Testing Affine-Invariant Properties

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Surveys: works with/of Eli Ben-Sasson, Elena Grigorescu, Tali Kaufman, Shachar Lovett, Ghid Maatouk, Amir Shpilka.

Property Testing

- … of functions from D to R:
  - Property $P \subseteq \{D \to R\}$
- Distance
  - $\delta(f,g) = \Pr_{x \in D}[f(x) \neq g(x)]$
  - $\delta(f,P) = \min_{g \in P}[\delta(f,g)]$
  - f is ε-close to g ($f \approx_\epsilon g$) iff $\delta(f,g) \leq \varepsilon$.
- Local testability:
  - P is (k, ε, δ)-locally testable if $\exists$ k-query test T
    - $f \in P \Rightarrow T^f$ accepts w.p. $1-\varepsilon$.
    - $\delta(f,P) > \delta \Rightarrow T^f$ accepts w.p. ε.
- Notes: want $k(\varepsilon, \delta) = O(1)$ for $\varepsilon,\delta = \Omega(1)$.

Classical Property Test: Linearity [BLR]

- Does $f(x+y) = f(x) + f(y)$, for all x, y?
- Variation (Affineness):
  - Is $f(x+y) + f(0) = f(x) + f(y)$, for all x, y?
  - (roughly $f(x) = a_0 + \sum_{i=1}^{n} a_i x_i$)
- Test: Pick random x,y and verify above.
- Obvious: f affine $\Rightarrow$ passes test w.p. 1.
- BLR Theorem: If f is δ-far from every affine function, then it fails test w.p. $\Omega(\delta)$.

- Ultimate goal of talk: To understand such testing results.

Affine-Invariant Properties

- Domain = K = $GF(q^n)$ (field with q^n elements)
- Range = $GF(q)$; q = power of prime p.
- P forms F-vector space.
- P <u>invariant</u> under affine transformations of domain.
  - Affine transforms? $x \mapsto a.x + b$, $a \in K^*$, $b \in K$.
  - Invariance? $f \in P \Rightarrow g_{a,b}(x) = f(ax+b) \in P$.
  - "affine permutation of domain leaves P unchanged".
- Quest: What makes affine-invariant property testable?

(My) Goals

- Why?
  - BLR test has been very useful (in PCPs, LTCs).
  - Other derivatives equally so (low-degree test).
  - Proof magical! Why did 3 (4) queries suffice?
  - Can we find other useful properties?

