Algebraic Property Testing

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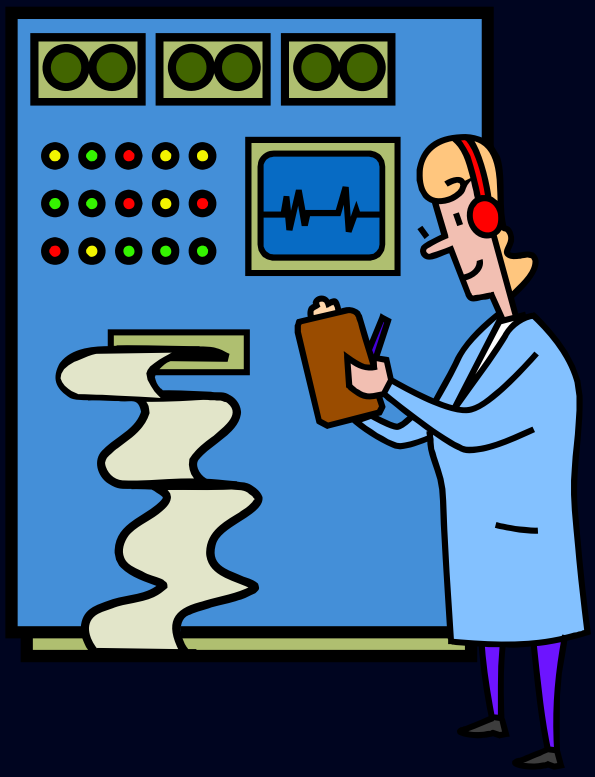
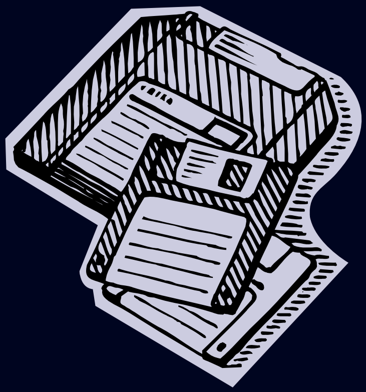
Joint work with Tali Kaufman (IAS).

