A Crash Course on Coding Theory

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Complexity in Coding Theory

Fundamental problems of coding theory

- Given a code, find/estimate its parameters.
  - Length - hopefully obvious.
  - Rate - also obvious for linear codes.
  - Distance!
- For a code C, solve the decoding problem.
  - Is the code given/fixed/well-known?
  - Is the distance of C known or not?
  - Is the received word close to C or not?
- Usually restrict to linear codes. Why?

Two basic themes. Many variants.

Hardness of decoding

Maximum likelihood decoding (MLD):

Given: Generator matrix G of linear code C.
Received vector y, error bound e.
Decide: $\exists c \in C$ s.t. $\Delta(c, y) \leq e$?

Thm: [Berlekamp, McEliece, van Tilborg '78]
MLD is NP-hard.

(MLD also called Nearest Codeword Problem in the CS literature.)

Proof

[Modern folklore] Reduction from Max CUT.

Max CUT:

Given: Graph $H = (V, E)$ and integer k.
Decide: $\exists S \subseteq V$ s.t.
edges from S to $\overline{S}$ is at least k?

The reduction:

- Let generator G be incidence matrix of H. ($[m, n, ?]$-code, where $n = |V|$, $m = |E|$).
- Let $r = 1^m$ and $e = m - k$.

Analysis:

- Code = characteristic vectors of cuts.
- Codeword closest to 1^m is the one with largest # of edges crossing cut.

Approximability

- Ok - so can't find nearest codeword.
- Can you even find a nearby codeword?
- Or even approximate the distance to nearest codeword?
- Approximability: General modern day concern.

Nearest Codeword Problem (NCP)

Given: Generator G, received vector y.
Goal: Find codeword $c \in C_G$ nearest to y.

Defn: An α-approximation algorithm to NCP is a polytime algorithm that, on input (G, y), outputs $c' \in C_G$ that satisfies

$$\Delta(c', y) \leq \alpha \Delta(c, y), \quad \forall c \in C_g$$

Theorem: $\forall\ \epsilon > 0$, NCP is not $2^{\log^{1-\epsilon} n}$-approximable, if P $\neq$ NP.

- Theorem combines: [Arora,Babai,Stern,Sweedyk '93] + [Dinur,Kindler,Safra '99].
- Our proof: Uses stronger assumptions Follows [Stern'93].

Proof

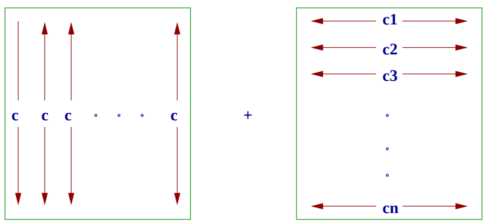
- Starting point: Know Max Cut is hard to approximate to within some $\alpha > 1$.
- Consequence: NCP is hard to approximate to within $\alpha > 1$, if P $\neq$ NP.
- Boosting the gap: Powering construction.

  Given $[n, k, ?]$-code C s.t. $\Delta(1^n, C) = e$.
  Can construct $[n^2, k(n+1), ?]$ code C^2
  s.t. $\Delta(1^{n^2}, C^2) = e^2$
- Conclude: NCP is not α-approximable, for any $\alpha < \infty$, if NP $\neq$ P, and is inapproximable to larger factors under stronger assumptions.

Powering

Codewords of C^2 are $n \times n$ matrices, constructed as follows.

For any collection of codewords $c, c_1, \ldots, c_n$ of C, C^2 contains the codewords c^2 drawn below:



Analysis: To pick codeword of largest weight, pick codeword c of large weight in C and the let $c_i = c$, if $(c)_i = 0$ and $c_i = \vec{0}$ otherwise.

Alternate routes

- [ABSS]+[DKS]: Go deeper into "PCPs" to get the hardness result. ("Tailormade" PCPs.)
- [Hastad]: Celebrated result on inapproximability of Max 3SAT actually goes through the NCP! Yields weaker result, but in many senses cleaner.