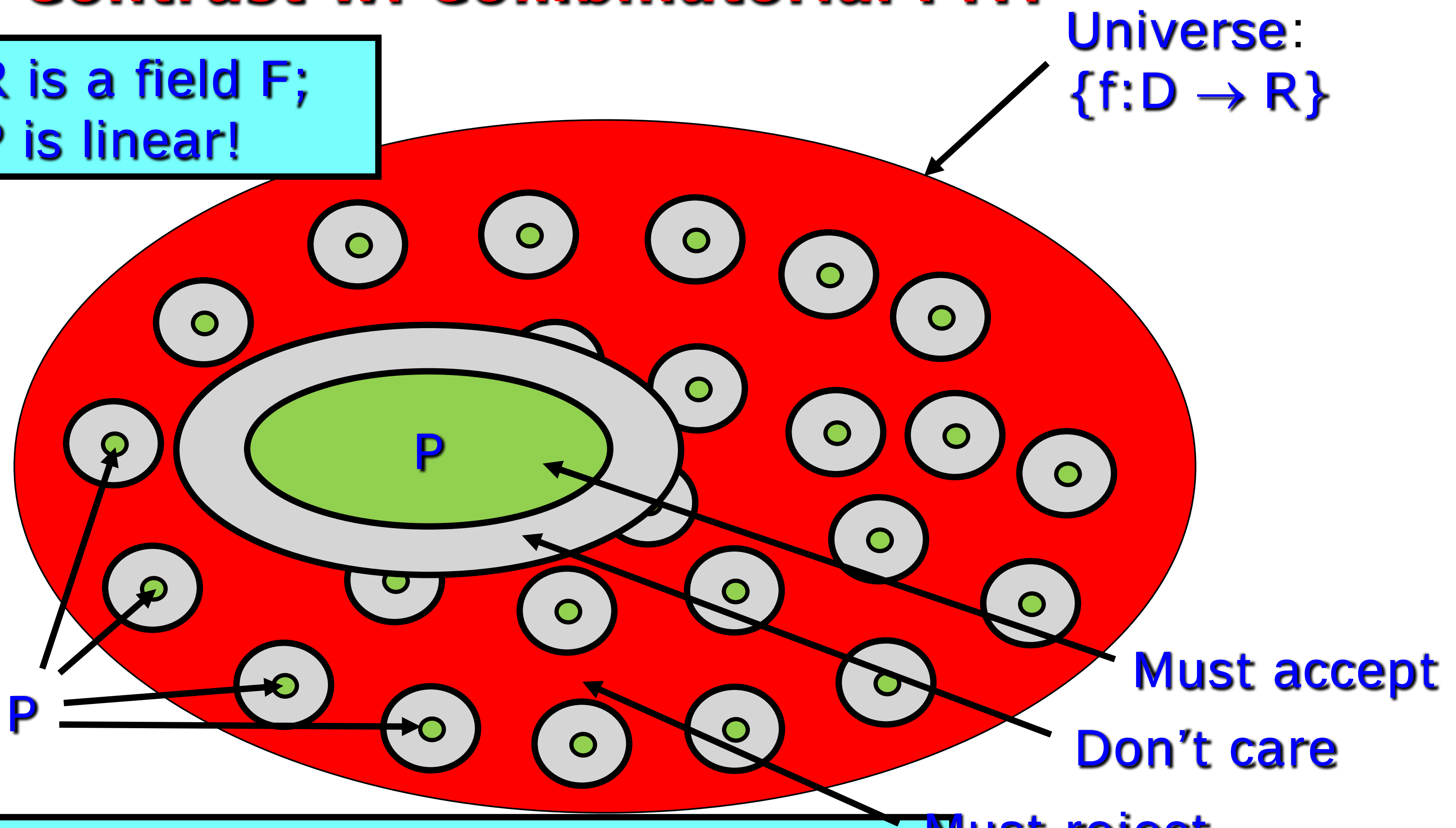
- Program:
  - Understand the proof better (using invariance).
  - Get structural understanding of affine-invariant properties, visavis local testability.
  - Get better codes/proofs?

Why's?

- Why Invariance
  - Natural way to abstract/unify common themes (in property testing).
  - Graph properties, Boolean, Statistical etc.?
- Why affine-invariance:
  - Abstracts linearity (affine-ness) testing.
  - Low-degree testing
  - BCH testing …
- Why F-vector space?
  - Easier to study (gives nice structure).
  - Common feature (in above + in codes).

Contrast w. Combinatorial P.T.

R is a field F;
P is linear!



Algebraic Property = Code! (usually)

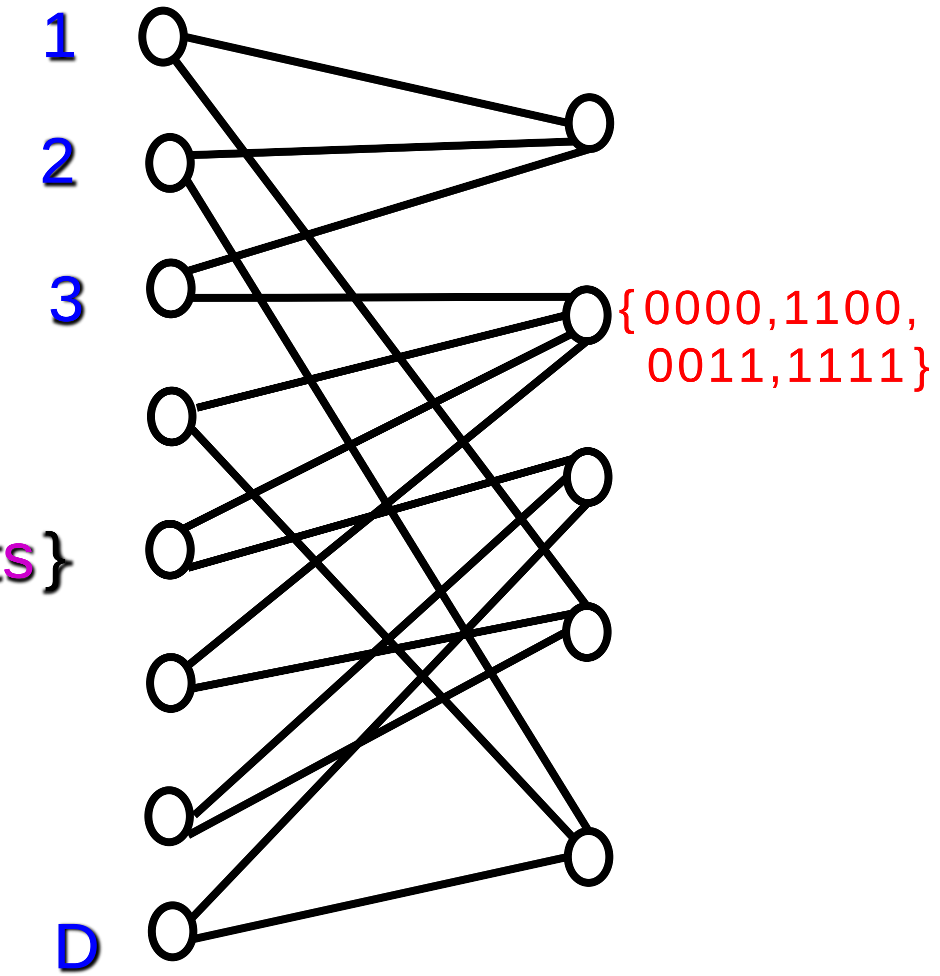
Basic Implications of Linearity [BHR]