Classical Data Processing



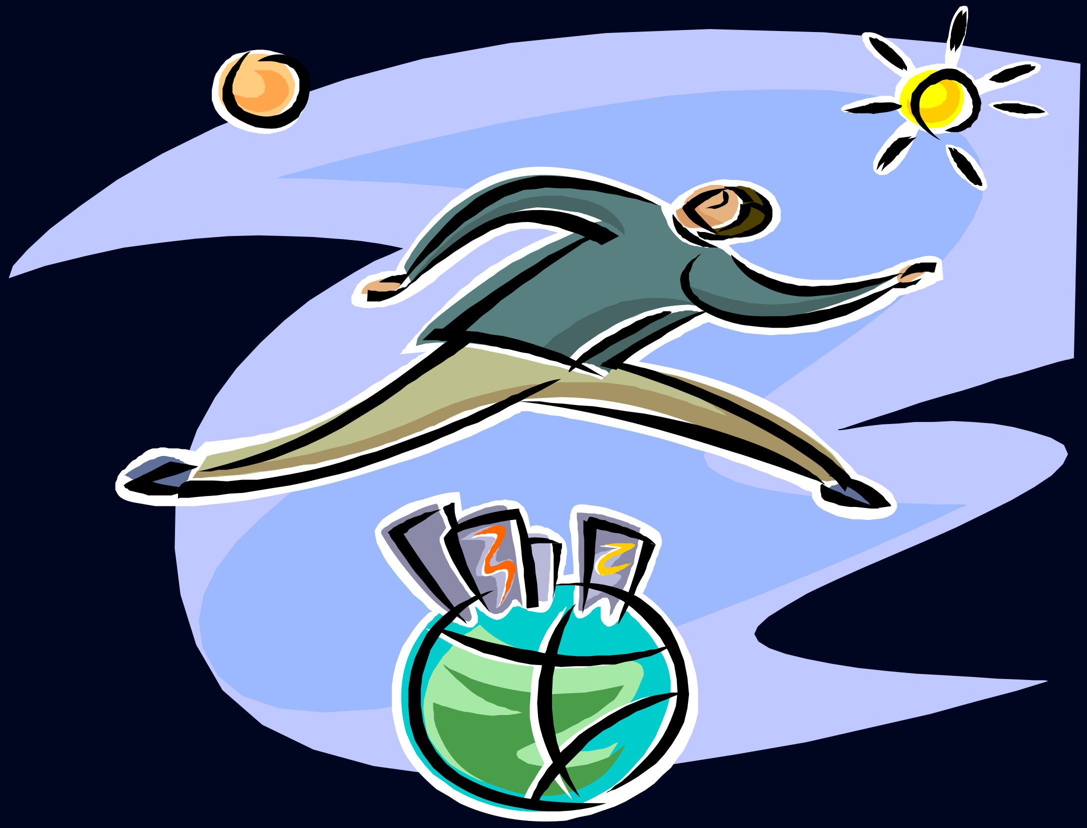
Big computers



Small Data

Modern Data Processing

Small computers



Enormous Data

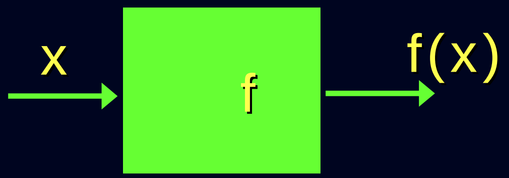
Needs new algorithmic paradigm

- Imbalance not a question of technology :
  - I.e., not because computing speeds are growing less fast than memory capacity.
- Imbalance is a function of expectations:
  - E.g., Users expect to be able to “analyze” the WWW, using a laptop. But WWW includes millions other such laptops.
- Need: Sublinear time algorithms
  - That “estimate” rather than “compute” some given function.

Property Testing

Data: $f : D \to R$

f given by a sampling box



Property: $\mathcal{F} \subseteq \{f : D \to R\}$

q-query Test: Samples f-box q times.

Accepts if $f \in \mathcal{F}$

Hope: ~~Rejects if $f \notin \mathcal{F}$~~ Impossible with $q \ll |D|$

Rejects if f is δ-far from $\mathcal{F}$

f δ-close to $\mathcal{F}$ if $\exists\, g \in \mathcal{F}$ s.t. $\Pr_{x \in D}[f(x) \neq g(x)] \leq \delta$

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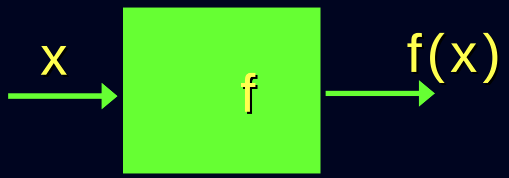

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Example: Linearity Testing

- [Blum, Luby, Rubinfeld '90]

$$D = \mathbb{F}_2^n ; R = \mathbb{F}_2$$

- Property = Linearity

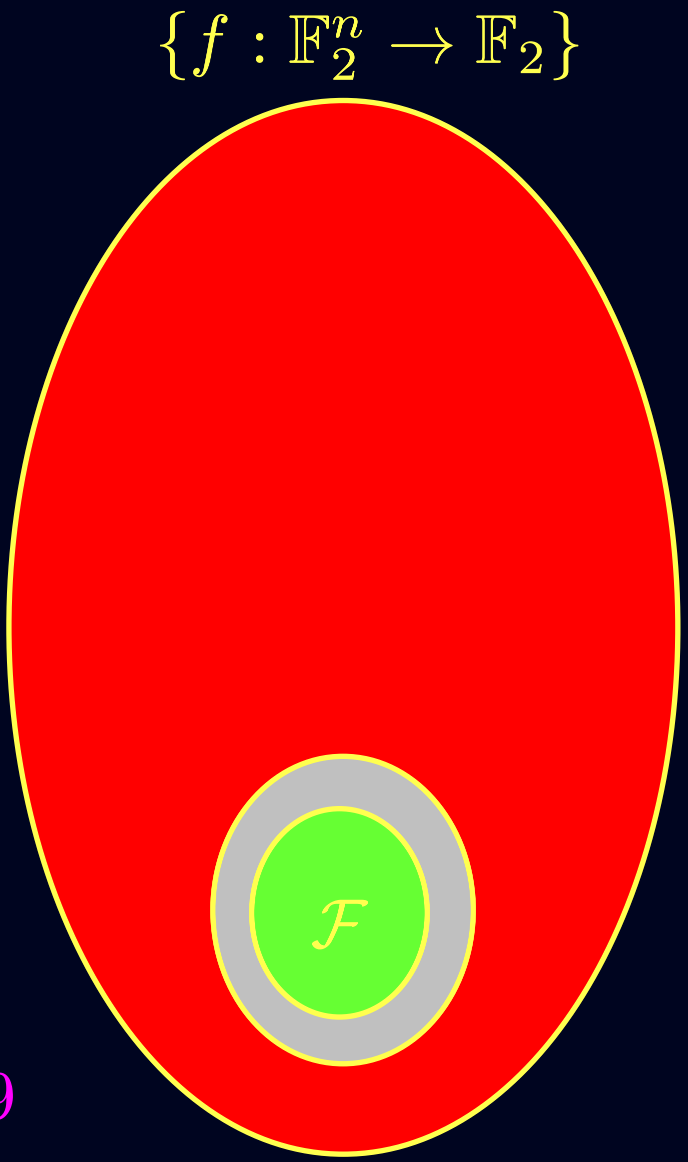
$$\mathcal{F} = \{f | \forall x, y f(x) + f(y) = f(x+y)\}$$

