Invariance in Property Testing

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Property Testing

- … of functions from D to R:
  - Property $P \subseteq \{D \to R\}$
- Distance
  - $\delta(f,g) = \Pr_{x \in D} [f(x) \neq g(x)]$
  - $\delta(f,P) = \min_{g \in P} [\delta(f,g)]$
  - f is ε-close to g ($f \approx_\epsilon g$) iff $\delta(f,g) \leq \varepsilon$.
- Local testability:
  - P is (k, ε, δ)-locally testable if $\exists$ k-query test T
    - $f \in P \Rightarrow T^f$ accepts w.p. $1-\varepsilon$.
    - $\delta(f,P) > \delta \Rightarrow T^f$ accepts w.p. ε.
- Notes: want $k(\varepsilon, \delta) = O(1)$ for $\varepsilon,\delta = \Omega(1)$.

Brief History

- [Blum,Luby,Rubinfeld – S'90]
  - Linearity + application to program testing
- [Babai,Fortnow,Lund – F'90]
  - Multilinearity + application to PCPs (MIP).
- [Rubinfeld+S.]
  - Low-degree testing
- [Goldreich,Goldwasser,Ron]
  - Graph property testing
- Since then … many developments
  - Graph properties
  - Statistical properties
  - …
  - More algebraic properties

Specific Directions in Algebraic P.T.

- More Properties
  - Low-degree ($d < q$) functions [RS]
  - Moderate-degree ($q < d < n$) functions
    - $q=2$: [AKKLR]
    - General q: [KR, JPRZ]
  - Long code/Dictator/Junta testing [BGS,PRS]
  - BCH codes (Trace of low-deg. poly.) [KL]

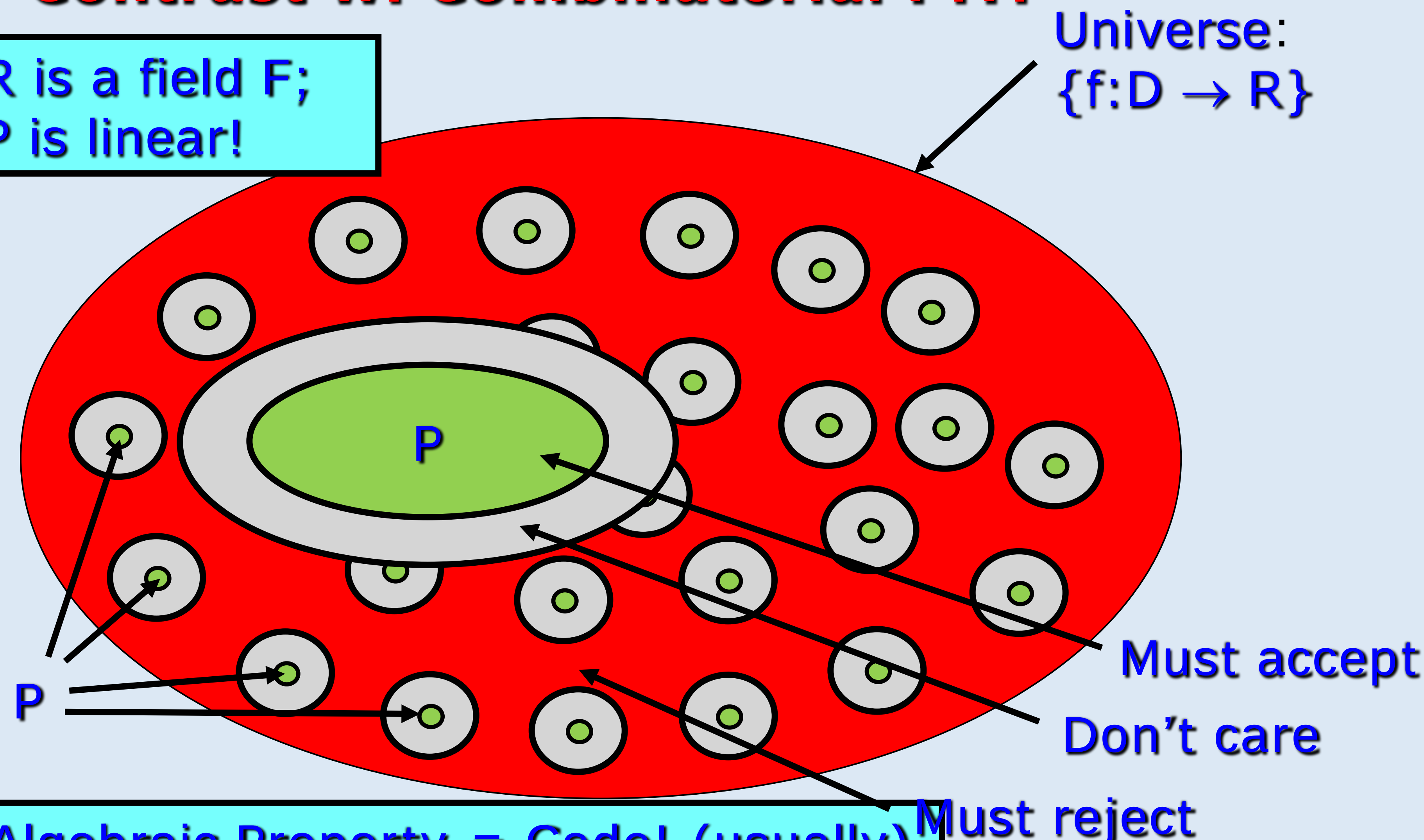
- Better Parameters (motivated by PCPs).
  - #queries, high-error, amortized query complexity, reduced randomness.