- If P is linear, then:
  - Tester can be made non-adaptive.
  - Tester makes one-sided error
    - ($f \in P \Rightarrow$ tester always accepts).
- Motivates:
  - Constraints:
    - k-query test => constraint of size k:
      - value of f at $\alpha_1, \ldots \alpha_k$ constrained to lie in subspace.
  - Characterizations:
    - If non-members of P rejected with positive probability, then P characterized by local constraints.
      - functions satisfying all constraints are members of P.

Pictorially

- f = assgm't to left
- Right = constraints
- Characterization of P:

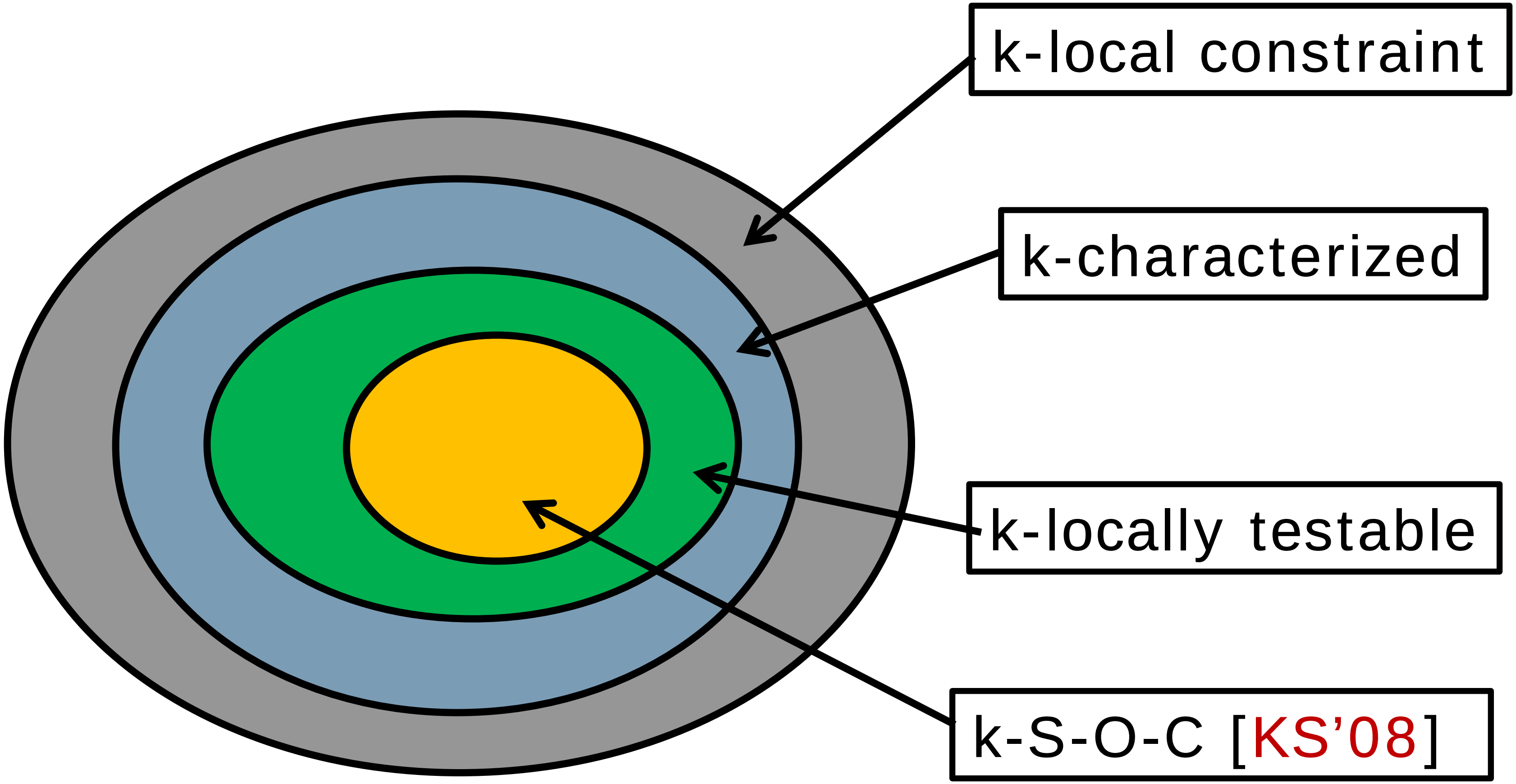
  P = {f sat. all constraints}



Back to affine-invariance: More Notes

- Why $K \to F$?
  - Very few permutations ($|K|^2$) !!
  - Still "2-transitive"
  - Includes all properties from F^n to F that are affine-invariant over F^n.
  - (Hope: Maybe find a new range of parameters?)
- Contrast with "linear-invariance" [Bhattacharyya et al.]
  - Linear vs. Affine.
  - Arbitrary P vs. F-vector space P
  - Linear over F^n vs. Affine over $K = GF(q^n)$.

Affine-invariance & testability



Goal of this talk

- Definition: Single-orbit-characterization (S-O-C)
- Known testable affine-invariant properties
  (all S-O-C!).
- Structure of Affine-invariant properties.
- Non testability results
- Open questions

Single-orbit-characterization (S-O-C)

- Many common properties are given by
  - (Affine-)invariance
  - Single constraint.
- Example: Affineness over $GF(2)^n$:
  - Affineness is affine-invariant.
  - $f(000000) - f(100000) \neq f(010000) - f(110000)$
- S-O-C: Abstracts this notion.
  - Suffices for testability [Kaufman+S'08]
  - Unifies all known testability results!!
  - Nice structural properties.

S-O-C: Formal Definition

- Constraint:
  - C = $(\alpha_1,\ldots,\alpha_k; V \subseteq F^k)$; $\alpha_i \in K$
  - C satisfied by f if
    - $(f(\alpha_1),\ldots,f(\alpha_k)) \in V$.

