilon\cdot c\cdot\delta)$$

$$\xrightarrow{\text{(ii)}} (16, 2, 2\epsilon)$$

$\boxtimes$

Why does Monday's Lecture $\Rightarrow$ II ?

Informally:

- PCP more than just a proof.
- Commitment to a specific proof.
- i.e., not just "$\exists a$ s.t. $C(a)=1$"

  but "here's $\Pi(a) \approx \tilde{\Pi}$ s.t. $\tilde{\Pi}[Q] = Q(a)$

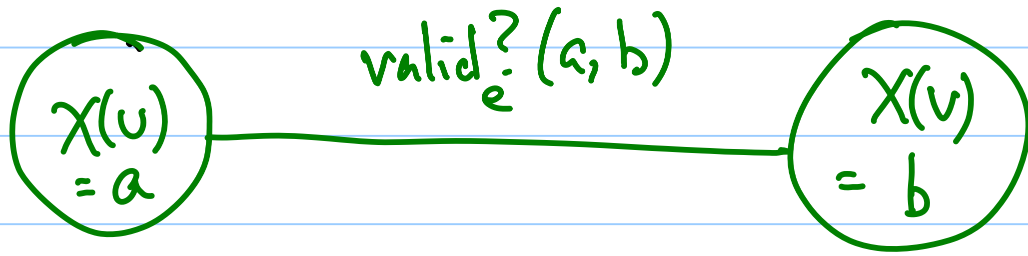
  s.t. $C(a) = 1$"

- Can extend to prove

  "here's $X(u)$, $X(v)$ s.t.

  $\text{valid}\big(X(u), X(v)\big) = 1$"

Formally : PCP Gadget

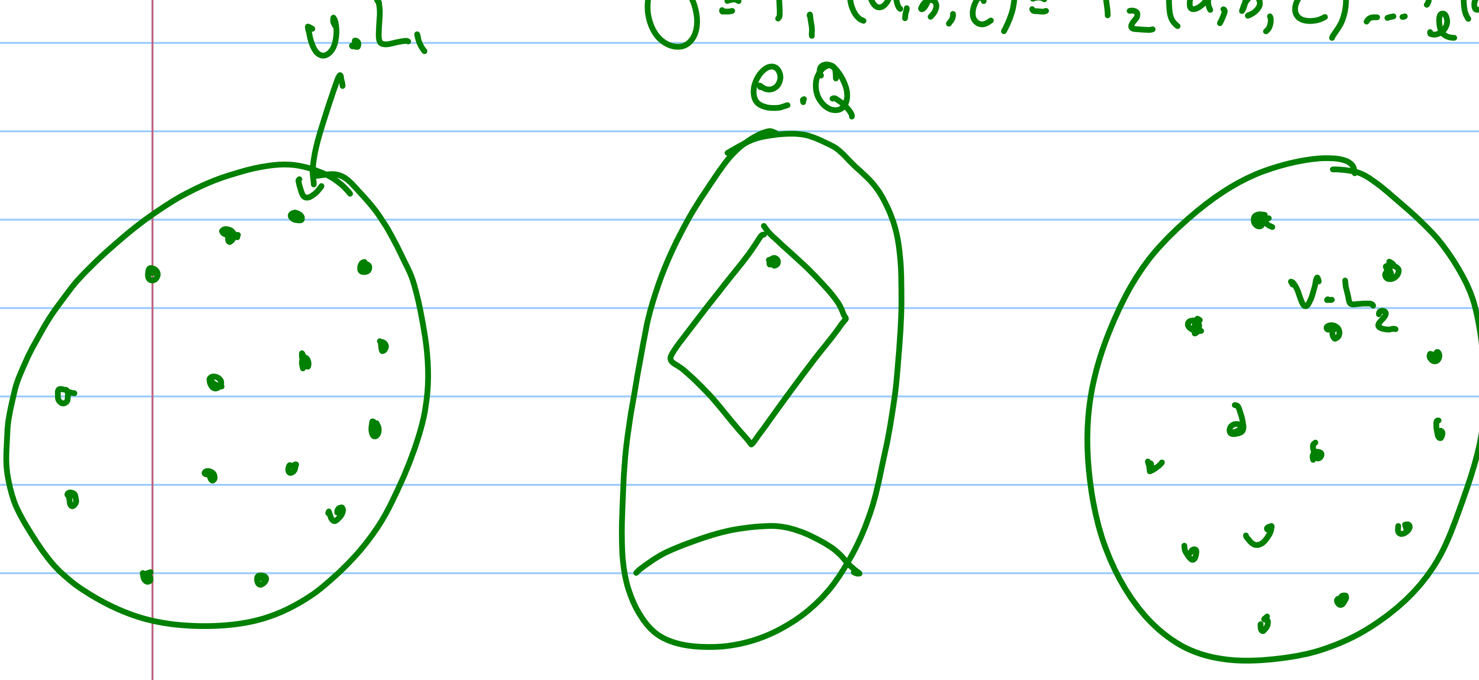


$\exists \underbrace{P_1 \ldots P_\ell}_{\text{degree } 2} (x_1 \ldots x_k, y_1 \ldots y_k, z_1 \ldots z_t)$ s.t.

