# Invariance in Property Testing

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Based on: works with/of Eli Ben-Sasson, Elena Grigorescu, Tali Kaufman, Shachar Lovett, Ghid Maatouk, Amir Shpilka.

# Property Testing

- Sublinear time algorithms:
  - Algorithms running in time o(input), o(output).
    - Probabilistic.
    - Correct on (approximation) to input.
    - Input given by oracle, output implicit.
  - Crucial to modern context
    - (Massive data, no time).
- Property testing:
  - Restriction of sublinear time algorithms to decision problems (output = YES/NO).
- Amazing fact: Many non-trivial algorithms exist!

# Example 1: Polling

- Is the majority of the population Red/Blue
  - Can find out by random sampling.
  - Sample size $/\propto$ margin of error
    - Independent of size of population

- Other similar examples: (can estimate other moments …)

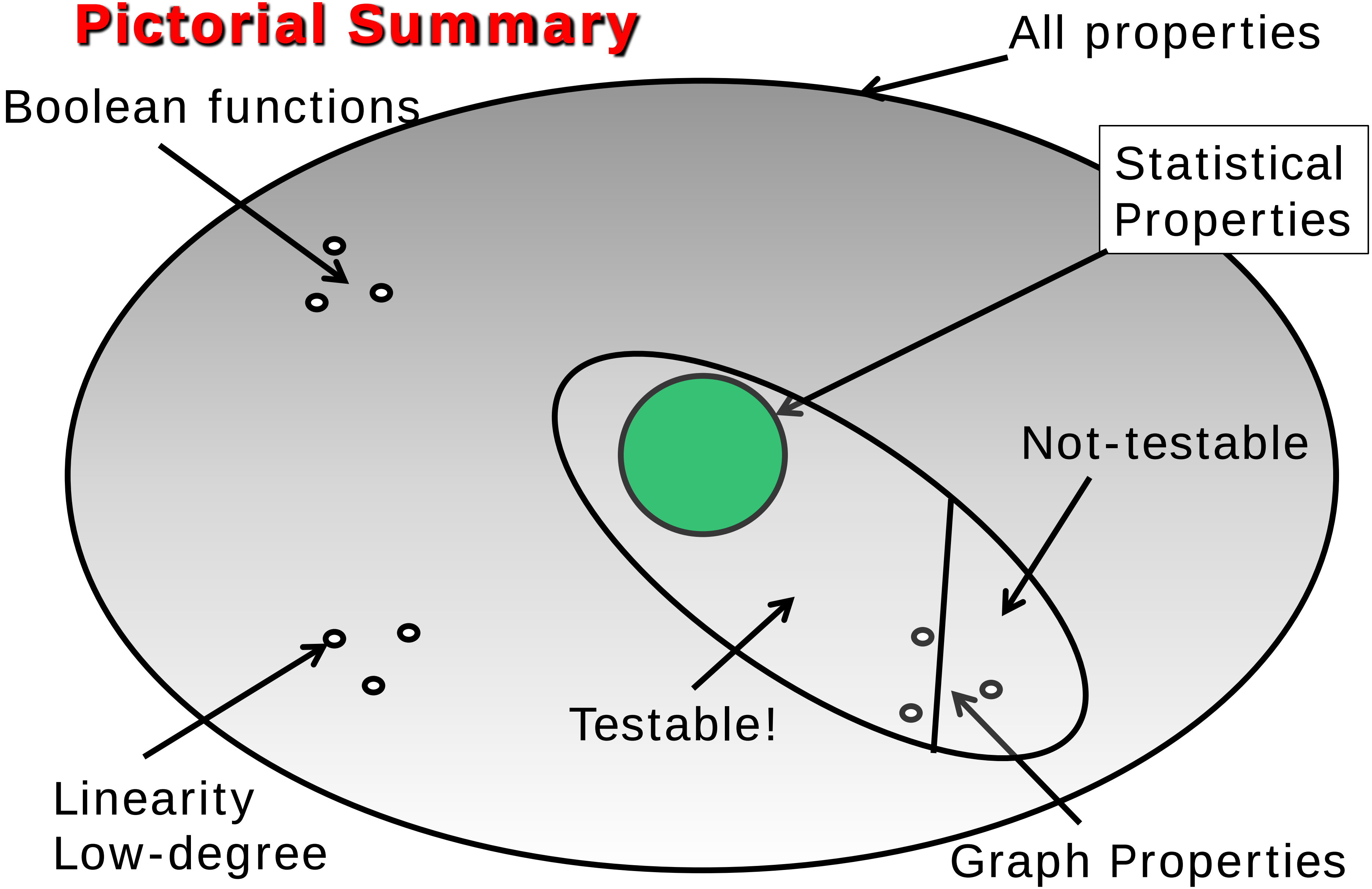
# Example 2: Linearity

- Can test for homomorphisms:
  - Given: $f: G \to H$ ($G,H$ finite groups), is $f$ essentially a homomorphism?
  - Test:
    - Pick $x,y$ in $G$ uniformly, ind. at random;
    - Verify f(x) ¢ f(y) = f(x ¢ y)
  - Completeness: accepts homomorphisms w.p. 1
    - (Obvious)
  - Soundness: Rejects f w.p prob. Proportional to its "distance" (margin) from homomorphisms.
    - (Not obvious, [BlumLubyRubinfeld'90])