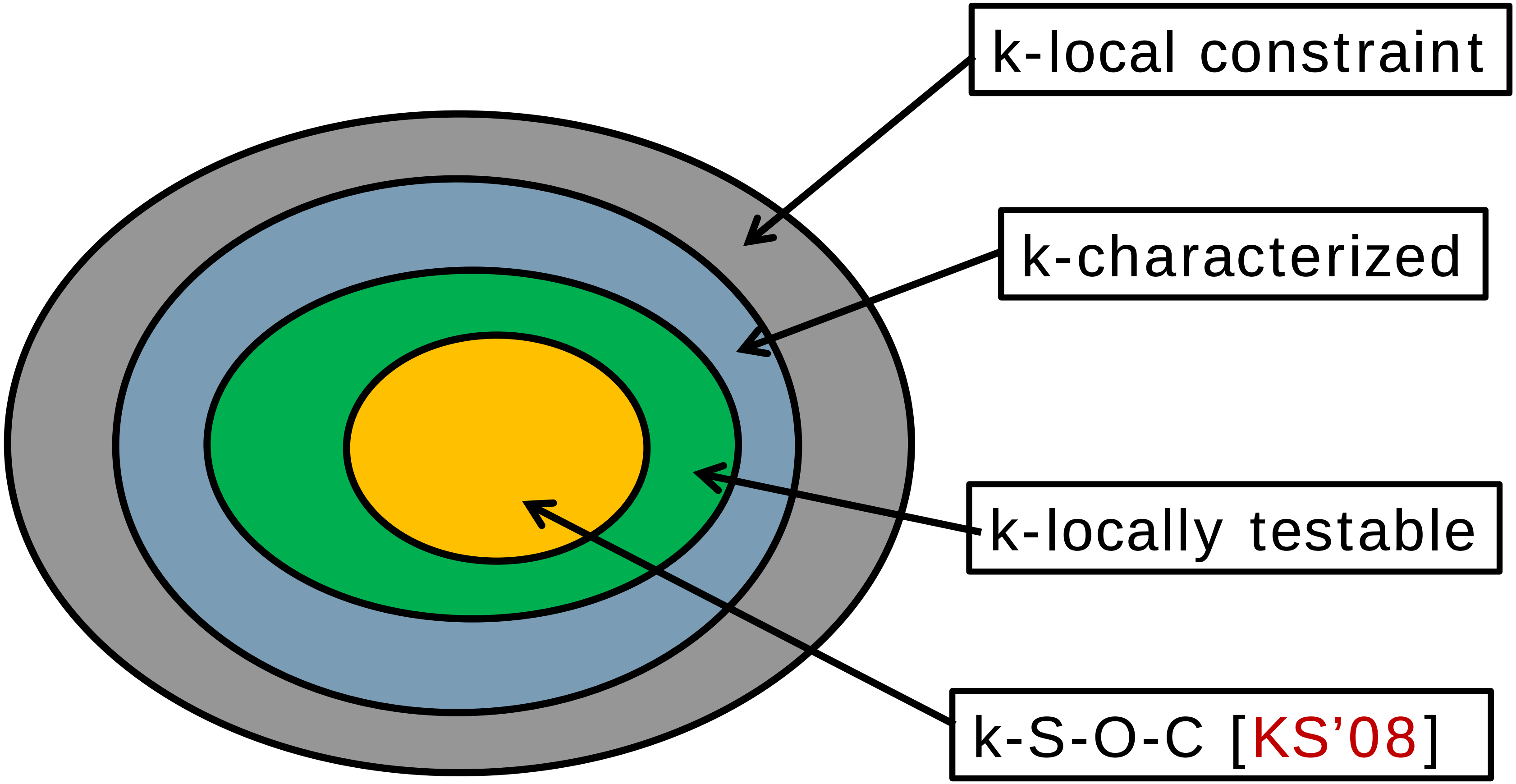
- Orbit of constraint = $\{C \circ \pi\}_\pi$, π affine.
  - $C \circ \pi = (\pi(\alpha_1),\ldots,\pi(\alpha_k); V)$.

- P has k-S-O-C, if orbit(C) characterizes P.

Known testable properties - 0

- Theorem [Kaufman-S.'08]:
  - If P has a k-S-O-C, then P is k-locally testable.

Affine-invariance & testability



Known testable properties - 0

- Theorem [Kaufman-S.'08]:
  - If P has a k-S-O-C, then P is k-locally testable.

- But who has k-S-O-C?
  - Affine functions:
    - over affine transforms of F^n
  - Degree d polynomials:
    - again, over affine transforms of F^n

Known testable properties - 1

- Reed-Muller Property:
  - View domain as F^n (n-variate functions)
  - Parameter d.
  - RM(d) = n-var. polynomials of degree $\leq$ d.
- Known to be $q^{O(d/q)}$-locally testable:
  - Test: Test if f restricted to O(d/q)-dimensional subspace is of degree d.
  - Analysis: [Kaufman-Ron] (see appendix 1).
- Single-Orbit?
  - Yes – naturally over affine transforms of F^n.
  - Yes – unnaturally over K (field of size F^n).

Known testable properties - 2

- Sparse properties:
  - Parameter t
  - $|P| \leq |K|^t$
- Testability:
  - Conditioned on "high-distance" [Kaufman-Litsyn, Kaufman-S.]. (no need for aff. inv.)
  - Unconditionally
    - [Grigorescu, Kaufman, S.], [Kaufman-Lovett] (for prime q).
    - Also S-O-C.