My concerns …

- Relatively few results …
  - Why can’t we get “rich” class of properties that are all testable?
  - Why are proofs so specific to property being tested?
- What made Graph Property Testing so well-understood?
- What is “novel” about Property Testing, when compared to “polling”?

Contrast w. Combinatorial P.T.

R is a field F;
P is linear!



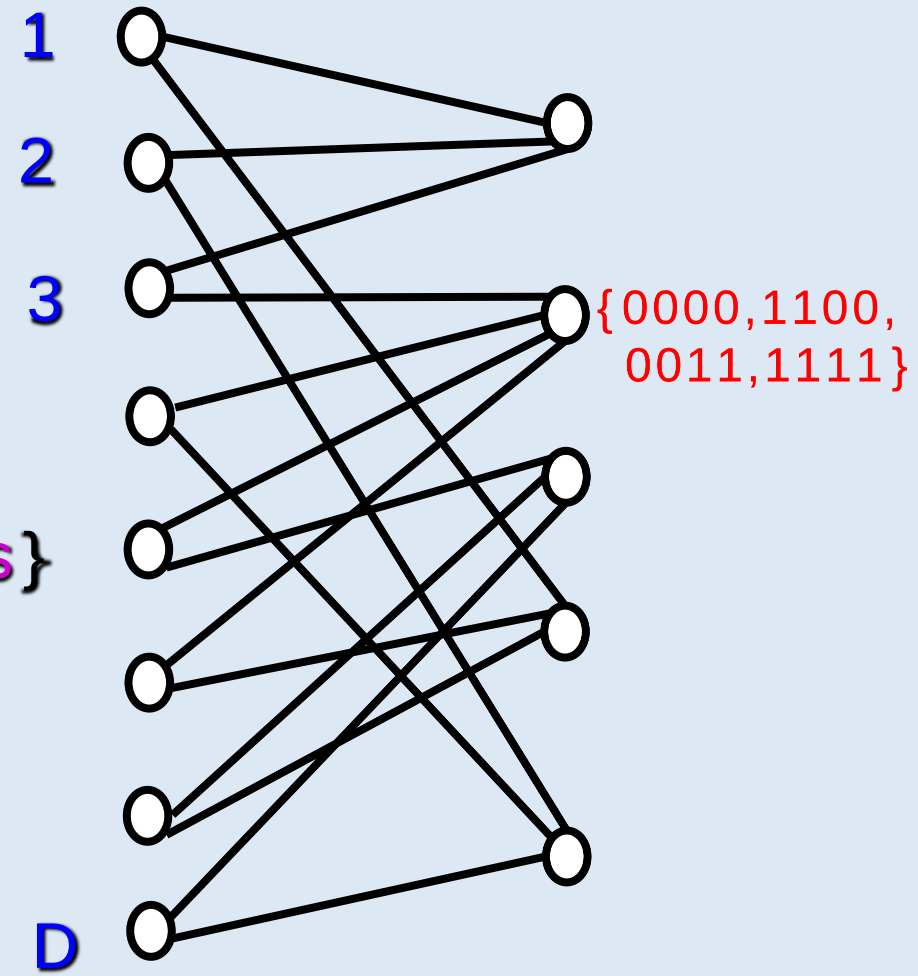
Algebraic Property = Code! (usually)