$\forall \; a = a_1 \ldots a_k \;,\; b = b_1 \ldots b_k$

$\text{valid}_e(a,b) \iff \exists \; c_1 \ldots c_t$ s.t.

$0 = P_1(a,b,c) = P_2(a,b,c) \ldots P_\ell(a,b,c)$



$L_1 = L_1(a)$ linear

$L_2 = L_2(b)$ linear

$Q = Q(a,b,c)$ quadratic

Constraints of type

I, II, III from last lecture

$$\begin{cases} \bullet\ \chi(e\cdot Q_1) + \chi(e\cdot Q_2) = \chi(e\cdot(Q_1+Q_2)) \\ \bullet\ \quad \vdots \\ \bullet \\ \bullet \\ \bullet \end{cases}$$

Additionally

- $\chi(U\cdot L) = \chi(e\cdot Q) + \chi(e\cdot(Q+\tilde{L}))$

$[\tilde{L}(a,b,c) = L(a)]$

- All constraints on 4 Boolean colors.
- if 99% satisfied then
  - $\{X(v.L)\}_L$ uniquely close to $\{L(a)\}_L$
  - $\{X(v.L)\}_L$ uniquely close to $\{L(b)\}_L$
  - $\text{valid}_e(a,b) = 1$.

Yields II with

$$\boxed{\delta = \frac{1}{100}.}$$

Towards I

- How do you amplify errors?
- How do you amplify anything?
  (even (V))
- To get weak amplification with linear reduction [weak ≡ increasing γ not k]:
  - Take random walk on G
  - Take conjunction of constraints on edges.
  - linear time if G is bounded degree.
  - [80's : AKS, CW, IZ]

    amplifies error if G expander.
  - ['88 : PY] Can transform G into bounded degree (expander).

[Dinur]'s Construction

- Fix constant t
- Let $B_v = B_v^t = \{ u \mid \delta(u,v) \le t \text{ in } G \}$

  δ: length of shortest path.

- Reduces $G = (V, E, \text{valid})$

$$\Downarrow$$

$$G' = G'_t = (V', E', \text{valid}')$$

- $V' = V$
- $E' = \{ (u,v) \mid \exists\, \overset{u}{\overset{\shortparallel}{w_1}} \ldots \overset{v}{\overset{\shortparallel}{w_\ell}}$ s.t. $(w_i, w_{i+1}) \in E$, $\frac{t}{2} \le \ell \le t \}$

  E': multiset

- $X' : u \longmapsto \{ X_u : B_u \to \{1..k\} \}$

new coloring of u is a function giving old coloring to B_u (neighborhood of u).

- $\text{Valid}'(X_u, X_v)$

  if • $\forall\ w \in B_u \cap B_v$

  $X_u(w) = X_v(w)$ and

  • $\forall\ (w_1, w_2) \in E, \quad w_1, w_2 \in B_u \cup B_v$

  $\text{Valid}_{w_1,w_2}\left(X_u(w_1), X_v(w_2)\right) = 1$.

- Analysis?
  - Technically simple variant of (V)
  - Conceptually brilliant!

III $\Rightarrow$ PCP Theorem ?

$$\underset{G_1}{\left(16, 2, \tfrac{1}{m}\right)} \Rightarrow \underset{G_2}{\left(16, 2, \tfrac{2}{m}\right)} \Rightarrow \; \vdots \; G_3$$

$$\Rightarrow \underset{G_{\log m}}{(16, 2, \epsilon)}$$

$\Big\}$ $\log m$ times

$$|G_i| = O(|G_{i-1}|) = c \cdot |G_{i-1}|$$

$$|G_{\log m}| = c^{\log m} \, |G|$$

$$= m^{\log c} \, |G| \, .$$

$\boxtimes$

Conclusions

- Full proof of PCP Theorem
  - (i) BLR Theorem [.75 lectures]
  - (ii) Amplification Theorem [1.25 lectures]

- Remember consequences to Inapproximability
  - Given 3CNF formula, hard to find assgmt. satisfying $\left(\frac{7}{8}+\epsilon\right)$ fraction of clauses
    [PCP, . . . , Håstad]

- Given n vertex graph with clique of size $n^{.99}$, can't find one of size $n^{.01}$.

- Given $n^{.01}$-colorable graph can't find $n^{.99}$-coloring.

- .... SET COVER

- MAX CUT
- VERTEX COVER
- Short vectors in lattices/codes
- TSP

⋮

(many important/hard questions)