Invariance in Property Testing

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Modern challenge to Algorithm Design

- Data = Massive; Computers = Tiny
  - How can tiny computers analyze massive data?
  - Only option: Design sublinear time algorithms.
    - Algorithms that take less time to analyze data, than it takes to read/write all the data.
    - Can such algorithms exist?

Yes! Polling …

- Is the majority of the population Red/Blue
  - Can find out by random sampling.
  - Sample size $\propto$ margin of error
    - Independent of size of population

- Other similar examples: (can estimate other moments …)

Recent "novel" example

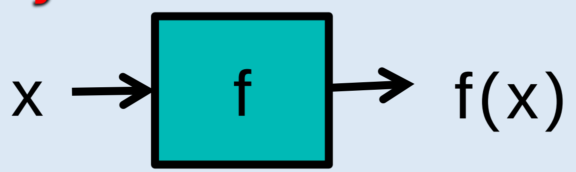
- Can test for homomorphisms:
  - Given: $f: G \to H$ (G,H finite groups), is f essentially a homomorphism?
  - Test:
    - Pick x,y in G uniformly, ind. at random;
    - Verify $f(x) \cdot f(y) = f(x \cdot y)$
  - Completeness: accepts homomorphisms w.p. 1
    - (Obvious)
  - Soundness: Rejects f w.p prob. Proportional to its "distance" (margin) from homomorphisms.
    - (Not obvious)

Property Testing

- Informally:
  - “Efficiently” test if “data” satisfies some “property”, in “essence”

[Blum, Luby Rubinfeld ’90]
[Rubinfeld, S. ’92; ‘96]
[Goldreich Goldwasser Ron ‘96]

- Formally:
  - Data: $f: D \to R$
  - Property: $P \subseteq \{g: D \to R\}$
  - Efficient: f given as a



- Tester should make few queries to f.
  - Essentially:
    - Accept $f \in P$ w.p. 1;
    - Reject f “far” from P w.h.p.

Distance: Far/Close

- Distance = (normalized) Hamming distance
  - $\delta(f,g) = \mathrm{Prob}_{x \in \mathcal{D}} [\ f(x) \neq g(x)\ ]$
  - $\delta(f,P) = \mathrm{Min}_{g \in \mathcal{P}}[\delta(f,g)]$

- (q, ϵ, δ)-tester for P:
  - Makes q queries to f.
  - Accepts w.p. probability ≈ 1 if $f \in P$
  - Reject w.p. probability ϵ if $\delta(f,P) \geq \delta$

- Ideally: $q = O(1)$ and $\epsilon(\delta) > 0, \quad \forall\, \delta > 0.$

[BLR] Lemma

- Let $\mathrm{Rej}(f) = \mathrm{Prob}_{x,y \in \mathbf{G}} [f(x) \cdot f(y) \neq f(x \cdot y)]$

- Lemma: If $\mathrm{Rej}(f) < 2/9$
  then $\delta(f, \mathrm{Hom}) = O(\mathrm{Rej}(f))$.

- Motivated by Program Checking:
  - E.g. to check if (complex) program multiplies matrices correctly:
    - Verify it is linear in each argument
    - Use this to check correctness.

Independently [Babai Fortnow Lund '90]

- Multilinearity testing: Is a function $f: F^m \to F$ essentially a degree 1 polynomial in each of the m variables?
  - Let $\mathrm{Rej}(f) = \mathrm{Prob}_\ell\ [f|_\ell$ is not affine]
    where ℓ is a random axis parallel line.
  - [BFL] Lemma:
    - If $\mathrm{Rej}(f) < 1/\mathrm{poly}(m)$, then
      $$\delta(f, \mathrm{MultiLin}) = O(\mathrm{Rej}(f)).$$

- Implications to Complexity (precursor to "Probabilistically Checkable Proofs")

Low-degree testing [Rubinfeld, S. '92-'96]

- Is a function $f: F^m \to F$ essentially a polynomial of degree d?
  - Let $\mathrm{Rej}(f) = \mathrm{Prob}_\ell\,[f|_\ell$ is not of degree $d]$
    where ℓ is a random line (not axis parallel).