Basic Implications of Linearity [BHR]

- If P is linear, then:
  - Tester can be made non-adaptive.
  - Tester makes one-sided error
    - ($f \in P \Rightarrow$ tester always accepts).
- Motivates:
  - Constraints:
    - k-query test => constraint of size k:
      - value of f at $\alpha_1, \ldots \alpha_k$ constrained to lie in subspace.
  - Characterizations:
    - If non-members of P rejected with positive probability, then P characterized by local constraints.
      - functions satisfying all constraints are members of P.

Pictorially

- f = assgm't to left
- Right = constraints
- Characterization of P:
  P = {f sat. all constraints}



Sufficient conditions?

- Linearity + k-local characterization
  $\Rightarrow$ k-local testability?

- [BHR] No!
  - Elegant use of expansion
  - Rule out obvious test; but also any test … of any "q(k)"-locality

- Why is characterization insufficient?
  - Lack of symmetry?

Example motivating symmetry

- Conjecture (AKKLR '96):
  - Suppose property P is a vector space over F_2;
  - Suppose its "invariant group" is "2-transitive".
  - Suppose P satisfies a k-ary constraint
    - $\forall f \in P,\ f(\alpha_1) + \cdots + f(\alpha_k) = 0.$
  - Then P is $(q(k), \epsilon(k,\delta),\delta)$-locally testable.
- Inspired by "low-degree" test over F_2. Implied all previous algebraic tests (at least in weak forms).

Invariances

- Property P invariant under permutation (function) $\pi: D \to D$, if

$$f \in P \Rightarrow f \circ \pi \in P$$

- Property P invariant under group G if

$\forall \pi \in G$, P is invariant under π.

- Can ask: Does invariance of P w.r.t. "nice" G leads to local testability?

Invariances are the key?

- “Polling” works well when (because) invariant group of property is the full symmetric group.
- Modern property tests work with much smaller group of invariances.
- Graph property ~ Invariant under vertex renaming.
- Algebraic Properties & Invariances?

Abstracting Algebraic Properties

- [Kaufman & S.]

- Range is a field F and P is F-linear.
- Domain is a vector space over F (or some field K extending F).

- Property is invariant under affine (sometimes only linear) transformations of domain.

- "Property characterized by single constraint, and its orbit under affine (or linear) transformations."

Invariance, Orbits and Testability

- Single constraint implies many
  - One for every permutation $\pi \in \mathrm{Aut}(P)$:
    - "Orbit of a constraint C"

      $$= \{C \circ \pi \mid \pi \in \mathrm{Aut}(P)\}$$
- Extreme case:
  - Property characterized by single constraint + its orbit: "Single orbit feature"
    - Most algebraic properties have this feature.
    - W.l.o.g. if domain = vector space over small field.

Example: Degree d polynomials

- **Constraint:** When restricted to a small dimensional affine subspace, function is polynomial of degree d (or less).
  - #dimensions $\leq$ d/(K - 1)
- **Characterization:** If a function satisfies above for every small dim. subspace, then it is a degree d polynomial.
- **Single orbit:** Take constraint on any one subspace of dimension d/(K-1); and rotate over all affine transformations.

Some results

- If P is affine-invariant and has k-single orbit feature (characterized by orbit of single k-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
  - Unifies previous algebraic tests (in weak form) with single proof.