# History (slightly abbreviated)

- [Blum,Luby,Rubinfeld – S'90]
  - Linearity + application to program testing
- [Babai,Fortnow,Lund – F'90]
  - Multilinearity + application to PCPs (MIP).
- [Rubinfeld+S.]
  - Low-degree testing; Formal definition.
- [Goldreich,Goldwasser,Ron]
  - Graph property testing; Systematic study.
- Since then … many developments
  - More graph properties, statistical properties, matrix properties, properties of Boolean functions …
  - More algebraic properties

# Pictorial Summary



# Some (introspective) questions

- What is qualitatively novel about linearity testing relative to classical statistics?

- Why are the mathematical underpinnings of different themes so different?

- Why is there no analog of "graph property testing" (broad class of properties, totally classified wrt testability) in algebraic world?

# Invariance?

- Property $P \subseteq \{f : D \to R\}$
- Property $P$ invariant under permutation (function) $\pi : D \to D$, if

$$f \in P \Rightarrow f \circ \pi \in P$$

- Property $P$ invariant under group $G$ if

$$\forall \pi \in G, P \text{ is invariant under } \pi.$$

- Observation: Different property tests unified/separated by invariance class.

# Invariances (contd.)

- Some examples:
  - Classical statistics: Invariant under all permutations.
  - Graph properties: Invariant under vertex renaming.
  - Boolean properties: Invariant under variable renaming.
  - Matrix properties: Invariant under mult. by invertible matrix.
  - Algebraic Properties = ?
- Goals:
  - Possibly generalize specific results.
  - Get characterizations within each class?
  - In algebraic case, get new (useful) codes?

# Abstracting Linearity/Low-degree tests

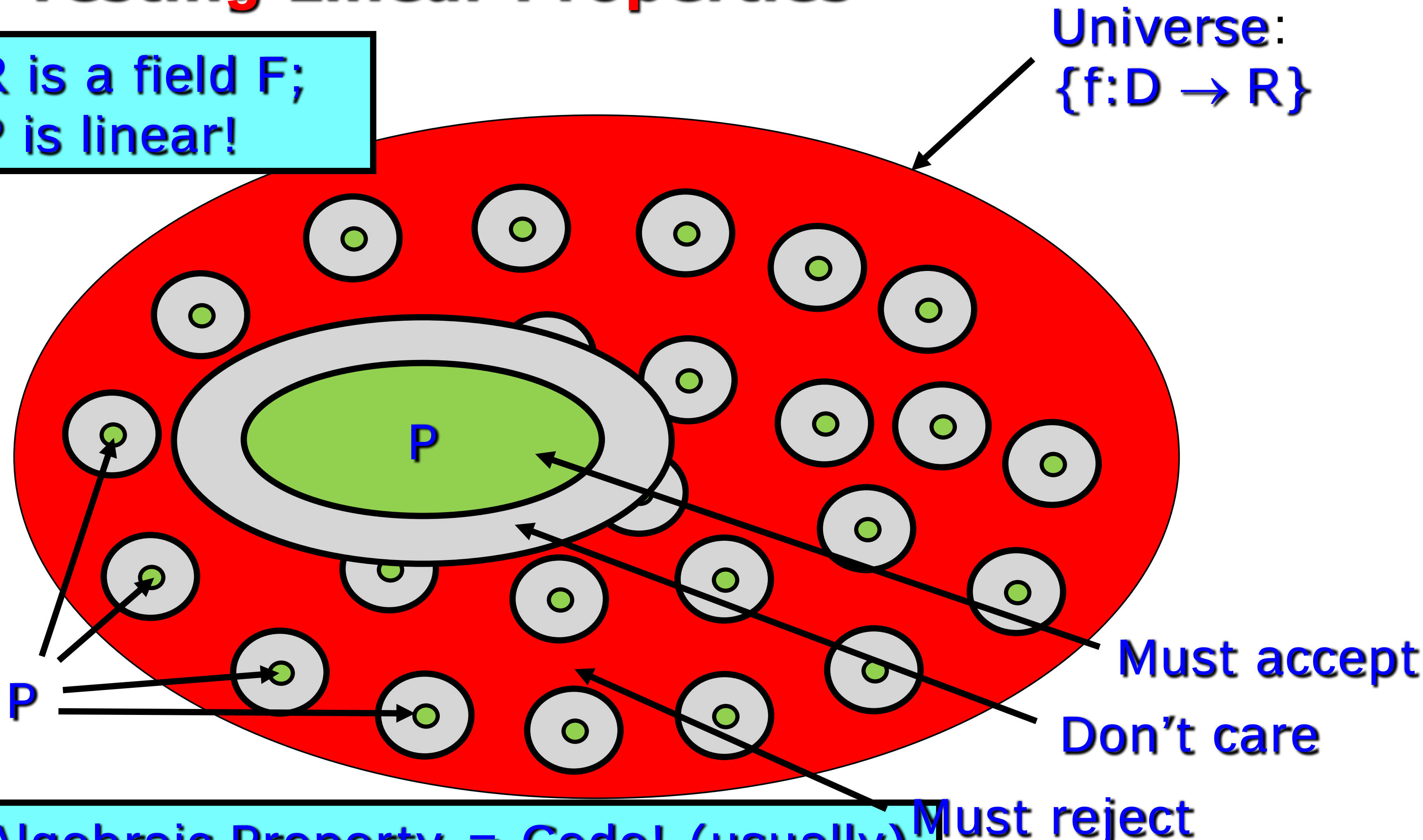
- Affine Invariance:
  - Domain = Big field ($GF(2^n)$)