  - Lemma ([ALMSS]):
    - $\exists\, \epsilon > 0$ s.t. $\forall\, d,m$, sufficiently large F
      if $\mathrm{Rej}(f) < \epsilon$
      then $\delta(f, \text{Degree-}d) = O(\mathrm{Rej}(f))$

Low-degree testing & Derivatives

- Let $f_a(x) = f(x+a) - f(a)$.
- Let $f_{a,b} = (f_a)_b$

- Let $Rej'(f) = E_{a,x}$ [$I(f_{a,a,a,\ldots}(x))$]
  - where $I(a) = 1$ if $a = 0$ and 0 otherwise.

- Variant of low-degree test implies that if the $(d+1)$ st derivative in random direction usually vanishes, then f is close to a degree d polynomial

Low-degree testing (Strong form)

- Is a function $f: F^m \to F$ essentially a polynomial of degree d?
  - Let $\rho(f) = \mathrm{Exp}_{\ell}\,[\,\delta(f|_{\ell},\ \text{Univ-Deg}(d))]$
    where ℓ is a random line.
    - Note: $\mathrm{Rej}(f)/F \le \rho(f) \le \mathrm{Rej}(f)$
  - Lemma ([ALMSS]):
    - $\exists\ \epsilon > 0$ s.t. $\forall$ d,m, sufficiently large F
      if $\rho(f) < \epsilon$
      then $\delta(f, \text{Degree-}d) = O(\rho(f))$

Low-degree testing (Stronger form)

- Is a function $f: F^m \to F$ essentially a polynomial of degree d?
  - Let $\rho(f) = \mathrm{Exp}_\ell\,[\,\delta(f|_{\ell}, \text{Univ-Deg}(d))]$ where ℓ is a random line.
    - Note: $\mathrm{Rej}(f)/F \le \rho(f) \le \mathrm{Rej}(f)$

  - Lemma (Arora + S. '97, Raz+Safra '97)
    - $\forall$ d,m, $\epsilon > 0$, sufficiently large F

      if $\rho(f) < 1 - \epsilon$

      then $\delta(f, \text{Degree-}d) = 1 - O(\epsilon)$

Motivations:

- [BLR] Linearity test: Program checking
- [BFL], [ALMSS]: Probabilistically checkable proofs
  - There exists a format for writing proofs that can be checked for correctness with constant queries and constant error probability
    - Uses low-degree testing & linearity testing.
- [GGR]: Should be studied for algorithm design.

1996-today

- Graph property testing [GGR, …, Alon, Shapira, Newman, Szegedy, Fisher]
    - Almost total understanding of graphical property testing … Regularity lemma.
    - Graph limits approach … (Borgs, Chayes, Lovasz, Sos, Szegedy, Vesztergombi)

- Algebraic Property Testing:
  - Many stronger results
  - Fewer new properties
    - [Alon-Kaufman-Krivelevich-Litsyn-Ron, Kaufman-Ron, Jutla-Patthak-Rudra-Zuckerman]
    - Low-degree testing over small fields (F_2)

Low-degree testing over GF(2)

- [AKKLR] = Alon-Kaufman-Krivelevich-Litsyn-Ron
- Let $F = F_2$
- Is a function $f: F^m \to F$ essentially a polynomial of degree d?
  - Let $Rej(f) = Prob_A\ [f|_A$ is a degree d poly$]$

    A is a random $(d+1)$-dim. affine subspace.
    - $U_{d+1}(f) = (\frac{1}{2} - Rej(f))^{2^{-d}}$
  - Lemma [AKKLR]
    - $\exists\ \epsilon > 0$ s.t. If $Rej(f) < \epsilon \cdot 2^{-d}$

      then $\delta(f,\text{Degree-}d) = O(Rej(f))$

    (Very weak "inverse Gowers" theorem)

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- Algebraic Property Testing:
    - Many stronger results
    - Fewer new properties
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My concerns …

- Why is the understanding of Algebraic Property Testing so far behind?
  - Why can't we get "rich" class of properties that are all testable?
  - Why are proofs so specific to property being tested.
- What made Graph Property Testing so well-understood?
- What is "novel" about Property Testing, when compared to "polling"?

Example

- Conjecture (AKKLR '96):
  - Suppose property P is a vector space over F_2;
  - Suppose its invariant group is 2-transitive.
  - Suppose P satisfies a k-ary constraint
    - $\forall f \in P,\ f(\alpha_1) + \cdots + f(\alpha_k) = 0.$
  - Then f is $(q(k), \epsilon(k,\delta),\delta(k))$-locally testable.
- Inspired by "low-degree" test over F_2. Implied all previous algebraic tests (at least in weak forms).