- Test: Pick random x, y

  Accept if $f(x) + f(y) = f(x+y)$

- Non-trivial analysis:

  f δ-far from $\mathcal{F}$ $\Rightarrow$ reject w.p. $2\delta/9$



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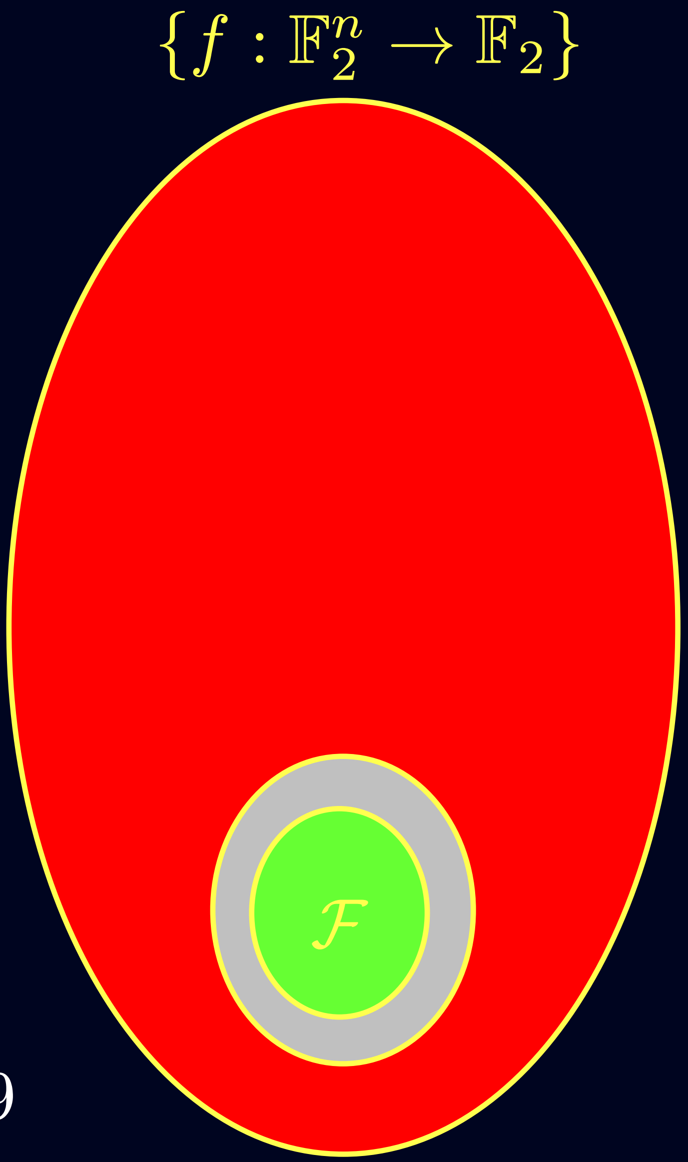

Property Testing: Abbreviated History

- Prehistoric: Statistical sampling
  - E.g., “Is mean/median at least X”.
- Linearity Testing [BLR’90], Multilinearity Testing [Babai, Fortnow, Lund ’91].
- Graph/Combinatorial Property Testing [Goldreich, Goldwasser, Ron ’94].
  - E.g., Is a graph “close” to being 3-colorable.
- Algebraic Testing [GLRSW,RS,FS,AKKLR,KR,JPSZ]
  - Is multivariate function a polynomial (of bounded degree).
- Graph Testing [Alon-Shapira, AFNS, Borgs et al.]
  - Characterizes graph properties that are testable.

This Talk

- Abstracting Algebraic Property Testing

- Generic Theorem:

  If $\mathcal{F} \subseteq \mathbb{F}^n \to \mathbb{F}$ is closed under addition,

  and under affine tranformations of the coordinates,

  and is locally characterized

  then it is testable.

- Motivations:
  - Generalizes, unifies previous algebraic works
  - Initiates systematic study of testability for algebraic properties
  - Sheds light on testing and invariances of properties.

Property Testing vs. “Statistics”

- Classical Statistics (Mean, Median, Quantiles):
  - Also run in sublinear time.
  - So what’s special about “linearity testing”?
- Classical statistics work on “symmetric” properties:

  $\mathcal{F}$ closed under arbitrary permutation on D.
- Linear functions closed under much smaller group of permutations:

  $\mathcal{F}$ closed under linear maps $L : \mathbb{F}_2^n \to \mathbb{F}_2^n$

  $$|D|^{\log |D|} \text{ such maps vs. } |D|!.$$
- Simlarly, graph properties have nice invariances

Is testing a corollary of invariance?

- Example 1: $D = R; \mathcal{F}_1 = \{Id | Id(x) = x\}$
  - Invariant group trivial. Testing easy.
- Example 2: $D = \{1, \ldots, n\}$; $R = \{1, \ldots, n^9\}$;
  $\mathcal{F}_2 = \{f | x < y \Rightarrow f(x) < f(y)\}$
  - Invariant group still trivial. No O(1) local tests.

- Conclusion: Testing not necessarily a consequence of invariance.
  - If we believe this to be the case for the linearity test, must prove it!!