    or vector space over small field ($GF(2)^n$).
  - Property invariant under affine transformations of domain ($x \mapsto A.x + b$)

- Linearity:
  - Range = small field ($GF(2)$)
  - Property = vector space over range.

# Testing Linear Properties

R is a field F;
P is linear!




Algebraic Property = Code! (usually)

# Why study affine-invariance?

- Common abstraction of properties studied in [BLR], [RS], [ALMSS], [AKKLR], [KR], [KL], [JPRZ].
  - (Variations on low-degree polynomials)

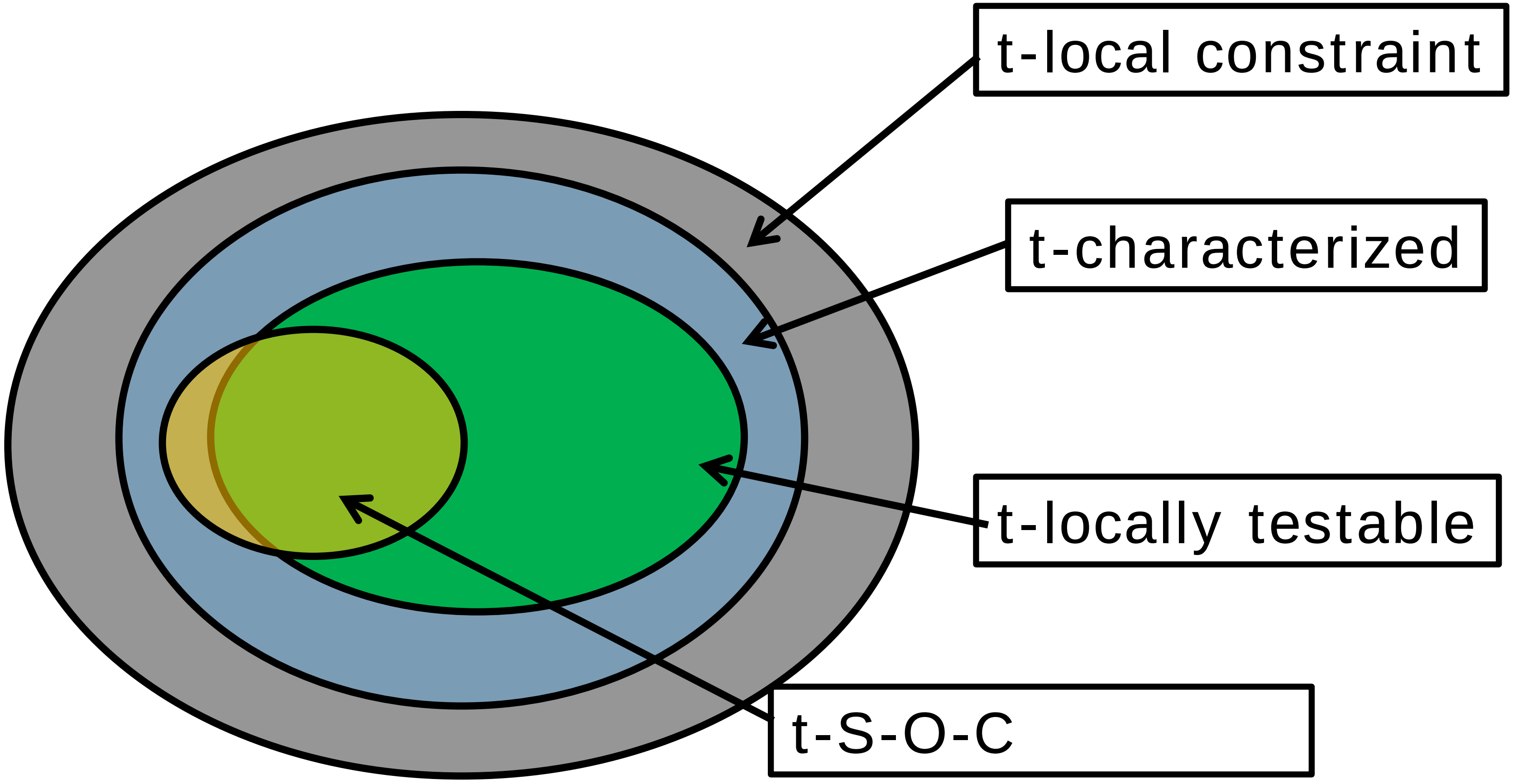
- Hopes
  - Unify existing proofs
  - Classify/characterize testability
  - Find new testable codes (w. novel parameters)