Known Testable Properties - 3

- Intersections:
  - $P_1 \cap P_2$ always locally testable, also S-O-C.
- Sums:
  - $P_1 + P_2$ $(= \{f_1 + f_2 \mid f_i \in P_i\})$
    - S-O-C iff P_1 and P_2 are S-O-C [BGMSS'11]
- Lifts [BMSS'11]
  - Suppose $F \subseteq L \subseteq K$.
  - $P \subseteq \{L \to F\}$ has k-S-O-C, with constraint C.
  - Then Lift_$\{L \to K\}(P)$ = property characterized by K-orbit(C).
  - By Definition: Lift(P) is k-S-O-C.

Known Testable Properties - ∞

- Finite combination of Lifts, Intersections, Sums of Sparse and Reed-Muller properties.
  - Known: They are testable (for prime q).
  - Open: Are they the only testable properties?
    - If so, Testability $\equiv$ Single-Orbit.
  - First target: n = prime:
    - no lifts/intersections; only need to show that every testable property is sum of sparse and Reed-Muller property.

Affine-Invariant Properties: Structure

Preliminaries

- Every function from $K \to K$, including $K \to F$, is a polynomial in $K[x]$
  - So every property P = {set of polynomials}.
  - Is set arbitrary? Any structure?
- Alternate representation:
  - $Tr(x) = x + x^q + x^{q^2} + \ldots + x^{q^{n-1}}$
  - $Tr(x+y) = Tr(x)+Tr(y)$; $Tr(\alpha x) = \alpha Tr(x)$, $\alpha \in F$.
  - $Tr: K \to F$.
  - Every function from $K \to F$ is $Tr(f)$ for some polynomial $f \in K[x]$.
  - Any structure to these polynomials?

Example

- $F = GF(2)$, $K = GF(2^n)$.
- Suppose P contains $Tr(x^{11} + x^3 + 1)$.
- What other functions must P contain (to be affine-invariant)?
- Claims:
  - Let $D = \{0,1,3,5,9,11\}$.
  - Then P contains every function of the form $Tr(f)$, where f is supported on monomials with degrees from D.
  - So $Tr(x^5), Tr(\alpha x^9+\beta x^5), Tr(x^{11}+x^5+x^3+x)+1 \in P$.
  - How? Why?

Structure - 1

- Definitions:
  - $Deg(P) = \{d \mid \exists f \in P, \text{ with } x^d \in supp(f)\}$
  - $Fam(D) = \{f: K \to F \mid supp(f) \subseteq D\}$

- Proposition: For affine-invariant property P

$$P = Fam(Deg(P)).$$

Structure - 2

- Definitions:
  - $\text{Shift}(d) = \{d, q.d, q^2.d, \ldots\} \bmod (q^n-1)$.
  - D is shift-closed if Shift(D) = D.
  - $e \le d : e = e_0 + e_1\, p + \ldots;$

    $d = d_0 + d_1\, p + \ldots;$

    $e \le d$ if $e_i \le d_i$ for all i.

  - $\text{Shadow}(d) = \{e \le d\};$
  - $\text{Shadow}(D) = \cup_{d \in D} \text{Shadow}(d)$.
  - D is shadow-closed if Shadow(D) = D.

Structure - 3

- Proposition: For every affine-invariant property P, $Deg(P)$ is p-shadow-closed and q-shift-closed.

  (Shadowing comes from affine-transforms;
  Shifts come from range being F).

- Proposition: For every p-shadow-closed, q-shift-closed family D, $Fam(D)$ is affine-invariant and

$$D = Deg(Fam(D))$$

Example revisited

- $Tr(x^{11} + x^3) \in P$
  - $Deg(P) \ni 11, 3$ (definition of Deg)
  - $Deg(P) \ni 11, 9, 5, 3, 1, 0$ (shadow-closure)
  - $Deg(P) \ni Tr(x^{11}), Tr(x^9)$ etc. (shift-closure).
  - $Fam(Deg(P)) \ni Tr(x^{11})$ etc. (definition of Fam).
  - $P \ni Tr(x^{11})$ ($P = Fam(Deg(P))$)

What kind of properties have k-S-O-C?

(Positive results interpreted structurally)

- If propery has all degrees of q-weight at most k then it is RM and has (q^k)-S-O-C:
  - q-weight(d) = $\sum_i d_i$,
    where $d = d_0 + d_1 q + \ldots$
- Also, if P = Fam(D) & D = Shift(S) for small shadow-closed S, then P is k(|S|)-S-O-C.
  - (Alternate definition of sparsity.)