Analysis of Invariance-based test

- Property P given by $\alpha_1,\ldots,\alpha_k$; $V \in F^k$
- $P = \{f \mid f(A(\alpha_1)) \ldots f(A(\alpha_k)) \in V, \forall$ affine $A: K^n \to K^n\}$
- $Rej(f) = Prob_A$ [$f(A(\alpha_1)) \ldots f(A(\alpha_k))$ not in V]
- Wish to show: If $Rej(f) < 1/k^3$,

  then $\delta(f,P) = O(Rej(f))$.

BLR Analog

- $\text{Rej}(f) = \Pr_{x,y} [f(x) + f(y) \neq f(x+y)] < \epsilon$
- Define $g(x) = \text{majority}_y \{\text{Vote}_x(y)\}$,
  where $\text{Vote}_x(y) = f(x+y) - f(y)$.
- Step 0: Show $\delta(f,g)$ small
- Step 1: $\forall x, \Pr_{y,z} [\text{Vote}_x(y) \neq \text{Vote}_x(z)]$ small.
- Step 2: Use above to show g is well-defined and a homomorphism.

BLR Analysis of Step 1

- Why is $f(x+y) - f(y) = f(x+z) - f(z)$, usually?

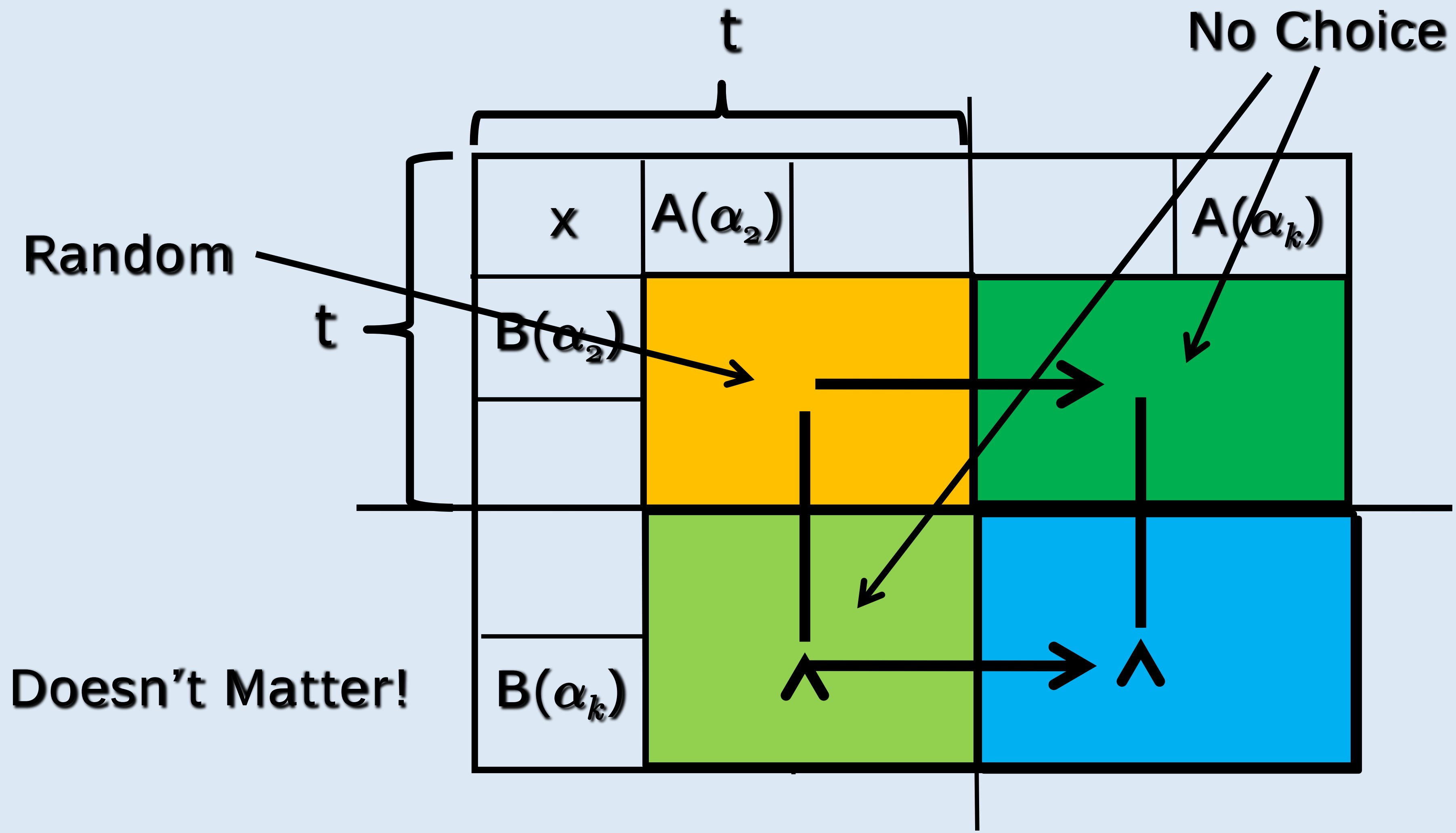
?	$f(z)$	$- f(x+z)$
$f(y)$	0	$-f(y)$
$- f(x+y)$	$-f(z)$	$f(x+y+z)$