- Rest of the talk: Brief summary of findings

# Basic terminology

- Local Constraint:
  - Example: $f(1) + f(2) = f(3)$.
  - Necessary for testing Linear Properties [BHR]
- Local Characterization:
  - Example: $\forall\, x, y,\ f(x) + f(y) = f(x+y) \Leftrightarrow f \in P$
  - Aka: LDPC code, k-CNF property etc.
  - Necessary for affine-invariant linear properties.
- Single-orbit characterization:
  - One linear constraint + implications by affine-invariance.
  - Feature in all previous algebraic properties.

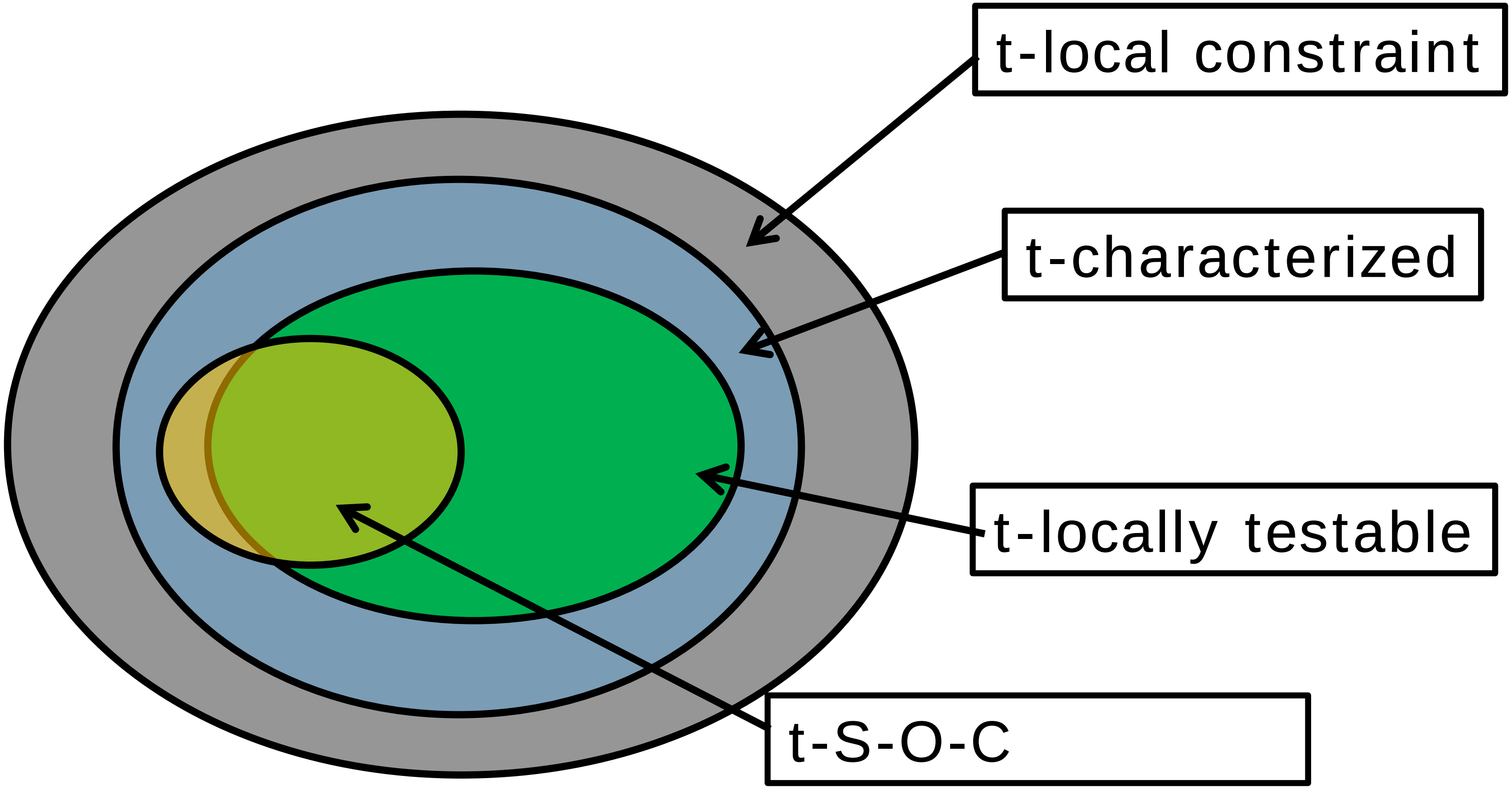
# Affine-invariance & testability



# State of the art in 2007

- [AKKLR]: t-constraint ⇒ t'-testable, for all linear affine-invariant properties?

# Affine-invariance & testability



# Some results

- [Kaufman+S.'07]: Single-orbit $\not\Rightarrow$ Testable.