- Other examples from Intersection, Sum, Lift.

What affine-invariant properties are not locally testable.

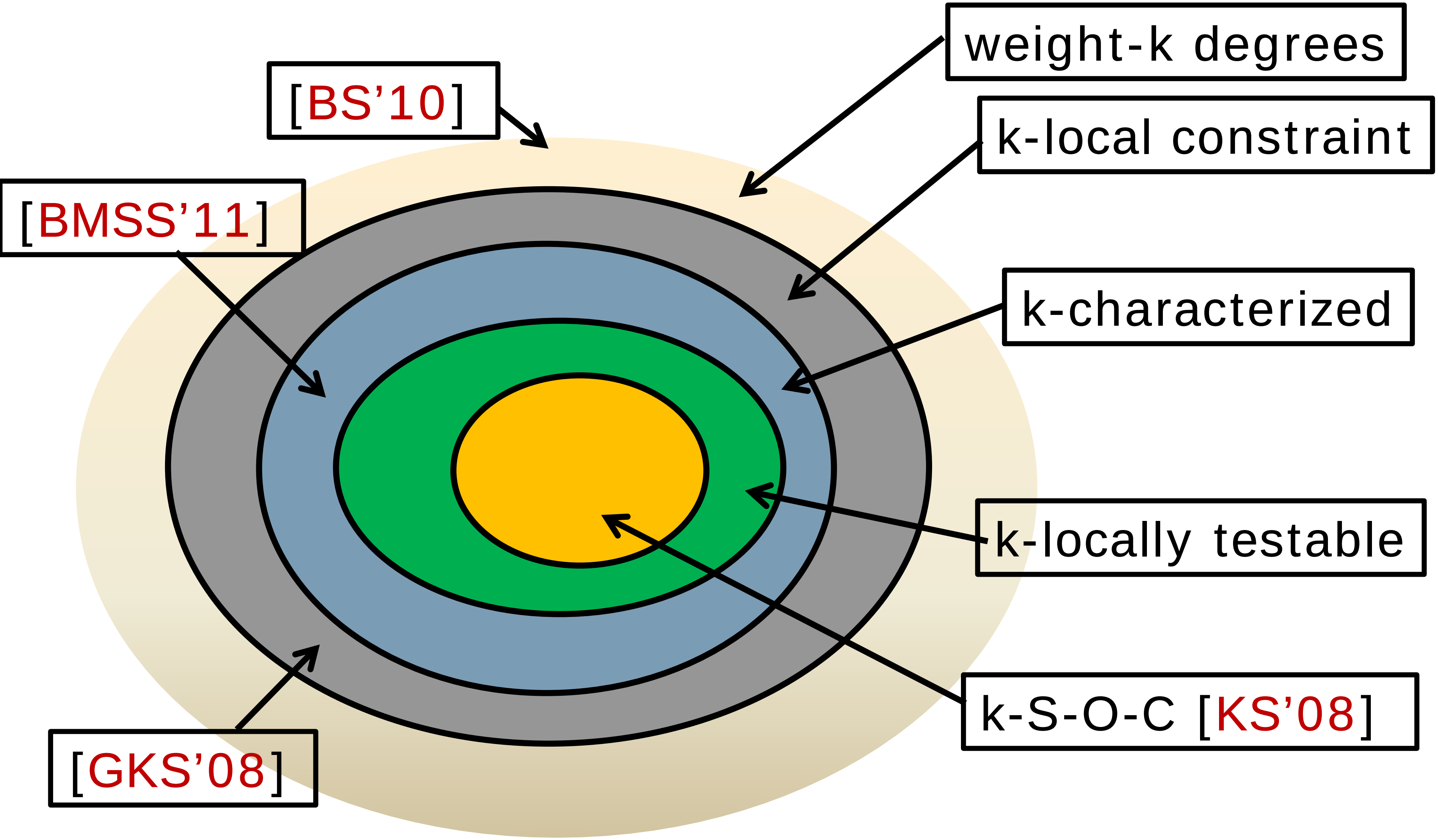
- Very little known.

- Specific examples:
  - GKS08: Exists a-i property with k-local constraint which is not k-locally characterized.

  - BMSS11: Exists k-locally characterized a-i property that is not testable.