Thoughts

- So something is hard! But what?
- If I throw a surprise code at you and ask you to decode, it will be hard! In fact, can even make the code linear, otherwise it will be a lot of effort to throw!
- Not in the usual spirit of problems we talk about.
- Still gives a useful application (inspiration?) — the McEliece cryptosystem.

The McEliece Cryptosystem

Public-Key Cryptosystem, inspired by the hardness of the NCP.

Key generation:
- Pick an $[n, k, d]_q$-AG code C, with t-error-locating pair A, B.
- Pick random permutation $\pi \in \{0,1\}^{n \times n}$
- Pick random non-singular $R \in \mathbb{F}_q^{k \times k}$.

Private Key: (A, B, π, R).

Public Key:
- Let $G \in \mathbb{F}_q^{k \times n}$ generate C.
- Let $G' = RG\pi$. (G' generates C with coordinates permuted by π).
- G' is public key.

McEliece Cryptosystem (contd.)

Encryption: (of message $m \in \mathbb{F}_q^k$)

- Pick $\eta \in \mathbb{F}_q^n$ of weight $\leq \frac{n-k}{2}$
- Let $mG' + \eta$ be its encryption.

Decryption:

- Given r, decode from $r\pi^{-1}$ using (A, B).

Belief: Hard to decode,
without knowledge of π, R.

Many ifs. Will return to this.

Decoding in the 80's

Which code? Distance d vs. Errors e?

Which code?	Known	Input
$e < \frac{d}{2}$	RS BCH AG	?
$e > d$	?	[BMV]

Updates from the 90's

Added new coordinates.

Which code?	Known	Fixed	Input
$e < \frac{d}{2}$	RS BCH AG	?	?
$\frac{d}{2} \leq e < d$	RS BCH AG	?	[DMS]
$e > d$	?	[BN]	[BMV]

Decoding vs. Preprocessing

- Positive results: For specific codes, $\exists$ algorithm that decodes efficiently (up to a limit on # errors).
- Negative results: There exists no algorithm that decodes all linear codes.
- Can we invert the quantifiers? "There exists a code, for which there is no efficient algorithm"?
- [Bruck & Naor '90] addressed this problem.

Decoding with preprocessing

Model:

- Allowed to preprocess the code. Preprocessing is computationally unbounded.
- But should not allow table lookup
- So preprocessing produces polysize circuit that decodes.

Challenge:

- Prev. NP-hardness had no "complexity" in received word - everything in code.
- Now we can't do the same.
- How to transfer the complexity?

Decoding with preprocessing (contd).

Reduction from Max CUT.

The Code:

- Let C_1 be code generated by incidence matrix of clique on n vertices.
- Let C be two-fold repetition of C_1. (Every edge of clique has two coordinates in the code - we call these "twins".)

Received word

- Map H to $r \in \{0,1\}^{n(n-1)}$.
- For every pair of vertices i, j do
  If (i, j) is an edge of H, then
  the twin pairs of r are equal to 1.
  else they are unidentical.

Decoding with preprocessing (contd).

Analysis:

- Codewords identical on twin coordinates.
- If r has different values, that amounts to saying "don't care".

Theorem: There exists a code C s.t. if it has a polynomial sized circuit decoding it, then $\mathrm{NP} = \mathrm{P}/\mathrm{poly}$.

Warning: Does not preserve approximations.

Decoding upto the minimum distance

- Positive results decode up to a certain bound on number of errors; and at least assume $e < d$.
- Negative results don't really mention distance of code!
- "These are certainly linear, but are they error-correcting codes?"
- Considered by [Dumer,Micciancio,S. '99]

Diameter Bounded Decoding (DBD):

Given: Generator G, vector r, integers $e < d$.
Promise: Code has distance at least d.
Decide: Is $\Delta(r, C_G) \leq e$?

Diameter Bounded Decoding

Theorem: DBD is NP-hard under randomized reductions.