# Proof: t-S-O-C ⇒ t-testable

- Property $P$ (k-S-O-C) given by $\alpha_1,\dots,\alpha_t$; $V \subseteq F^t$
- $P = \{f \mid f(A(\alpha_1)) \dots f(A(\alpha_t)) \in V,\ \forall$ affine $A: K^n \to K^n\}$
- $\mathrm{Rej}(f) = \mathrm{Prob}_A$ [ $f(A(\alpha_1)) \dots f(A(\alpha_t))$ not in $V$ ]
- Wish to show: If $\mathrm{Rej}(f) < 1/t^3$,

  then $\delta(f,P) = O(\mathrm{Rej}(f))$.

# Proof: BLR Analog

- $Rej(f) = Pr_{x,y} [\, f(x) + f(y) \neq f(x+y)] < \epsilon$
- Define $g(x) = \text{majority}_y \{Vote_x(y)\}$,
  where $Vote_x(y) = f(x+y) - f(y)$.
- Step 0: Show $\delta(f,g)$ small
- Step 1: $\forall$ x, $Pr_{y,z} [Vote_x(y) \neq Vote_x(z)]$ small.
- Step 2: Use above to show g is well-defined and a homomorphism.

# Proof: BLR Analysis of Step 1

- Why is $f(x+y) - f(y) = f(x+z) - f(z)$, usually?

| | | |
|---|---|---|
| ? | $f(z)$ | $-f(x+z)$ |
| $f(y)$ | 0 | $-f(y)$ |
| $-f(x+y)$ | $-f(z)$ | $f(x+y+z)$ |

# Proof: Generalization

- $g(x) = \bar{\beta}$ that maximizes, over $A$ s.t. $A(\alpha_1) = x$,

  $\Pr_A [\bar{\beta}, f(A(\alpha_2)),\ldots,f(A(\alpha_k)) \in V]$

- Step 0: $\delta(f,g)$ small.

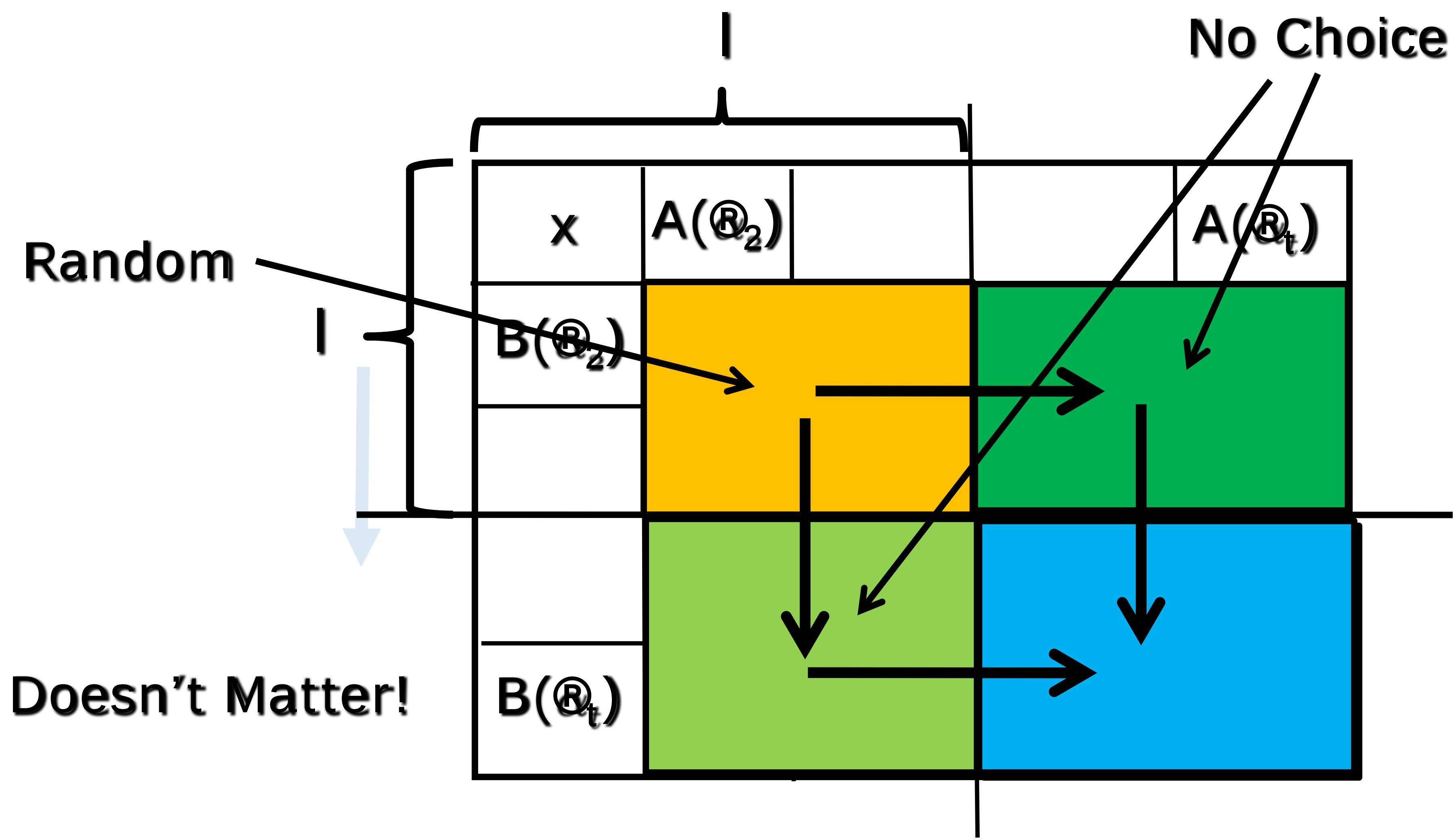
- $\text{Vote}_x(A) = \bar{\beta}$ s.t. $\bar{\beta}, f(A(\alpha_2))\ldots f(A(\alpha_k)) \in V$