- BSS'10: If $wt(d) \geq k$ for some d in $Deg(P)$, then P does not have a k-local constraint.

Affine-invariance & testability



Quest in lower bound

- Given degree set D (shadow-closed, shift-closed) prove it has no S-O-C.

- Equivalently: Prove there are no

  $\lambda_1 \ldots \lambda_k \in F$, $\alpha_1 \ldots \alpha_k \in K$ such that

  - $\sum_{i=1}^{k} \lambda_i \alpha_i^d = 0$ for every $d \in D$.
  - $\sum_{i=1}^{k} \lambda_i \alpha_i^d \neq 0$ for every minimal $d \notin D$.

Pictorially

$$M(D) = \begin{pmatrix} \alpha_1^{\,d} & \alpha_2^{\,d} & \cdots & \alpha_k^{\,d} \end{pmatrix}$$

Is there a vector $(\lambda_1,\ldots,\lambda_k)$ in its right kernel?

Can try to prove "NO" by proving matrix has full rank.

Unfortunately, few techniques to prove non-square matrix has high rank.

Non-testable Property - 1

- AKKLR (Alon,Kaufman,Krivelevich,Litsyn,Ron) Conjecture:
  - If a linear property is 2-transitive and has a k-local constraint then it is testable.
  - [GKS'08]: For every k, there exists affine-invariant property with 8-local constraint that is not k-locally testable.
  - $P = \mathrm{Fam}(\mathrm{Shift}(\{0,1\} \cup \{1+2, 1+2^2, \ldots, 1+2^k\}))$.

Proof (based on [BMSS'11])

- $F = GF(2)$; $K = GF(2^n)$;
- $P_k = \mathrm{Fam}(\mathrm{Shift}(\{0,1\} \cup \{1 + 2^i \mid i \in \{1,\ldots,k\}\}))$
- Let $M_i = \begin{pmatrix} \alpha_1^{\;2} & \alpha_2^{\;2} & \cdots & \alpha_k^{\;2} \\ & & & \\ \alpha_1^{\;2^i} & \alpha_2^{\;2^i} & \cdots & \alpha_k^{\;2^i} \end{pmatrix}$
- If $\mathrm{Ker}(M_i) = \mathrm{Ker}(M_{i+1})$, then $\mathrm{Ker}(M_{i+2}) = \mathrm{Ker}(M_i)$
- $\mathrm{Ker}(M_{k+1})$ = would accept all functions in P_{k+1}
- So $\mathrm{Ker}(M_i)$ must go down at each step, implying Rank(M_{i+1}) > Rank(M_i).

Stronger Counterexample

- GKS counterexample:
  - Takes AKKLR question too literally;
  - Of course, a non-locally-characterizable property can not be locally tested.

- Weaker conjecture:
  - Every k-locally characterized affine-invariant (2-transitive) property is locally testable.
  - Alas, not true: [BMSS]

[BMSS] CounterExample

- Recall:
  - Every known locally characterized property was locally testable
  - Every known locally testable property is S-O-C.
  - Need a locally characterized property which is (provably) not S-O-C.
  - Idea:
    - Start with sparse family P_i.
    - Lift it to get Q_i (still S-O-C).
    - Take intersection of superconstantly many such properties. $Q = \cap_i Q_i$

Example: Sums of S-O-C properties

- Suppose $D_1 = Deg(P_1)$ and $D_2 = Deg(P_2)$
- Then $Deg(P_1 + P_2) = D_1 \cup D_2$.
- Suppose S-O-C of P_1 is C_1: $f(a_1) + \ldots + f(a_k) = 0$; and S-O-C of P_2 is C_2: $f(b_1) + \ldots + f(b_k) = 0$.
- Then every $g \in P_1 + P_2$ satisfies:

  $$\sum_{i,j} g(a_i\, b_j) = 0$$

- Doesn't yield S-O-C, but applied to random constraints in orbit(C_1), orbit(C_2) does!
  - Proof uses $wt(Deg(P_1)) \leq k$.

Concluding

- Affine-invariance gives nice umbrella to capture algebraic property testing:
  - Important (historically) for PCPs, LTCs, LDCs.
  - Incorporates symmetry.
- Would be nice to have a complete characterization of testability of affine-invariant properties.
  - Understanding (severely) lacking.
- Know:
  - Can't be much better than Reed-Muller.
  - Can they be slightly better? YES!

Thank You!