Generalization

- $g(x) = \beta$ that maximizes, over A s.t. $A(\alpha_1) = x$,
  $\Pr_A[\beta, f(A(\alpha_2)), \ldots, f(A(\alpha_k)) \in V]$
- Step 0: $\delta(f,g)$ small.
- $\mathrm{Vote}_x(A) = \beta$ s.t. $\beta, f(A(\alpha_2)) \ldots f(A(\alpha_k)) \in V$
  (if such β exists)
- Step 1 (key): $\forall x$, whp $\mathrm{Vote}_x(A) = \mathrm{Vote}_x(B)$.
- Step 2: Use above to show $g \in P$.

Matrix Magic?

Say $A(\alpha_1)$ … $A(\alpha_t)$ independent; rest dependent



Some results

- If P is affine-invariant and has k-single orbit feature (characterized by orbit of single k-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
    - Unifies previous algebraic tests with single proof.

- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some $q = q(K,k)$)
    - (explains the AKKLR optimism)

Results (contd.)

- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some $q = q(K,k)$)
- Proof Ingredients:
  - Analysis of all affine invariant properties.
  - Rough characterization of locality of constraints, in terms of degrees of polynomials in the family.

- Infinitely many (new) properties …

More details

- Understanding invariant properties:
  - Recall: all functions from K^n to F are Traces of polynomials
    - ($Trace(x) = x + x^p + x^{p^2} + \ldots + x^{q/p}$ where $K = F_q$ and $F = F_p$)
  - If P contains $Tr(3x^5 + 4x^2 + 2)$; then P contains $Tr(4x^2)$ …
  - So affine invariant properties characterized by degree of monomials in family.
  - Most of the study … relate degrees to upper and lower bounds on locality of constraints.

Some results

- If P is affine-invariant over K and has a single k-local constraint, then it is has a q-single orbit feature (for some q = q(K,k))
  - (explains the AKKLR optimism)
- Unfortunately, q depends inherently on K, not just F … giving counterexample to AKKLR conjecture [joint with Grigorescu & Kaufman]
- Linear invariance when P is not F-linear:
  - Abstraction of some aspects of Green's regularity lemma … [Bhattacharyya, Chen, S., Xie]
  - Nice results due to [Shapira]

More results

- Invariance of some standard codes
  - E.g. "dual-BCH": Have k-single orbit feature! So are "more uniformly" testable.

    [Grigorescu, Kaufman, S.]
- Side effect: New (essentially tight) relationships between $Rej_{AKKLR}(f)$ and $\delta(f, \text{Degree-}d)$ over F_2 [with Bhattacharyya, Kopparty, Schoenebeck, Zuckerman]

More results (contd.)

- Invariance of some standard codes
- Side effect: New (essentially tight) relationships between $Rej_{AKKLR}(f)$ and $\delta(f, \text{Degree-}d)$ over F_2

- One hope: Could lead to "simple, good locally testable code"?
  - (Sadly, not with affine-inv. [Ben-Sasson, S.])

- Still … other groups could be used? [Kaufman+Wigderson]

Conclusions

- Invariance seems to be a nice perspective on "property testing" …
  - Certainly helps unify many algebraic property tests.
  - But should be a general lens in sublinear time algorithmics.

Thanks