  (if such $\bar{\beta}$ exists)

- Step 1 (key): $\forall\, x$, whp $\text{Vote}_x(A) = \text{Vote}_x(B)$.
- Step 2: Use above to show $g \in P$.

## Proof: Matrix Magic?

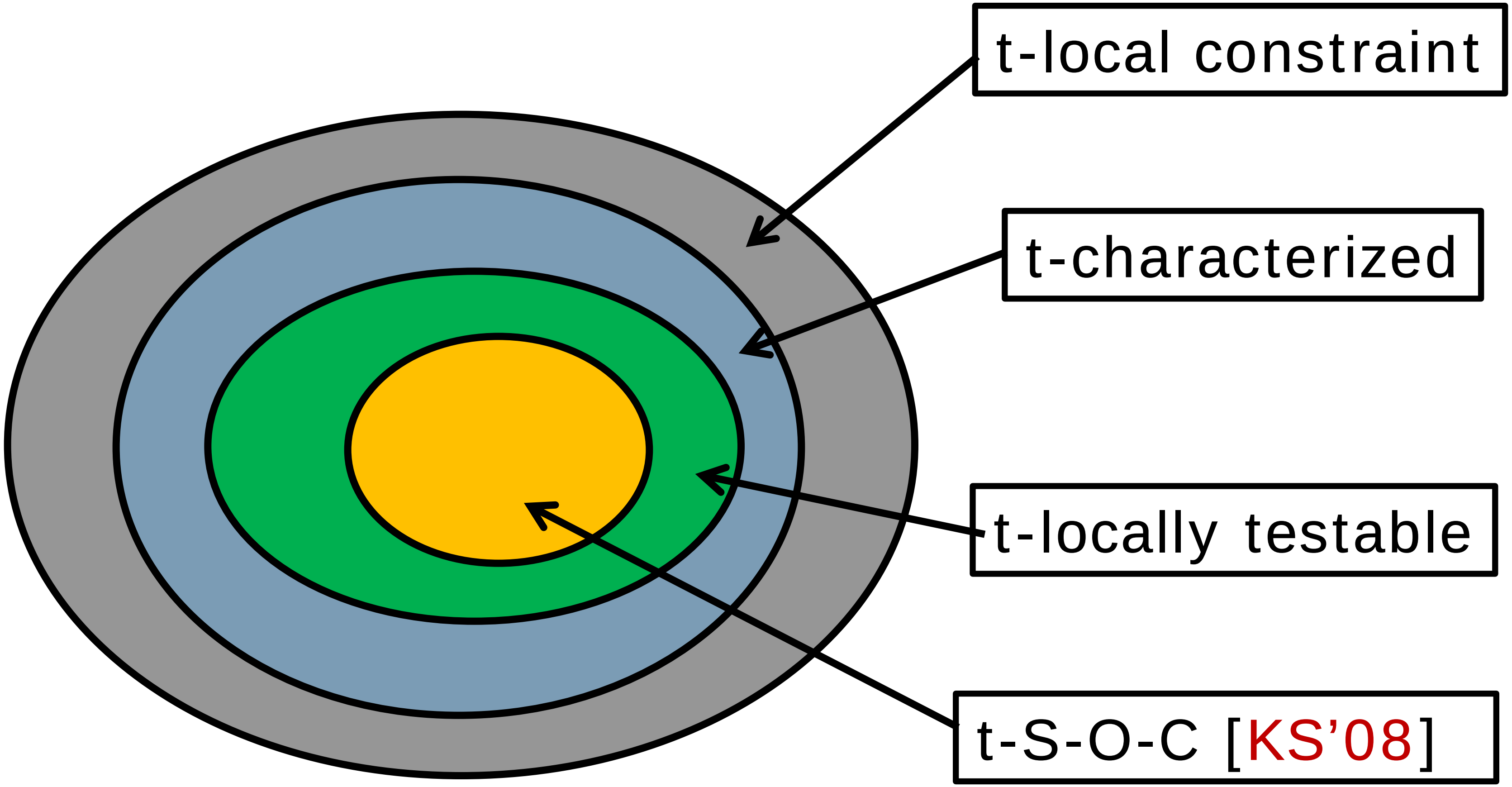
Say A(®$_1$) … A(®$_l$) independent;
rest dependent



# Some results

- [Kaufman+S.'07]: Single-orbit $\not\Rightarrow$ Testable.