Invariances

- Property P invariant under permutation (function) $\pi: D \to D$, if

$$f \in P \Rightarrow f \circ \pi \in P$$

- Property P invariant under group G if for all $\pi \in G$, P is invariant under π.

Invariances are the key?

- "Polling" works well when (because) invariant group of property is the full symmetric group.
- Modern property tests work with much smaller group of invariances.
- Graph property ~ Invariant under vertex renaming.
- Algebraic Properties & Invariances?

Abstracting Algebraic Properties

- [Kaufman & S.]

- Range is a field F and P is F-linear.
- Domain is a vector space over F (or some field K extending F).

- Property is invariant under affine (sometimes only linear) transformations of domain.

- "Property characterized by single constraint, and its orbit under affine (or linear) transformations."

Example: Degree d polynomials

- Constraint: When restricted to a small dimensional affine subspace, function is polynomial of degree d (or less).
  - #dimensions $\leq d/(K-1)$

- Characterization: If a function satisfies above for every small dim. subspace, then it is a degree d polynomial.

- Single orbit: Take constraint on any one subspace of dimension $d/(K-1)$; and rotate over all affine transformations.

Some results

- If P is affine-invariant and has k-single orbit feature (characterized by orbit of single k-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
  - Unifies previous algebraic tests (in weak form) with single proof.

Analysis of Invariance-based test

- Property P given by $\alpha_1,\ldots,\alpha_k$; $V \in F^k$
- $P = \{f \mid f(A(\alpha_1)) \ldots f(A(\alpha_k)) \in V, \forall$ affine $A: K^n \to K^n\}$
- $Rej(f) = Prob_A$ [$f(A(\alpha_1)) \ldots f(A(\alpha_k))$ not in V]
- Wish to show: If $Rej(f) < 1/k^3$,

  then $\delta(f,P) = O(Rej(f))$.

BLR Analog

- $\mathrm{Rej}(f) = \Pr_{x,y}[\, f(x) + f(y) \neq f(x+y)] < \epsilon$

- Define $g(x) = \mathrm{majority}_y \{\mathrm{Vote}_x(y)\}$,
  where $\mathrm{Vote}_x(y) = f(x+y) - f(y)$.

- Step 0: Show $\delta(f,g)$ small

- Step 1: $\forall\, x$, $\Pr_{y,z}[\mathrm{Vote}_x(y) \neq \mathrm{Vote}_x(z)]$ small.

- Step 2: Use above to show g is well-defined and a homomorphism.

BLR Analysis of Step 1

- Why is $f(x+y) - f(y) = f(x+z) - f(z)$, usually?