Comments:

- Proof adaptation of proof of [Ajtai] (and its simplification due to [Micciancio]) of NP-hardness of Shortest Lattice Vector Problem.
- Proof only uses instances with $\Delta(r, C) \leq e$ or $\Delta(r, C) > d$ and yields $e = d/(2 - \epsilon)$.

Review Ajtai-Micciancio (AM) proof

1. Combinatorial Step: Construct
   - $C = [n_1, k, d]_q$ code.
   - Vector $\vec{x} \in \mathbb{F}_q^{n_1}$ s.t.
     $\forall \vec{c} \in C, \quad \Delta(\vec{x}, \vec{c}) \leq \frac{d}{1.99}$.
2. Starting Point:
   Hard instance of Nearest Codeword Problem [ABSS]
   - $(B, \vec{v}, \leq \frac{d}{100}, > d)$.
   $(B = [n_2, k, d']_q$ code, $\vec{v}$.)
3. Endpoint:
   Paste to get hard instance of decoding.
   - $(C \circ B, \vec{x} \circ \vec{v}, \leq \frac{d}{1.99} + \frac{d}{100}, > d)$
   - $C \circ B$ has distance at least d.

Pasting

1. Strings = simple concatenation.
2. Codes = also concatenation .

$$C \circ B \text{ has matrix } \begin{bmatrix} C \\ - \\ B \end{bmatrix}$$

Codewords of $C \circ B$ are concatenations of codewrds from C and B.

$$[n_1, k, d_1]_q \circ [n_2, k, d_2]_q \Rightarrow [n_1 + n_2, k, d_1 + d_2]_q$$

Combinatorial Step: Details

Not Possible.

1. Combinatorial Step': Construct
   - $C = [n_1, l, d]_q$ code.
   - Vector $\vec{x} \in \mathbb{F}_q^{n_1}$ s.t.
     for many $\vec{c} \in C, \quad \Delta(\vec{x}, \vec{c}) \leq \frac{d}{1.99}$
   - Further construct $A \in \mathbb{F}_q^{k \times n_1}$ s.t. $A(S) = \mathbb{F}_q^k$.
     (where $S = \{\vec{c} \in C \text{ s.t. } \Delta(\vec{c}, \vec{x}) \leq \frac{d}{1.99}\}$)
2. Endpoint': Output
   $(C \circ (BAC), \vec{x} \circ \vec{v}, \leq \frac{d}{1.99} + \frac{d}{100}, > d)$

Analysis

Good Case:

- $\exists \vec{z} \in \mathbb{F}_q^k$ s.t. $\Delta(\vec{v}, B\vec{z}) \leq \frac{d}{100}$.
- $\exists \vec{y} \in \mathbb{F}_q^l$ s.t. $\Delta(\vec{x}, C\vec{y}) \leq \frac{d}{1.99}$ and $AC\vec{y} = \vec{z}$.
- Then $\Delta((C \circ (BAC)) \cdot \vec{y}, \vec{x} \circ \vec{v}) \leq \frac{d}{1.99} + \frac{d}{100}$.

Bad Case: Second part of codewords of $C \circ BAC$ are still codewords of B and hence not close to $\vec{v}$.

The linear transform A

- Recall goal:
  - Have large set $S \subseteq \mathbb{F}_q^n$.
  - Want $A : \mathbb{F}_q^n \to \mathbb{F}_q^k$ s.t. $A(S) = \mathbb{F}_q^k$.
  - Familiar problem in complexity.
  - Most natural idea: Pick $A \in \mathbb{F}_q^{n \times k}$ at random and hope it works.
  - Simple application of Chebycheff shows it works w.h.p. if $|S| \geq q^{2k}$.

Lower bound for list decoding

- Recall new goal: Construct
  - Code C of distance $\geq d$.
  - Vector $\vec{x}$ with exponentially many codewords of C at distance $\frac{d}{1.99}$ from it.
- I.e., want List-Decode$(C, \vec{x}, \frac{d}{1.99})$ to have exponential output size.
- Possible?