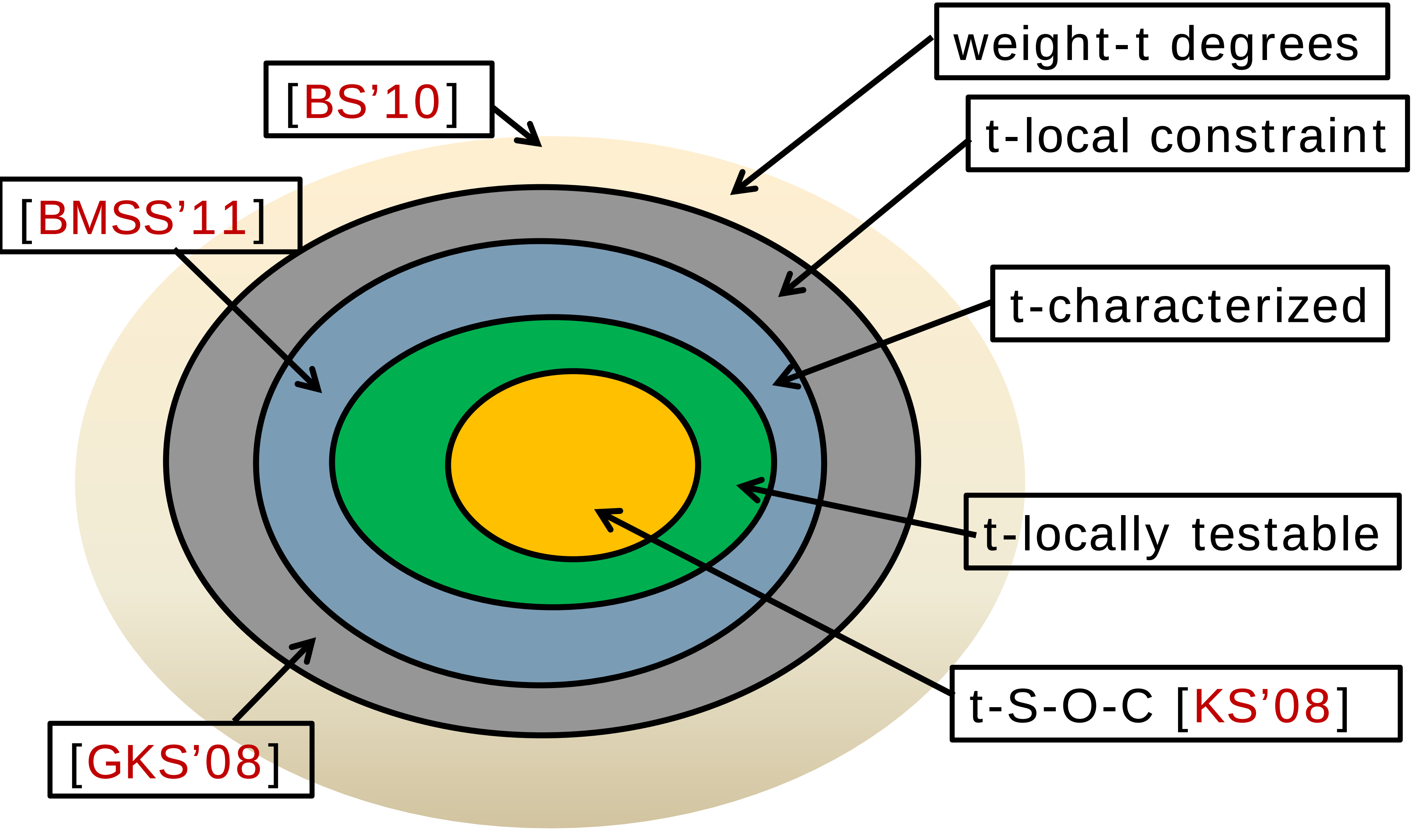
# Affine-invariance & testability



# Some results

- [Kaufman+S.'07]: Single-orbit $\Rightarrow$ Testable.
  - Unifies known algebraic testing results.
  - Converts testability to purely algebraic terms.
  - Yields "Constraints = Char. = Testability" for vector spaces over <u>small fields</u>.
  - Left open: Domain = <u>Big field</u>.
  - $\exists$ Many "non-polynomial" testable properties
- [GKS'08]: Over big fields, Constraint ≠ Char.
- [BMSS'11]: Over big fields, Char ≠ Testability.
- [BGMSS'11]: Many questions/conjectures outlining a possible characterization of affine-invariant properties.

# Affine-invariance & testability



# State of knowledge over big fields

- All known testable properties are S-O-C.

- If $|K| = |F^n|$, then the class of degree d n-variate polynomials is $(|F|^{d+1})$- S-O-C over K.