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Part II: Formal Definitions & Results

Linear Invariance

$\mathcal{F}$ is Linear Invariant if

- $\mathcal{F}$ is a linear subspace (of $\mathbb{F}^{|\mathbb{F}|^n}$)
- $f \in \mathcal{F}$ and $L : \mathbb{F}^n \to \mathbb{F}^n$ linear $\Rightarrow f \circ L \in \mathcal{F}$

(Affine Invariance defined similarly)

- Examples:
  - Linear functions,
  - n-variate polynomials of degree $\leq d$,
  - homogenous polynomialsof degree d,
  - $\mathcal{F}_1 + \mathcal{F}_2$

Testing, constraints, characterizations

- Suppose $\mathcal{F}$ has a k-query test.
- Then members of $\mathcal{F}$ satisfy a k-local constraint.

Constraint: $C = (x_1, \ldots, x_k \in \mathbb{F}^n$; subspace $V \subseteq \mathbb{F}^k)$

$\forall\, f \in \mathcal{F}, \quad f$ satisfies C i.e., $\langle f(x_1), \ldots, f(x_k) \rangle \in V$

- E.g., in the linear case: $f(\alpha) + f(\beta) = f(\alpha + \beta)$

$$C = (\alpha, \beta, \alpha + \beta; V = \{000, 011, 101, 110\})$$

Characterization: $\mathcal{C} = \{C_1, \ldots, C_m\}$

$f \in \mathcal{F} \quad \Leftrightarrow f$ satisfies $\mathrm{C}_1, \ldots, C_m$.

(Linear-Invariant) Algebraic Characterizations

- Characterizations require many constraints!
- Linear (affine) invariance turns one constraint into many.

$C = (x_1, \ldots, x_k; V)$ constraint and L linear (affine)
$\Rightarrow C \circ L = (L(x_1), \ldots, L(x_k); V)$ is also a constraint.

- (Linearity) Example:

$(L(\alpha), L(\beta), L(\alpha + \beta); V)$ constraint for linearity

- Algebraic Characterization: Single constraint C s.t.

$\{C \circ L | L$ linear (affine) $\}$ characterize $\mathcal{F}$.

Main Theorems

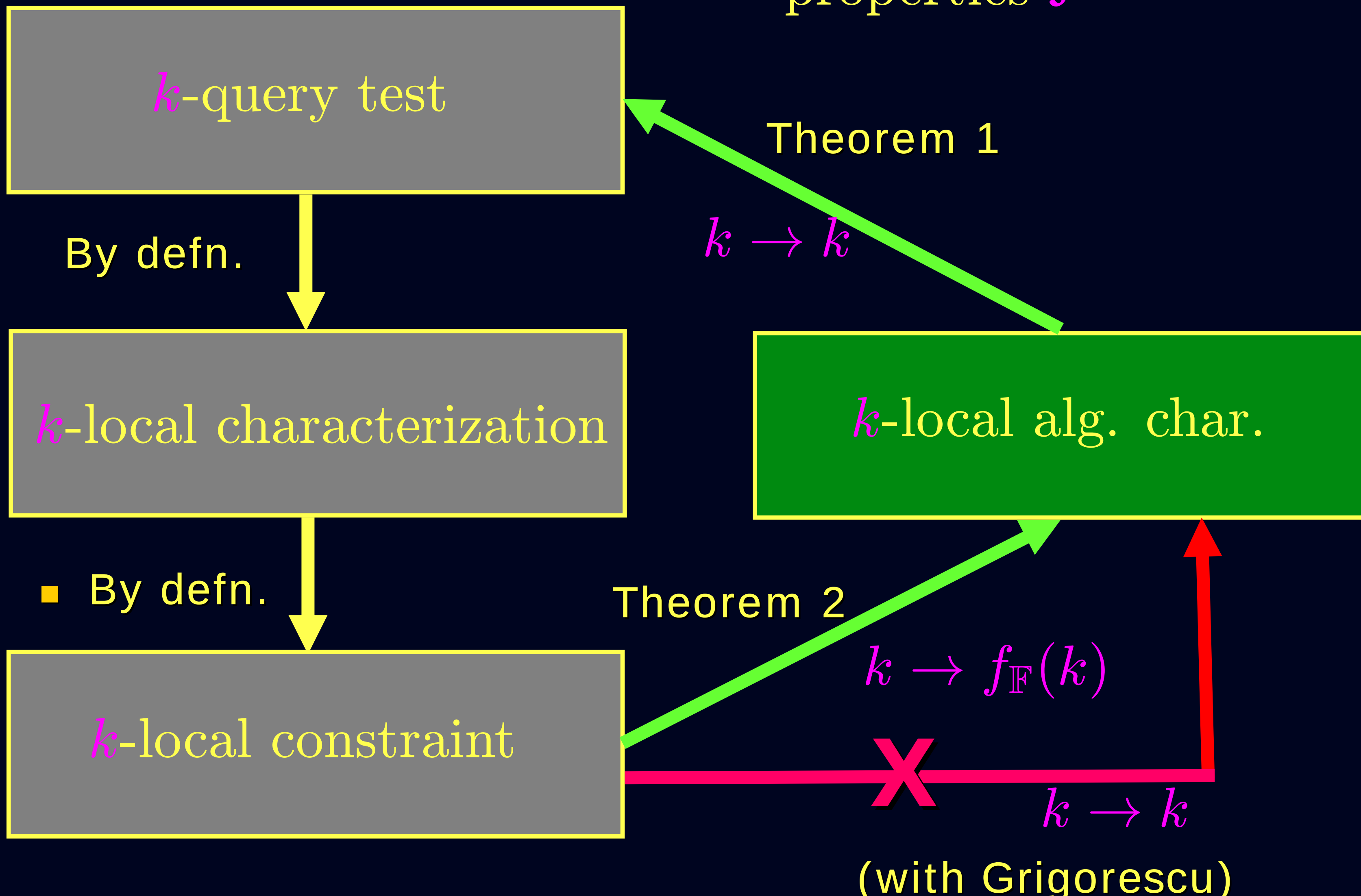
Theorem: $\mathcal{F}$ affine-invariant
and has k-local algebraic characterization,
implies it has a k-query property test.