How to get $C, \vec{x}$?

- Adleman-Ajtai Lattice: Too number-theoretic.
- Random C and $\vec{x}$: Distance of $\vec{x}$ to C should be same as distance between two distinct vectors of C.
- Random C and carefully chosen $\vec{x}$: Unclear.
- Arbit r and C chosen carefully wrt r: Unclear.

Picking $C, \vec{x}$

- Pick a code C that does better than random code!
  - Example Reed-Solomon Code.
  - Gives $[n, k, n-k]_q$ code, for any $k \leq n \leq q$.
  - Random code weaker. E.g. gives only $[n, n-n^\epsilon, \frac{1}{2-\epsilon}n^\epsilon]_q$ code.
- Pick vector $\vec{x}$ at random.
- Expected number of vectors at distance at most $\frac{1}{2-\epsilon/2}n^\epsilon$ is exponentially large!

Done? Not yet.

An Inverted Markov Inequality

- Positive r.v. Expectation large. Want to sample so that prob. of finding small value is small.
- Markov's bounds r.v. from above!
- Graph-theoretic formulation: Bipartite graph.
  - Left vertices = codewords
  - Right vertices = all vectors (space of $\vec{x}$)
  - Edge between $\vec{c}$ and $\vec{x}$ if they are within distance $\frac{d}{1.99}$.
- Expectation bound $\Rightarrow$ Average right degree large D.

- Sampling lemma [A-M]: Pick random edge and output its right vertex. Then Prob [degree of output $\leq \delta D] \leq 1-\delta$

In our case: Pick random codeword and then introduce $\frac{d}{1.99}$ errors at random. Output this corrupted word.

Summary

Given: instance $(B, \vec{v}, \leq \frac{d}{100}, > d)$ of Nearest Codeword Problem

Pick Reed Solomon code C of distance d and $n \approx d^{100}$.

Pick random vector $\vec{x}$ at distance $\frac{d}{1.99}$ from 0^n.

Output $(C \circ BAC, \vec{x} \circ \vec{v}, \leq \frac{d}{1.99} + \frac{d}{100}, > d)$.

Thm: DBD is NP-hard.

Minimum distance

- So far only focussed on the decoding question.
- What about the Distance of the code?
- Complexity undetermined till late 90's.
- Finally resolved [Vardy '97] - NP-complete indeed.
- Subsequently embellished with inapproximability [DMS '99]. (Reduction from DBD.)

MinDist
Given: Generator G. Task: Find distint codewords $c_1, c_2 \in C_G$ that minimize $\Delta(c_1, c_2)$.

Reducing DBD to MinDist

- Take a hard instance of DBD, i.e., (C, r) s.t. $\Delta(r, C) \leq 2d/3$ or $\Delta(r, C) \geq d$.
- Consider $C' = C + r$.
  Either $\Delta(C') \leq 2d/3$ or $\Delta(C') \geq d$.
- NP-hard to distinguish.

Theorem: MinDist is hard to approximate to within a factor of $3/2$, unless NP $= \mathrm{RP}$.

But can now take tensor products of the code with itself and boost hardness result.

Theorem: MinDist is hard to approximate to within any constant factor, unless NP $= \mathrm{RP}$. (Stronger results possible under stronger assumptions.)

Open Questions

Still - more open than closed!

- Can we certify just the good codes?
- Can we decode just the good codes?
- Show hardness of decoding RS Code?
- Hardness of decoding up to half the minimum distance?
- Hardness of decoding up to the minimum distance for a fixed code.
- Is there a worst-case to average-case connection here?
- Security of the McEliece Cryptosystem (implies all of the above?)

Topics we did not cover

- Convolutional codes. (Also Tree codes, and trellises.)
- Quantum error-correcting codes.
- Cyclic codes.
- Additive codes.
- Stuff that I don't know about.

Acknowledgments

Amin Shokrollahi + Venkatesan Guruswami.