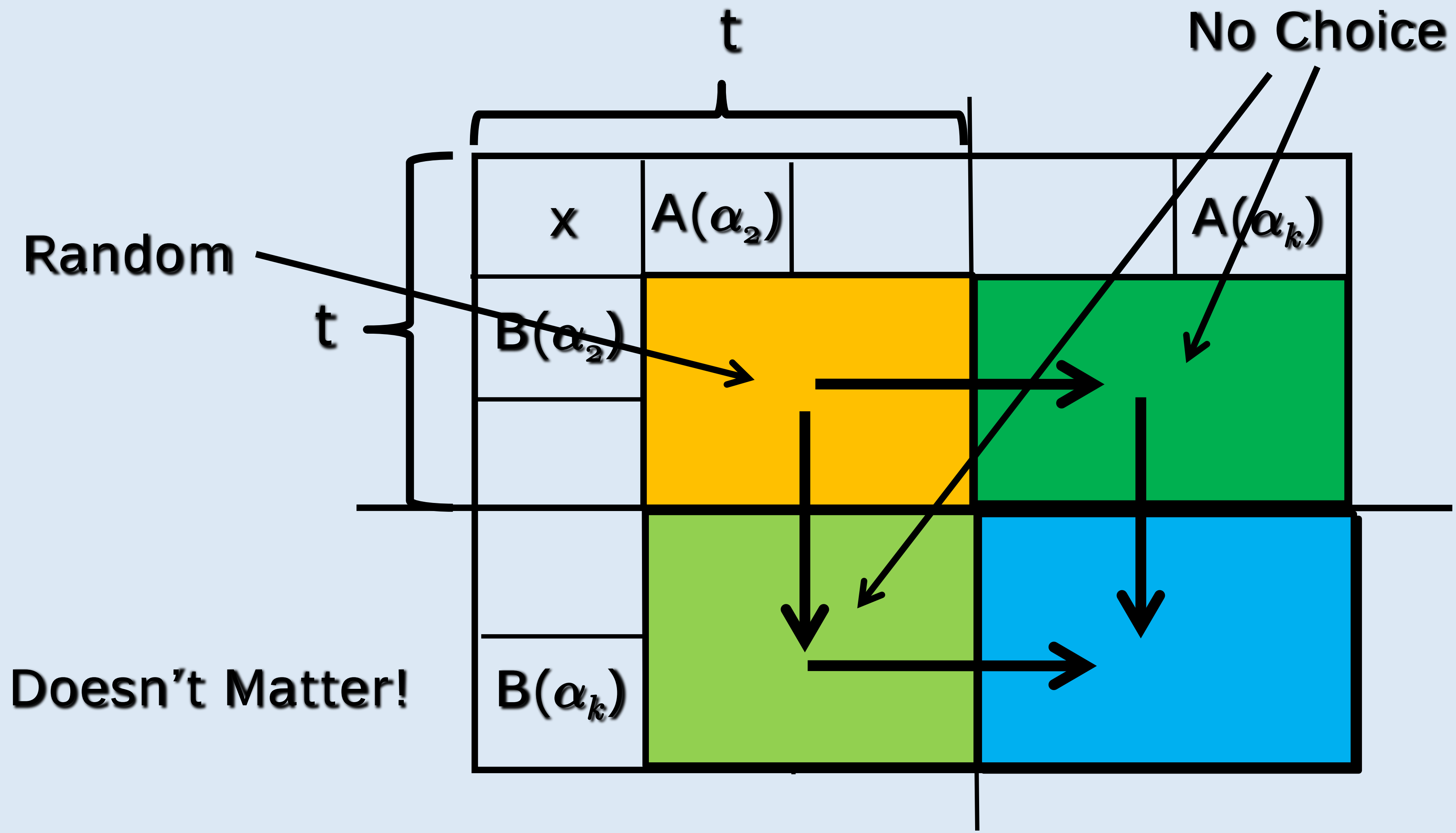
?	$f(z)$	$- f(x+z)$
$f(y)$	0	$-f(y)$
$- f(x+y)$	$-f(z)$	$f(x+y+z)$

Generalization

- $g(x) = \beta$ that maximizes, over A s.t. $A(\alpha_1) = x$,
  $\Pr_A [\beta, f(A(\alpha_2)),\ldots,f(A(\alpha_k)) \in V]$
- Step 0: $\delta(f,g)$ small.
- $\mathrm{Vote}_x(A) = \beta$ s.t. β, $f(A(\alpha_2))\ldots f(A(\alpha_k)) \in V$
  (if such β exists)
- Step 1 (key): $\forall\, x$, whp $\mathrm{Vote}_x(A) = \mathrm{Vote}_x(B)$.
- Step 2: Use above to show $g \in P$.

Matrix Magic?

Say $A(\alpha_1) \ldots A(\alpha_t)$ independent; rest dependent



Some results

- If P is affine-invariant and has k-single orbit feature (characterized by orbit of single k-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
  - Unifies previous algebraic tests with single proof.
- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some $q = q(K,k)$)
  - (explains the AKKLR optimism)

Some results

- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some q = q(K,k))
    - (explains the AKKLR optimism)
- Unfortunately, q depends inherently on K, not just F … giving counterexample to AKKLR conjecture [joint with Grigorescu & Kaufman]
- Linear invariance when P is not F-linear:
    - Abstraction of some aspects of Green's regularity lemma … [Bhattacharyya, Chen, S., Xie]
    - Nice results due to [Shapira]

More results

- Invariance of some standard codes (BCH etc.):
  - Have k-single orbit property! So duals are testable. [Grigorescu, Kaufman, S.]
- Side effect: New (essentially tight) relationships between $\mathrm{Rej}_{AKKLR}(f)$ (=½ + Gowers norm$^{2^d}$) and δ(f,Degree-d). [with Bhattacharyya, Kopparty, Schoenebeck, Zuckerman]
- One hope: Could lead to "simple, good locally testable code"?
  - (Sadly, not with affine-inv. [Ben-Sasson, S.])
- Still … other groups could be used? [Kaufman+Wigderson]

Conclusions

- Invariance seems to be a very nice perspective on "property testing" …
- (Needs Harmonic Analysis ☺)
- Hope: Can lead to interesting, new results?

Thanks