- Unifies, simplifies, and extends previous algebraic tests

Theorem: $\mathcal{F}$ affine-invariant
and has k-local constraint
$\Rightarrow$ $\mathcal{F}$ has $f_{\mathbb{F}}(k)$-local algebraic characterization.

Pictorially

For affine-invariant properties $\mathcal{F}$



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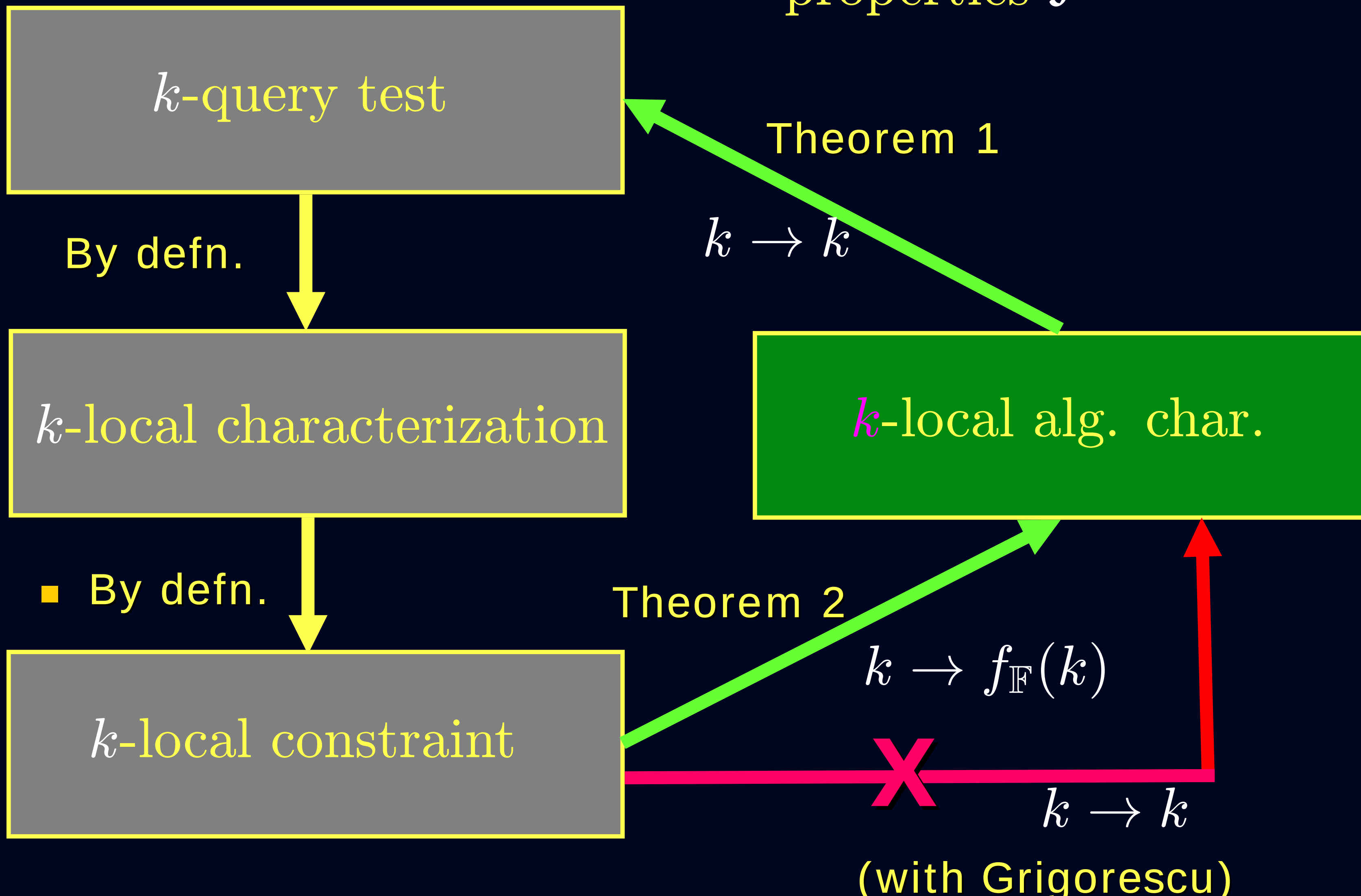

Part III: BLR (and our) analysis

BLR Analysis: Outline

- Have f s.t. $\Pr_{x,y}[f(x) + f(y) \neq f(x+y)] = \delta < 2/9$.
  Want to show f close to some $g \in \mathcal{F}$.
- Define $g(x) =$ most likely $_y\{f(x+y) - f(y)\}$.
- If f close to $\mathcal{F}$ then g will be in $\mathcal{F}$ and close to f.
- But if f not close? g may not even be uniquely defined!
- Steps:
  - Step 0: Prove f close to g
  - Step 1: Prove "most likely" is overwhelming majority.
  - Step 2: Prove that g is in $\mathcal{F}$.