- [Kaufman-Lovett] If $P \subseteq \{K \rightarrow F_p\}$ has only $|K|^t$ members, then P is k(t,p)-S-O-C.

- Sums, Intersections, and "Lifts" of S-O-C properties are S-O-C.

# Quest in lower bound

- Proposition: For every affine-invariant property P, there exists a set of degrees D s.t.

  P = {polynomials supported on monomials in D}

- Quest: Given degree set D (shadow-closed, shift-closed) prove it has no S-O-C.

- Equivalently: Prove there are no

  $\lambda_1 \ldots \lambda_k \in F$, $\alpha_1 \ldots \alpha_k \in K$ such that

  - $\sum_{i=1}^{k} \lambda_i \alpha_i^{d} = 0$ for every $d \in D$.
  - $\sum_{i=1}^{k} \lambda_i \alpha_i^{d} \neq 0$ for every minimal $d \notin D$.

# Pictorially

$$M(D) = \left( \text{®}_1^{d} \quad \text{®}_2^{d} \quad \cdots \quad \text{®}_k^{d} \right)$$

Is there a vector $(\text{¸}_1,\ldots,\text{¸}_k)$ in its right kernel?

Can try to prove "NO" by proving matrix has full rank.

Unfortunately, few techniques to prove non-square matrix has high rank.

# Structure of Degree sets

- Let $D$ = degree set ($P$).

- $D$ Shift closed: Range = $F_q$ and $d \in D \Rightarrow q.d \in D$.

- $D$ Shadow closed: Let $p$ = char($q$) and $d$ in $D$.
  Then every $e$ in $p$-shadow of $d$ is also in $D$.

  - $e$ in $p$-shadow of $d$ if every digit in base $p$ expansion is smaller.

# Non-testable Property - 1

- AKKLR (Alon,Kaufman,Krivelevich,Litsyn,Ron) Conjecture:
  - If a linear property is 2-transitive and has a $k$-local constraint then it is testable.
  - [GKS'08]: For every $k$, there exists affine-invariant property with 8-local constraint that is not $k$-locally testable.
  - Range = $GF(2)$; Domain = $GF(2^n)$
  - $P = \mathrm{Fam}(\mathrm{Shift}(\{0,1\} \cup \{1+2, 1+2^2, \ldots, 1+2^k\}))$.

# Proof (based on [BMSS'11])

- $F = GF(2)$; $K = GF(2^n)$;
- $P_k = \mathrm{Fam}(\mathrm{Shift}(\{0,1\} [\cup \{1 + 2^i \mid i \in \{1,\ldots,k\}\}))$
- Let $M_i =$ $$\begin{pmatrix} \circledR_1^{\,2} & \circledR_2^{\,2} & \cdots & \circledR_k^{\,2} \\ \\ \circledR_1^{\,2^i} & \circledR_2^{\,2^i} & \cdots & \circledR_k^{\,2^i} \end{pmatrix}$$
- If $\mathrm{Ker}(M_i) = \mathrm{Ker}(M_{i+1})$, then $\mathrm{Ker}(M_{i+2}) = \mathrm{Ker}(M_i)$
- $\mathrm{Ker}(M_{k+1})$ = would accept all functions in $P_{k+1}$
- So $\mathrm{Ker}(M_i)$ must go down at each step, implying $\mathrm{Rank}(M_{i+1}) > \mathrm{Rank}(M_i)$.

# Stronger Counterexample

- GKS counterexample:
  - Takes AKKLR question too literally;
  - Of course, a non-locally-characterizable property can not be locally tested.

- Weaker conjecture:
  - Every $k$-locally characterized affine-invariant (2-transitive) property is locally testable.
  - Alas, not true: [BMSS]

# [BMSS] CounterExample

- Recall:
  - Every known locally characterized property was locally testable
  - Every known locally testable property is S-O-C.
  - Need a locally characterized property which is (provably) not S-O-C.
  - Idea:
    - Start with sparse family $P_i$.
    - Lift it to get $Q_i$ (still S-O-C).
    - Take intersection of superconstantly many such properties. $Q = \cap_i Q_i$

# Example: Sums of S-O-C properties

- Suppose $D_1 = Deg(P_1)$ and $D_2 = Deg(P_2)$
- Then $Deg(P_1 + P_2) = D_1 \cup D_2$.
- Suppose S-O-C of $P_1$ is $C_1$: $f(a_1) + \ldots + f(a_k) = 0$; and S-O-C of $P_2$ is $C_2$: $f(b_1) + \ldots + f(b_k) = 0$.
- Then every $g \in P_1 + P_2$ satisfies:

  $\sum_{i,j} g(a_i\ b_j) = 0$
- Doesn't yield S-O-C, but applied to random constraints in orbit($C_1$), orbit($C_2$) does!
  - Proof uses $wt(Deg(P_1)) \leq k$.

# Hopes

- Get a complete characterization of locally testable affine-invariant properties.

- Use codes of (polynomially large?) locality to build better LTCs/PCPs?
  - In particular move from “domain = vector space” to “domain = field”.

- More broadly: Apply lens of invariance more broadly to property testing.

# Thank You!