BLR Analysis: Step 0

- Define $g(x) =$ most likely $_y\{f(x+y) - f(y)\}$.

Claim: $\Pr_x[f(x) \neq g(x)] \leq 2\delta$

– Let $B = \{x | \Pr_y[f(x) \neq f(x+y)f(y)] \geq \frac{1}{2}\}$

– $\Pr_{x,y}[$linearity test rejects $|x \in B] \geq \frac{1}{2}$

$\Rightarrow \Pr_x[x \in B] \leq 2\delta$

– If $x \notin B$ then $f(x) = g(x)$

$\mathrm{Vote}_x(y)$

BLR Analysis: Step 1

- Define $g(x) =$ most likely $_y\{f(x+y) - f(y)\}$.
- Suppose for some x, $\exists$ two equally likely values.
  Presumably, only one leads to linear x, so which one?
- If we wish to show g linear,
  then need to rule out this case.

Lemma: $\forall\, x$, $\Pr_{y,z}[\mathrm{Vote}_x(y) \neq \mathrm{Vote}_x(z))] \leq 4\delta$

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?	$f(y)$	$-f(x+y)$
$f(z)$	$f(y+z)$	$-f(y+2z)$
$-f(x+z)$	$-f(2y+z)$	$f(x+2y+2z)$

Prob. Row/column sum non-zero $\leq \delta$.

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Prob. Row/column sum non-zero $\leq \delta$.

BLR Analysis: Step 2 (Similar)

Lemma: If $\delta < \frac{1}{20}$, then $\forall\, x, y,\ g(x) + g(y) = g(x+y)$

$g(x)$	$g(y)$	$-g(x+y)$
$f(z)$	$f(y+z)$	$-f(y+2z)$
$-f(x+z)$	$-f(2y+z)$	$f(x+2y+2z)$

Prob. Row/column sum non-zero $\leq 4\delta$.

Our Analysis: Outline

- f s.t. $\Pr_L[\langle f(L(x_1),\ldots,f(L(x_k))\rangle \in V] = \delta \ll 1$.
- Define $g(x) = \alpha$ that maximizes
  $\Pr_{\{L|L(x_1)=x\}}[\langle \alpha, f(L(x_2)),\ldots,f(L(x_k))\rangle \in V]$

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Same as before

- Steps:
  - Step 0: Prove f close to g
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  - Step 2: Prove that g is in $\mathcal{F}$.

$\mathrm{Vote}_x(L)$

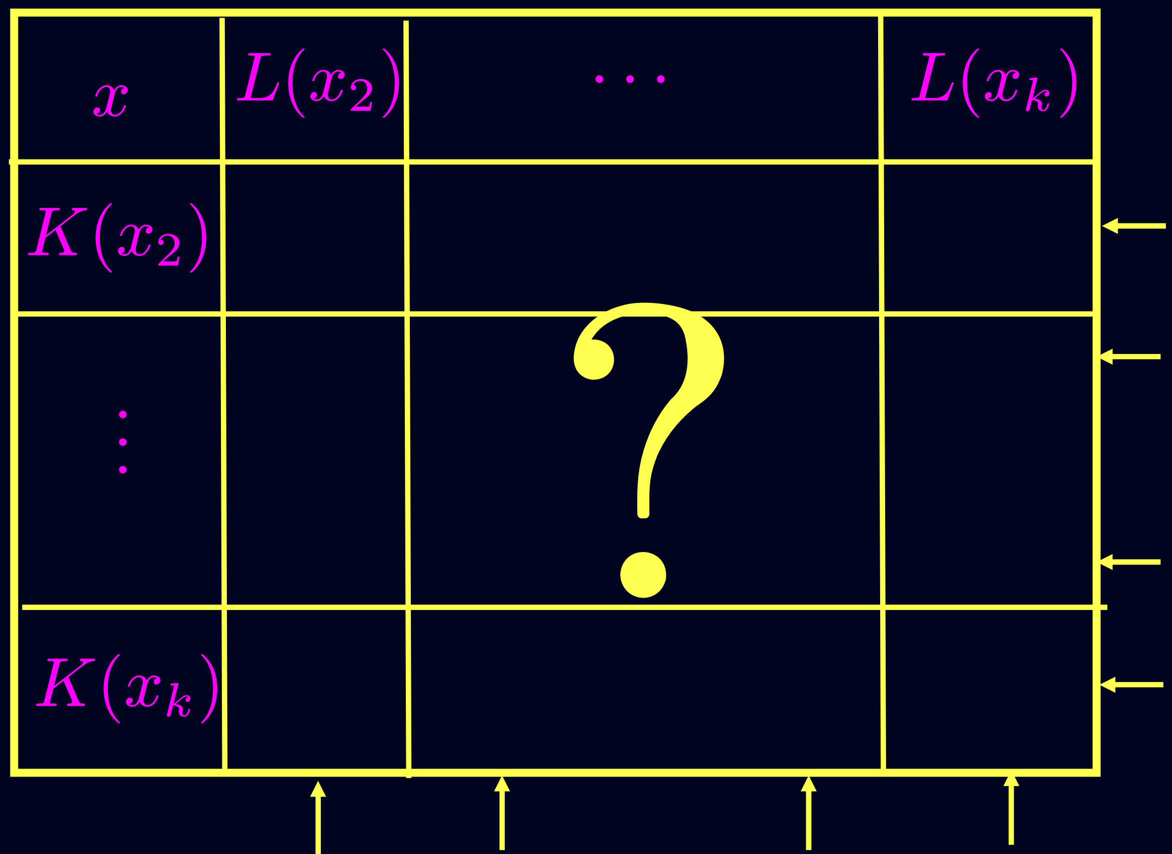
Matrix Magic?

- Define $g(x) = \alpha$ that maximizes
  $\Pr_{\{L \mid L(x_1)=x\}}[\langle \alpha, f(L(x_2)), \ldots, f(L(x_k))\rangle \in V]$

Lemma: $\forall\, x$, $\Pr_{L,K}[\mathrm{Vote}_x(L) \neq \mathrm{Vote}_x(K))] \leq 2(k-1)\delta$

x	$L(x_2)$	$\cdots$	$L(x_k)$
$K(x_2)$			
$\vdots$		?	
$K(x_k)$			

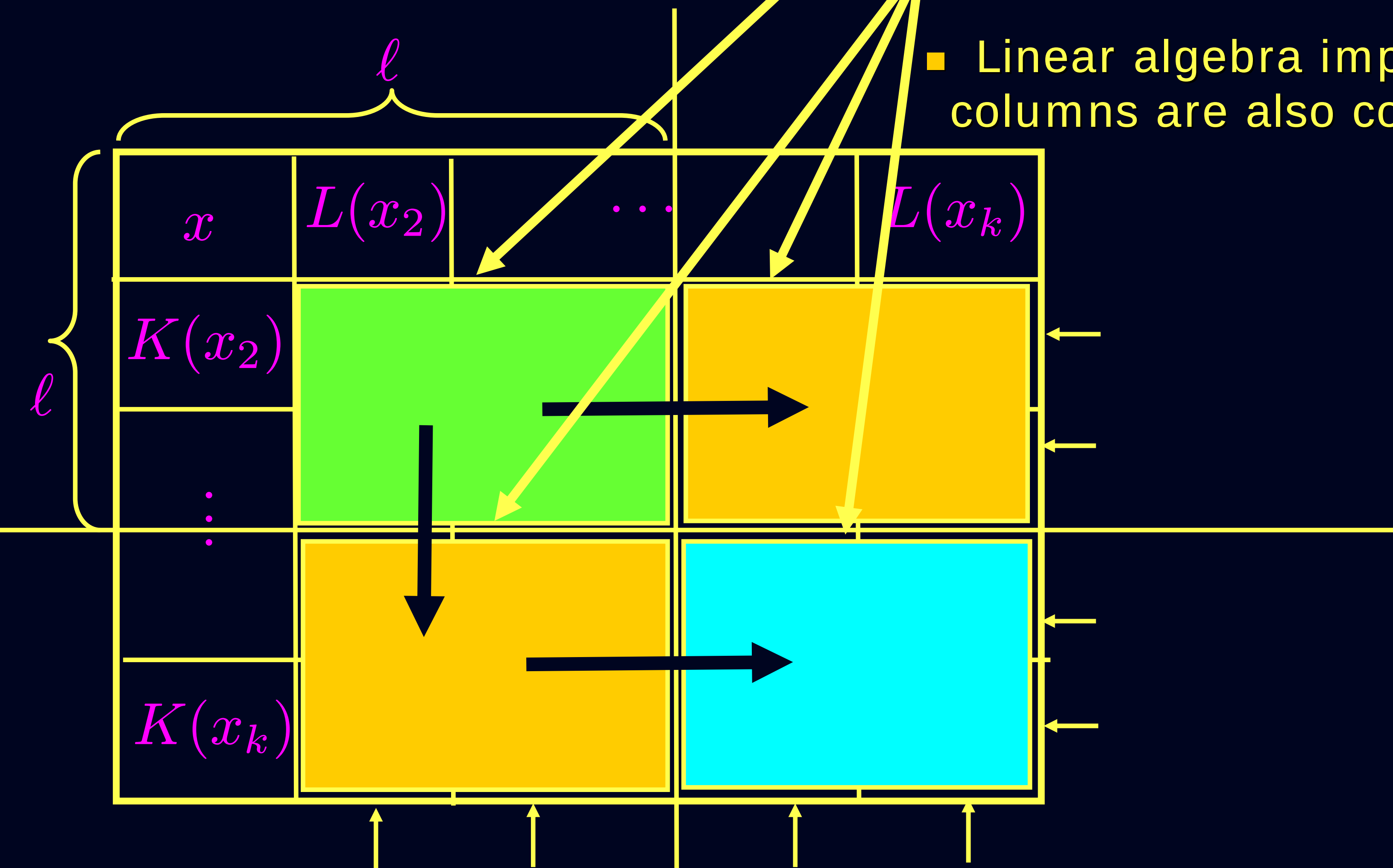
Matrix Magic?



- Want marked rows to be random constraints.
- Suppose $x_1, \ldots, x_\ell$ linearly independent; and rest dependent on them.

Matrix Magic?

- Fill with random entries
- Fill so as to make constraints
- Linear algebra implies final columns are also constraints.



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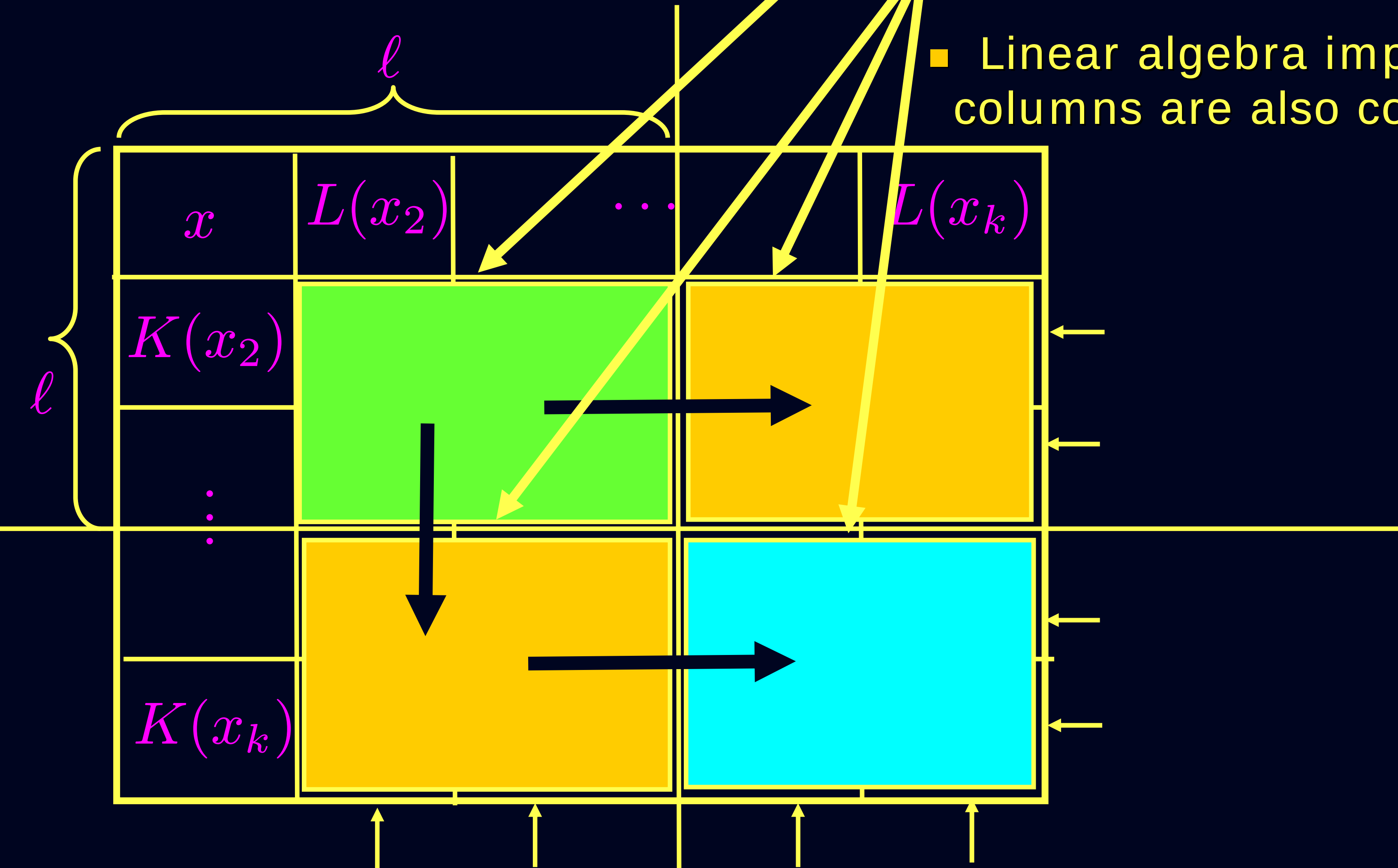

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Conclusions

- Linear/Affine-invariant properties testable if they have local constraints.
- Gives clean generalization of linearity and low-degree tests.
- Future work: What kind of invariances lead to testability (from characterizations)?